SET-MEMBERSHIP NON-LINEAR OBSERVERS WITH APPLICATION TO VEHICLE LOCALISATION

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Abstract

This communication proposes to re-consider the localization problem (i.e. the configuration estimation) of a moving car as the state estimation of a discrete-time non-linear process. The use of set-theoretic concepts for state estimation is presented under the form of the computation of the direct and reverse images of a set by a function using interval analysis.

One simplification of the computation of the direct image is shown for a class of non-linear systems and enables to reduce the computational complexity of the algorithm. The considered application belongs to this class and the method is applied to experimental data.

1 Introduction

This communication presents an application of set-membership estimation to the state estimation of discrete-time non-linear systems. The method follows the approach introduced by [1],[2] and introduces a technical improvement to reduce its computational complexity. A real world non-linear application illustrates the proposed method: the dynamic localization of a car equipped with GPS and angular encoders used by the ABS.

By virtue of a new way to formulate classical computation problems, set-membership estimation gives new insights and new answers to old questions.

Basically, set-membership estimation finds the solution of systems of inequalities by defining the *set of all possible solutions*.

A state-observation problem is naturally formulated as a system of inequalities when the measurement noise is defined by a bounded error model: $\{|y_{\text{mes}} - y_{\text{true}}| \leq \beta\}$. As finding the exact solution is impossible in the general case, approximate solutions have been defined. The most popular ones are called ellipsoidal algorithms that compute an outer ellipsoidal envelop of the solution. This kind of methods introduced by [3] for parameter identification have further been developed under the form of an algorithm structurally comparable to Kalman filtering [4] for the state estimation of linear stationnary systems. For non-linear systems, the use of interval computation and topological concepts led to the definition of the SIVIA algorithm [5]. Basically, this algorithm brackets the solution of a non-linear system of inequalities included inside an initial search orthotopic domain between two subpavings (a subpaving is a finite union of boxes inside the search domain). The solution found by this way is guaranteed and both its existence and its accuracy depends upon the convenient definition of an inclusion function. We claim that this guarantee has the same importance that existence and unicity results in the classical state estimation theory [6]. Existence of the solution is answered by the guarantee to find (and compute) any possible solution inside a search domain chosen by the user. Unicity is not required a priori and this has been shown as an advantage for state estimation problems such as mobile localization of mobile robots inside indoor environments.

This paper is organized as follows: the first section introduces the practical problem of localization as the state estimation of a non-linear process. With this aim in mind, section 3 presents first the principle of set-membership estimation applied to the recursive state estimation. Computational aspects are then considered and a strategy to decrease the complexity is proposed for a particular class of systems. The considered application belongs to this case. Experimental results obtained by this way are shown in the last section and compared with those of Extended Kalman Filtering that is the classical method used in this case.

2 Practical problem statement

Let us consider the localisation problem of a usual car with 4 wheels (Fig. 1) in a 2D world. Localisation consists of estimating the configuration $q = [x, y, \theta]^{\mathrm{T}}$ where the characteristic point **C** is the centre of the rear axle of a vehicle. The choice of the reference frame is part of the definition of the task to be performed. The one discussed in this work is the navigation on a road network, then the reference frame used in this case is a local (x,y) tangential plan to the earth surface (French Lambert I conic projection). This system of coordinates is the projection in the local frame of the WGS84 conventional reference frame used by the GPS.

The physical sensors considered in this paper are the two rear ABS encoders and a natural 8-channel GPS receiver. The measurements they provide are the elementary rotation $(d\xi_{rl}, d\xi_{rr})$ of each wheel and the GPS location (x, y).

