A Road Reduction Method using Multi-Criteria Fusion

Maan E. El Najjar, Philippe Bonnifait

Heudiasyc UMR 6599. Université de Technologie de Compiègne.
BP 20529, 60205 Compiegne Cedex, France.
Tel. 33 (0)3 44 23 44 23 – Fax. 33 (0)3 44 23 44 77
maan.el-najjar@hds.utc.fr

Abstract

To localize a vehicle on a map, a reliable road selection system is essential. For this purpose, this article presents a credibilist multi-criteria association algorithm that performs data association between the infrastructure information (the map) and the noisy measurements of two sensors (DGPS, Odometer). The algorithm takes into account the inaccuracy, the uncertainty and the redundancy of the data. The multi-criteria fusion process is realized using Belief Theory and Dempster-Shafer's rule. A local strategy is developed to allot believes to two criteria. Experimental results show that the credibilist roads around an estimated position are well selected.

Keywords : Map-Matching, Multi-Sensor Fusion, Belief Theory, Uncertainty Modeling, Vehicle Localization, GPS, GIS.

1 Introduction

Advanced on-vehicle sensors, such as radar or video cameras, have made great advances in giving a vehicle some local representation of its situation with respect to the road and the other vehicles. However, these sensors are inherently limited by their range. For example, an obstacle detection system may fail because of a sharp bend. Using an accurate digital map with DGPS may be a solution to enable the provision of advance warning to drivers of features that are beyond current visibility.

Any perceptive method, dedicated to a safety application, must provide judicious information in order to take a decision. This perception function is generally realized with a set of homogeneous or heterogeneous sensors. It provides the decision module an image of the physical environment observed. This representation cannot be perfect because it is built with data from inaccurate and uncertain information sources. Moreover, the representation of the environment is perhaps erroneous if the information source is degraded, or if it is subject to harmful external influences. In all these cases, the system has to consider the inaccuracy and the uncertainty of the data and the reliability of the sensors.

Current commercially-available maps, such as those provided by Navigation Technologies Inc., have a good coverage for many countries around the world. Their accuracy and details will be improved in the next 5-10 years. Many studies have been carried out in this field as exemplified by (Rogers, 2000).

In this paper, we anticipate the potential performance of the positioning systems by using a DGPS receiver and a precise map "Géoroute" provided by the French National Institute of Geography (IGN). The relative precision of the IGN data-base is better than 5 meters and that of the DGPS is close to one meter (with more than 5 satellites in view and in an optimal configuration). Moreover, we use the ABS sensors to help the DGPS receiver when the satellite constellation is degraded (Bonnifait et al. 2001).

The goal of this research is to develop a robust method to localize the vehicle. We focus on the selection of the roads (also called Road Reduction Filter) by using a data fusion technique based on Belief Theory and a fuzzy information representation. To carry out this fusion process, many architectures are possible: centralized (global processing), decentralized (local processing), open-loop or closed-loop (i.e. the history of the trajectory of the vehicle is used or not). Usually, the choice of one architecture rather than another depends on precision, sensitivity to measurements degradation, computing complexity and load of communication. We propose a decentralized an open-loop architecture because it is a good response to the constraints of information conservation and reliable fusion. In fig. 1, x and y correspond to the position co-ordinates, q is the heading of the vehicle, P_{xy} is the co-variance matrix of the vector $(x,y,q)^t$, v is the speed and S_i is the *i*th segment of a road center-line.

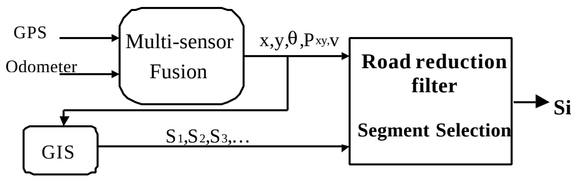

Fig. 1. Architecture of the selection process.

In order to speed up the treatments (a map contains thousands of roads), we apply a filter which selects the road segments that are located within a radius of 100 meters. The center of the circle is the estimate of the current position (x, y).

The paper is organized as follows. In section 2, an Extended Kalman Filter that fuses the DGPS and the odometer is described. Thanks to it, an estimation of the heading angle is computed with its associate variance. The Road Reduction Method is presented in section 3. The strategy proposed fuses two criteria using the Belief theory. The last section is dedicated to analyze several experimental results.