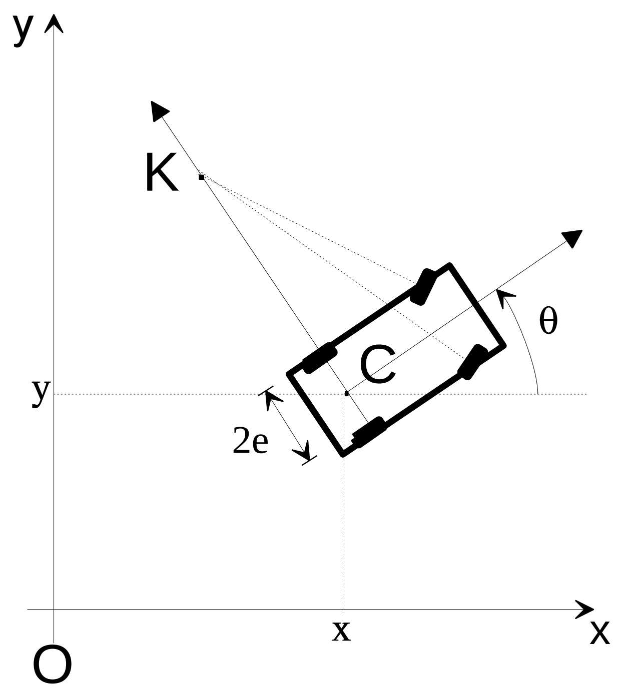

Figure 1: 4 wheels model

2.1 Odometry: evolution model

The common models used for vehicle control and observation are so-called cart or bicycle models. In the bicycle case, the two front (resp. rear) wheels are represented by a single fictive wheel whereas the cart model considers only the motions of the rear wheels. The bicycle model implicitly assumes that the yaw rate of the vehicle is measured by a specific sensor, either by a gyrometer fixed to the body or by an angle encoder tied to the driving-wheel together with the knowledge of the velocity. We prefer then use the cart model since the odometric data $(\dot{\xi}_{rl}, \dot{\xi}_{rr})$ is accessible on most modern cars: no additional sensors are needed.

This model is obtained by assuming that the axis of the rear-wheels is fixed with respect to the body of the car and that the wheels are rolling without slipping. As a consequence, the speed of their center is perpendicular to their common axle. It yields the following differential relations:

$$\left\{\begin{array}{lcl} \dot{x} & = & v.\cos\theta \\ \dot{y} & = & v.\sin\theta \\ \dot{\theta} & = & R.(\dot{\xi}_{rr} - \dot{\xi}_{rl})/(2e) \\ v & = & R.(\dot{\xi}_{rl} + \dot{\xi}_{rr})/2 \end{array}\right. \qquad (1)$$

Since, for computational reasons, state estimation will be performed in discrete time we propose to consider the following model (2) that is argued on the fact that the trajectory of the characteristic point **C** is locally circular [7].

$$\left\{\begin{array}{lcl} x_{k+1} & = & x_k + d_{k+1}\cos(\theta_k + r_{k+1}/2) \\ y_{k+1} & = & y_k + d_{k+1}\sin(\theta_k + r_{k+1}/2) \\ \theta_{k+1} & = & \theta_k + r_{k+1} \end{array}\right. \qquad (2)$$

with $\left\{\begin{array}{lcl} d_k & = & R.(d\xi_{rl} + d\xi_{rr})/2 \\ r_k & = & R.(d\xi_{rr} - d\xi_{rl})/(2.e) \end{array}\right.$

where R is the radius of the wheels and e the half-track.

Note that in both models (1) and (2), the odometric measurements are considered as inputs. The state is the configuration $[x, y, \theta]^T$ to be estimated.

2.2 Observation model

The GPS measures directly the position (x_k, y_k) of the car, nevertheless the basic accuracy is not good (around 10 meters since May 1, 2000). As the measurement errors are temporally correlated, the accuracy of the displacement ($\Delta x_k^{GPS} = x_k^{GPS} - x_{k-1}^{GPS}$, $\Delta y_k^{GPS} = y_k^{GPS} - y_{k-1}^{GPS}$) is much better (around 15 centimeters).

Considering the evolution model (2), the relation between the displacement measurement $(\Delta x_k^{GPS}, \Delta y_k^{GPS})$, the state (x, y, θ) and the inputs (d_k, r_k) is:

$$\left\{\begin{array}{lcl} \Delta x_k^{GPS} & = & d_k\cos(\theta_k - r_k/2) \\ \Delta y_k^{GPS} & = & d_k\sin(\theta_k - r_k/2) \end{array}\right. \qquad (3)$$

Finally, the observation equation we will consider in the sequel is:

$$z_k = \left\{\begin{array}{lcl} x_k^{GPS} & = & x_k \\ y_k^{GPS} & = & y_k \\ \Delta x_k^{GPS} & = & d_k\cos(\theta_k - r_k/2) \\ \Delta y_k^{GPS} & = & d_k\sin(\theta_k - r_k/2) \end{array}\right. \qquad (4)$$

One can notice that both the evolution equation and the observation equation are non linear. The usual way to attack this problem consists to deny it by using local linearization. In the sequel, we propose to show how the use of set-membership estimators makes it possible to avoid any approximation of the discrete-time non-linear model (2) and (4).

3 Theory and practice of Set-Membership state Estimation

Set-membership estimation consists of finding, in the state space, the domain that is compatible with both the measurements and a given state space model of the system. This methodology has first been introduced within the framework of Bounded Error Estimation [3],[8] applied to linear systems. In such a case, the inaccuracy of the observation is modeled by defining, around each scalar measurement, a bounded interval inside which the true value certainly lies. The result of this estimation is expressed by an ellipsoid that certainly contains the true value of the estimated parameters. This is a characteristic of Set-membership estimation that no additional criterion (probability or other) is introduced and that all points inside the feasible state space domain are equally candidate to be the true value. In the realm of linear stationnary systems, this technique has led to a state estimation algorithm that is a real alternative to Kalman filtering [4].

The general case of non-linear systems has been revisited using *interval arithmetic*. Interval arithmetic has first been introduced [9] to model the inaccuracy of computer computations resulting from the finite length of the binary representation of numbers. Further, it has been used for global optimization [10] and parameter estimation [5]. The essential result of this last research is a set inversion algorithm SIVIA[1] devoted to compute the solution $\mathcal{X}$ of a general non linear system (5).

$$\left|\begin{array}{l}\mathcal{X}=\{x; \exists\, y\in\mathcal{Y};\, y=f(x)\},\\ f \text{ is a computable application from } \mathbb{R}^n \text{ into } \mathbb{R}^m,\\ \text{the test } \{y\in\mathcal{Y}\} \text{ is computable.}\end{array}\right. \quad (5)$$

This algorithm has been used fruitfully in model identification or robust control design. Following this research axis, Kieffer et al. [1] have recently extended this technique to the recursive solution of state estimation. The principle of the solution consists, like for the classical Kalman filtering, to alternate the evolution and observation stages.

3.1 Principle of set-membership estimation

For pedagogical purpose, we propose to introduce the characteristics of set-membership estimation by an academic example. It will show how work the estimation and evolution stages and how their results are expressed.

Consider the linear model (6) where $x_k \in \mathbb{R}^2$ and u_k is a measured scalar input and z_k is the observation.

$$\begin{array}{l}\left\{\begin{array}{lcl}x_{1,k+1} & = & x_{1,k}+ax_{2,k}\\ x_{2,k+1} & = & x_{2,k}+bu_k\end{array}\right.\\ z_k = g.x_{1,k}\end{array} \quad (6)$$

The noise of the sensor is modeled in bounded error form and we propose to consider also the calibration error on the gain $g > 0$.

$$|z - g.x_1| \leq 1 \quad (7)$$

$$|g - g_o|/g_o \leq 5\% \quad (8)$$

With this observation model, a given value of the state $x_k \in \mathbb{R}^2$ yields a measurement $z_k \in \mathbb{R}$ whose value is indeterminate inside a known interval[2]

$\mathbf{z}_k \subset \mathbb{R}$.

Conversely, a scalar measurement m_k issued by the sensor is interpreted as a feasible interval $\mathbf{m}_k \subset \mathbb{R}$ that corresponds to a feasible domain $D_k \subset \mathbb{R}^2$ that certainly contains the true state x_k.

The evolution stage consists of finding the domain E_{k+1} that can be reached at time $k+1$ using the evolution model (6) and knowing the domain X_k at time k and the input domain U_k.