2 Sensor Fusion of DGPS and Odometry

Consider a car like vehicle. The mobile frame ***M*** is chosen with its origin *M* attached to the center of the rear axle (Fig. 2). The x-axis is aligned with the longitudinal axis of the car.

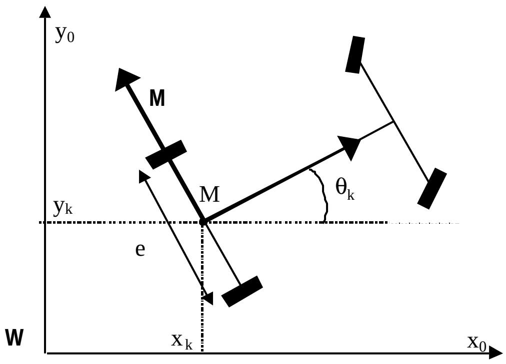

Fig. 2. Mobile frame attached to the car.

A time t_k, the vehicle position is represented by the (x_k,y_k) cartesian coordinates of *M* in a world frame ***W***. The heading angle is denoted θ_k. If the road is perfectly planar and horizontal and if the motion is locally circular, the evolution model is expressed by:

$$\begin{cases} x_k = x_{k+1} + \Delta.\cos(q_k + w/2) \\ y_k = y_{k+1} + \Delta.\sin(q_k + w/2) \\ q_k = q_{k+1} + w \end{cases} \quad (1)$$

where *D* is the length of the circular arc followed by *M*, *w* the elementary rotation of the mobile frame. These values are computed using the ABS measurements of the rear wheels. Let denote $u=[D, w]^t$ and $X=[x, y, \theta]^t$. Eq. (1) can be rewritten as:

$$X_{k+1} = f(X_k, u_k) + \alpha_k \quad (2)$$

When a DGPS position *Y* is available, a correction of the odometric estimation is performed using an Extended Kalman Filter (EKF).

$$Y = \begin{bmatrix} x_{DGPS} \\ y_{DGPS} \end{bmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ q \end{pmatrix} + b_k \quad (3)$$

α_k and β_k are respectively the model noise and the measurement noise. The covariance matrix Q_β of the DGPS measurement can be estimated in real time using the NMEA sentence "GST" provided by the AgGPS132 receiver that we have used in the experiments (see the appendix). Therefore, the noise β_k is not stationary.

This architecture can be seen as a “loosely coupled fusion system”. The heading θ is not directly measured since the GPS is used as a position sensor. Nevertheless, by studying the state observability of the non linear system, one can verify that the observability condition is verified when the speed of the car is non-zero.

The EKF has been tested on 4.5 km long run (see Fig 3).

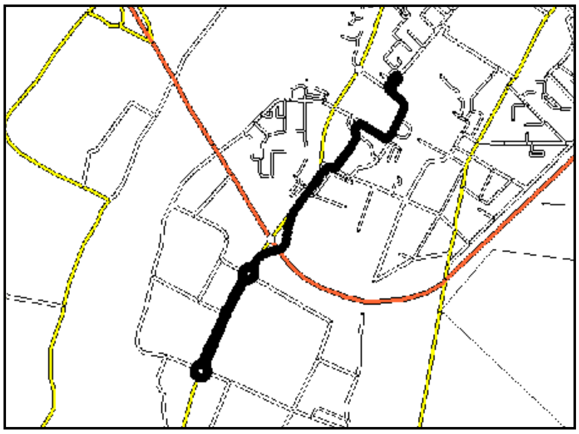

Fig. 3. Top view of the experiment.

The behavior of the filter is characterized by the study of the *x* and *y* innovations (differences between the DGPS measurements and the predicted measurements). It can be seen on Fig. 4 that the *x* innovation is zero mean. Moreover, the autocorrelation corresponds to the one of a white noise. This indicates that the filter is correctly tuned.