Consider for instance an interval $U_k = [\underline{u}_k, \overline{u}_k]$ and a generalized interval: $X_k = \{x_k = [x_{1,k}x_{2,k}]^T; x_{1,k} \in [\underline{x}_{1,k}, \overline{x}_{1,k}]$ and $x_{2,k} \in [\underline{x}_{2,k}, \overline{x}_{2,k}]\}$.

Using the evolution model (6), one gets the domain E_{k+1} displayed on Fig. 2.

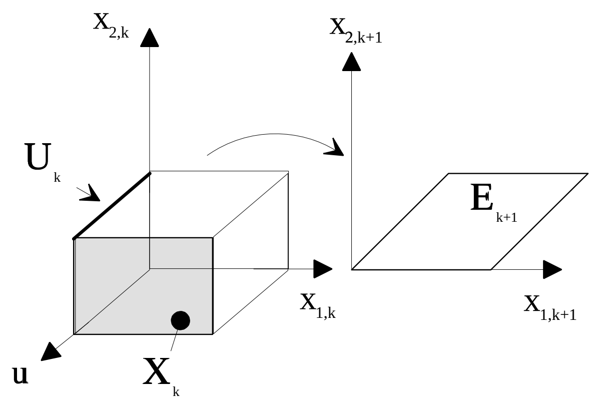

Figure 2: Evolution of X_k in E_{k+1} using the process model

The observation stage consists of finding the intersection X_k between the current estimated domain E_k in the state space and the domain D_k compatible with the measurement z_k.

In our example, the domain D_k is a band. Its intersection with the domain E_k gives the trapezium X_k (See Fig. 3)

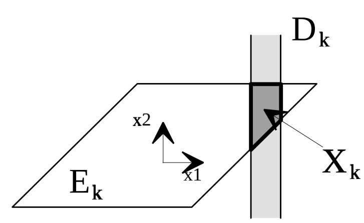

Figure 3: Estimation of X_k using E_k and the measurement

This technique is easily extendable to the non-linear observation equation: let imagine now that the previous sensor doesn't give the sign of the measurement then, we have $z = |g.x|$.

The inversion of this equation gives now two bands. The intersection with the prediction E_k can give two disconnected domains.

In this case, the method deals successfully with the non-uniqueness of the solution: two domains aren't discernable since they produce the same output. This academic example shows briefly a phenomenon that has been encountered in the real problem of absolute robot localization [2],[11].

[1] Sivia : Set Inversion Via Interval Analysis

[2] $\mathbf{z}_k = \left\{\begin{array}{l}[-1+0.95g_o.x_{1,k}\ ;\ 1+1.05g_o.x_{1,k}] \text{ for } x_{1,k}\geq 0\\ [-1+1.05g_o.x_{1,k}\ ;\ 1+0.95g_o.x_{1,k}] \text{ for } x_{1,k}< 0\end{array}\right.$

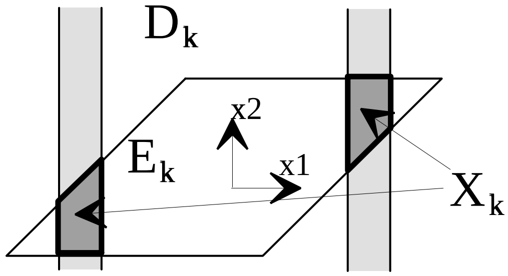

Figure 4: Observation stage using $z = |g.x|$

As a conclusion, consider now the general non-linear process (9), set-membership state estimation alternates two stages (evolution and observation). The result of each stage is a (possibly disconnected) set. Two operations are necessary to perform this: the computation of the image of a set in the evolution stage and the computation of the inverse-image of a set in the observation stage.

$$\left\{ \begin{array}{rcl} x_{k+1} & = & f(x_k, u_k) \\ z_k & = & h(x_k) \end{array} \right. \qquad (9)$$

3.2 computational aspects

A technique to represent domains of any form consists to bracket them between two sets called *subpavings* that are finite unions of boxes, where a box $\mathcal{B}$ is a generalized interval $\mathcal{B} \subset \mathbb{R}^n$; $\mathcal{B} = \{x \in \mathbb{R}^n ;\ \forall i = 1 \ldots n\ ,\ x_i\ \in [\underline{x}_i, \overline{x}_i]\}$.

The outer subpaving encloses the true domain and the inner subpaving when it is not empty, guarantees that the true domain exists. In the sequel, we shall use the interval arithmetic and inclusion fonctions $f_{[]}$ of usual functions f. An inclusion function makes a correspondance between any generalized interval (box) $\mathbf{X}$ and another box $\mathbf{Y}$ that contains the exact image $f(\mathbf{X})$. In the sequel, inclusion function will be designed as convergent, i.e. $\lim_{w(\mathbf{X}) \to 0} w(f_{[]}(\mathbf{X})) = 0$ with $w : \mathbb{R}^n \to \mathbb{R}$; $w(\mathbf{x}) = \sup_{A \in \mathbf{x}, B \in \mathbf{x}} ||\overrightarrow{AB}||$.

Paradoxically, computing the inverse image of a set is easier than computing the direct image.

3.2.1 Inverse problem

It is dealt with using the SIVIA algorithm the principle of which will be shown by reference to the figure 5. It gives an exterior and an interior subpaving of the state space with a given precision.

Let a function $h : X \to Y = h(X)$ and a domain $\mathcal{Y}$, we search to determinate the domain $\mathcal{X}$ of the antecedents of points included in $\mathcal{Y}$.

Using inclusion function forall box in X, it is possible to evaluate an including box $h_{[]}(X) \supset h(X)$.

If this domain is included in $\mathcal{Y}$ (for instance Y3 on Fig. 5), the initial box (here X3) is put in the list of the inner subpaving,

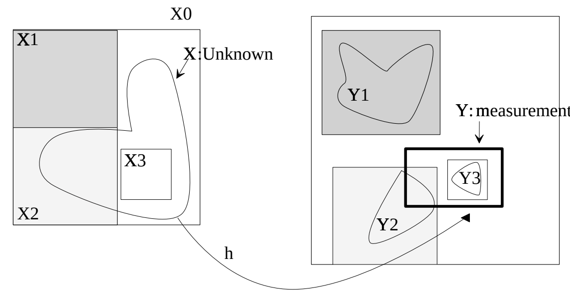

Figure 5: Sivia algorithm

and if $f(X)$ does not intersects $\mathcal{Y}$ (for instance X1), then it is classed in the outer subpaving. In the last case (X2), the box is divised into two parts unless its larger dimension is smaller than a given precision threshold.

3.2.2 Direct problem

After the previous observation stage (or after the initialization), the feasible state space domain is bracketed between two subpavings (that are list of boxes). Let us use again inclusion functions; it is possible to compute an outer envelop of each box. Some problems arise at this level. The first one is the overvaluation of the solution when one computes the image of a large box: on Fig. 6 the direct image of the larger box on the left is the dotted rectangle on the right, that is not a tight enclosure of the true domain (parallelogram). The second one arises from cutting the initial box into smaller boxes: the number of small boxes is increased exponentially with the dimension of the number of variable. The last one is still a complexity problem and stems from the overlapping of the images of the smaller boxes by the inclusion function.

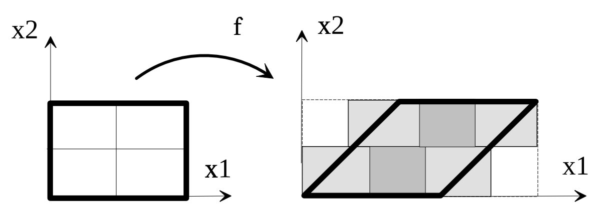

Figure 6: Evolution stage with cutting in all directions

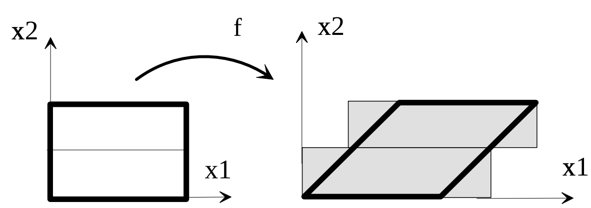

Figure 7: Evolution stage with cutting in the pertinent direction

To reduce the complexity, we try to optimize the cutting stage. For that, remark on Fig. 6 and Fig. 7 that it is unnecessary to cut the initial box in the direction of variables that appear only once in the evolution equation (x_1 in the Eq. 6).

This property can be proved easily:

Let f such that x_i appears only in $f_j(x)$ and only once in $f_j(x)$ ($f_{[]j}(x)$ is a minimal including function [12] for the variable x_i).

Let $\mathrm{x} = \begin{bmatrix} [\underline{x}_1 \ , \ \overline{x}_1] \\ \vdots \\ [\underline{x}_n \ , \ \overline{x}_n] \end{bmatrix}$ be a box.

Let us cut this box in the x_i direction:

$$\mathbf{x}_i = \underbrace{[\underline{x}_i \ , \ x_i]}_{x_i^L} \cup \underbrace{[x_i \ , \ \overline{x}_i]}_{x_i^R}$$

then:

$$f_{[]}([x^L]) = \begin{bmatrix} \dots \\ f_{[]j}([x^L]) \\ \dots \end{bmatrix} \quad f_{[]}([x^R]) = \begin{bmatrix} \dots \\ f_{[]j}([x^R]) \\ \dots \end{bmatrix}$$

if $k \neq j$, $f_{[]k}([x^L]) = f_{[]k}([x^R])$ because x_i is not an argument of f_k.

if $k = j$, $f_{[]k}([x^L]) \cup f_{[]k}([x^R]) = f_{[]k}([x])$ because $f_{[]k=j}$ is a *minimal function* for x_i.

Finally:

$$f_{[]}([x^L]) \cup f_{[]}([x^R]) = f_{[]}([x])$$

The cutting in the x_i direction is therefore unnecessary.

This remark is of paramount important in the considered application since the variables x and y appear only once in the evolution equation (2). The number of variables that must be cut is only 3 (θ_k, d_k, r_k) whereas it was initially 5 $(x_k, y_k, \theta_k, d_k, r_k)$.

4 Application to the accurate localization problem

In this paragraph, the car localisation problem exposed in section 2 is solved using a Set-Membership Estimator (SME). Experimental results obtained with a laboratory-car are compared with the usual Extended Kalman Filter (EKF) estimation technique. The measurements of the ABS encoders and the ones of the GPS were stored while the car was moving. The sampling frequency was $1Hz$ and the maximum speed of the car was $50km/h$. The estimations were computed afterwards. The "real" trajectory (Fig. 8) was obtained by fusing the GPS data with the data of a fixed receiver (kinematic differential GPS with phase treatment). This DGPS referenced trajectory is used to compare the results of each method. Its precision is in the order of two meters.

The EKF was tuned with the usual assumptions: the measurement errors are zero-mean and temporally uncorrelated. The second assumption is violated here since the GPS errors have

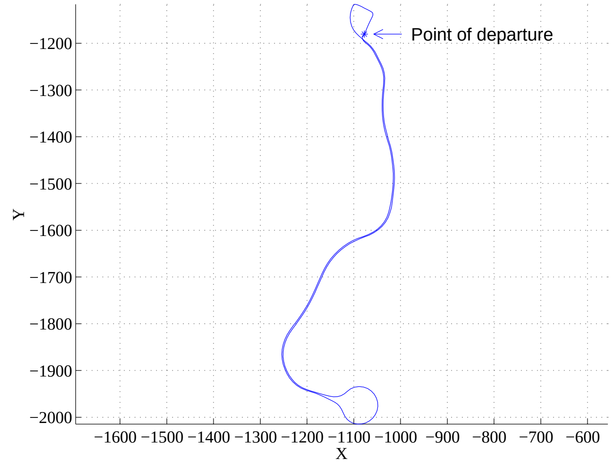

Figure 8: Experimental 2.1 km long trajectory

a correlation time approximately equal to two hours (for a first order Auto-Regressive model). Please recall that this assumption is not necessary for the SME.

Moreover, because of the non-negligible thinness of the tires, the instantaneous value of the half-track "e" is not well-known. This phenomenon is easy to take into account with the SME formalism, as it has been shown for the sensitivity "g" in the section 3.1. As the EKF needs a scalar value, this source of disturbance is neglected.

At each sampling time, the EKF computes an estimation of the state together with the corresponding covariance matrix. On the other hand, the SME determines interval bounds of each component of the state. In order to compare the results of the two estimation processes, we propose to consider the 3σ-bounds (99 % confidence bounds) of the EKF.

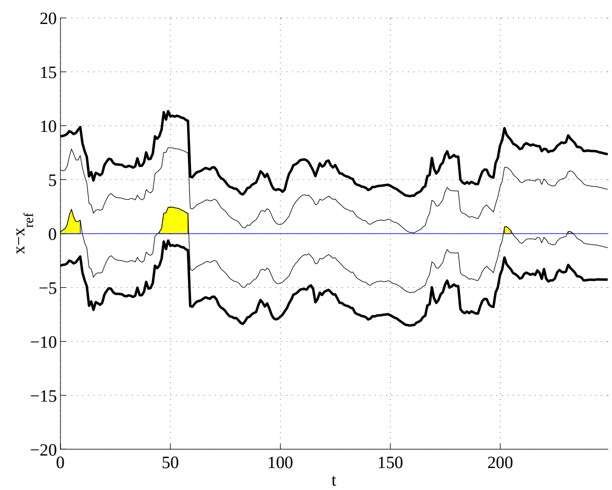

Figure 9: Confidence bounds of the x estimations

Figures 9 and 10 report the *errors* of the two estimators as regarding the x and y components. The SME estimations are plotted with thick lines. The thin lines correspond to the EKF. This results shown that the real value of the state vector goes out sometimes of the 3σ-bound of the EKF (when zero is not inside the EKF generated interval). This phenomenon indicates a failure of the algorithm based on the EKF. On the other hand, the SME interval estimation is always consistent with the real value. Moreover, one can notice that the real value is rarely at the centre of the interval estimation.

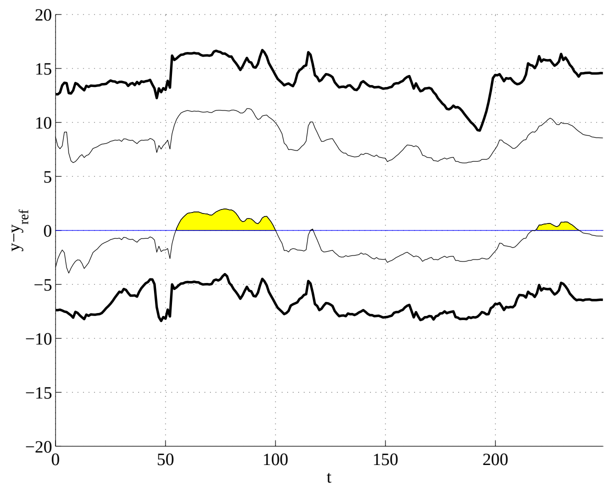

Figure 10: Confidence bounds of the y estimations

5 Conclusion

This communication has shown how set-membership concepts and techniques gives an intuitive and efficient solution for the state estimation of non-linear systems. The simplicity of the solution is particularly remarkable if one reminds that state estimation of non-linear systems is considered as a difficult subject. The strength of such a solution is reinforced by the fact that interval analysis makes it possible to build algorithms that yields guaranteed solutions under the form of an inclusion between two subpavings. The corresponding drawback of this approach is the computational complexity that is exponential in the number of coupled variables. The technical remark pointed out in this communication enables to reduce the computational complexity for some particular structures of non-linear systems among which the practical car localization problem that we have considered. In the application to real world data, we have shown that the estimated state domains were always consistent with the experimental data, that the estimated heading was less noisy than that given by the algorithms performed by commercial GPS.

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