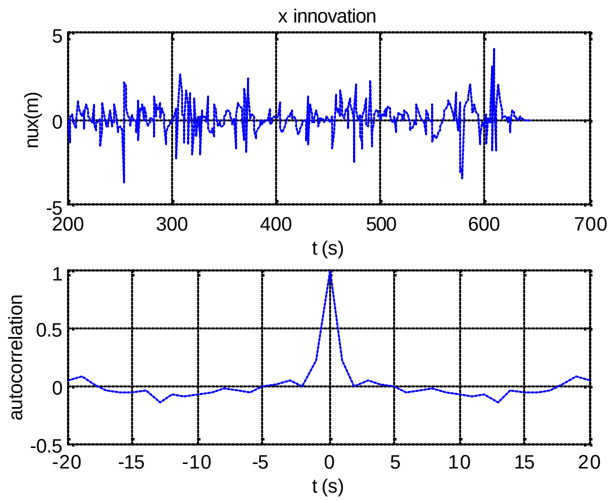

Fig. 4. x innovation of the EKF.

Finally, thanks to the odometric model, the EKF estimates continuously the X vector even when the signal of the satellites is blocked by bridges, tunnels, buildings, etc.

3 Road Selection using Multi-Criteria Fusion

Map matching techniques vary from those using simple point data, integrated with optical gyro and velocity sensors (Kim, 1996), to those using more complex mathematical techniques such as Kalman filters (Tanaka et al. 1990, Betaille and Bonnifait 2000). Systems that use only geometric information utilize the "shape" of line segments (road center-lines) that define the road network (Bernstein et al., 1998).

The first step is to determine which road center-lines are candidates for the vehicle's location (see Fig 5).

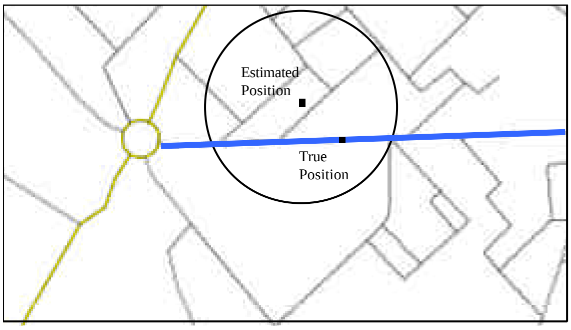

Fig. 5. Candidates roads around the estimated position.

The shortest Euclidean distance from the estimated position to each road segment is computed. It is not simply a matter of finding the line segment nearest to the estimated position. This will often give an incorrect result. For example in figure 5, the vehicle is on the highlighted road, but this is not the nearest road line segment to the estimated position.

The method proposed fuses several criteria using the Belief Theory for the road selection process. As the application is related to road safety, only geometrical criteria are used because they are not influenced by human errors. This means, for example, that a criterion such as *the speed of the vehicle is in agreement with the speed limitation* is not considered.

The two criteria chosen in this article can be formulated as follows:

1 - the vehicle location is close to a segment of the neighborhood;

2 - segments on which the vehicle can be located are those which have an angle close to the heading of the vehicle. This criterion is adapted with the estimated 3σ bound of the heading and the speed of the car.

The Belief Theory needs the affectation of elementary probabilistic masses defined on [0,1]. The mass notion is very near to the probabilistic mass notion, exception that it is not shared only on single hypotheses but it is possible to attribute a mass for an union of hypotheses: this is the main difference with Bayesian theory.

The frame of discernment $Q=\{H_1, H_2, \ldots, H_n\}$ is composed of exclusive and exhaustive hypotheses $H_i Ç H_j = Æ$, *"i¹j*, each of them being a solution of the problem.

The mass assignment is computed on the definition referential 2^{Θ}.

$2^{\Theta}=\{Æ, H_1, H_2, \ldots H_n, H_1 È H_2, \ldots, H_i È H_j È H_k È H_l È \ldots H_m, \ldots, Q\}$.

This distribution is a function of the knowledge about the source. The whole mass obtained is called "basic mass assignment". The sum of these masses is equal to 1. Each expert (or each source of information) defines a mass assignment according to its opinion about the situation.

The frame of discernment that we use is Q = {*Yes*, *No*, *Perhaps*} corresponding the answer of the following question: *is this segment the good one?*

To build functions of mass assignment corresponding to Q, we propose to consider the inaccuracy of the various information sources (DGPS, Odometer and Geographical Information System - GIS) and physical observations (for example, a car with a 40 m/s speed cannot be orthogonal to the direction of the segment). With this approach, information sources (criteria) are worked out from sensors.

The problem of mass assignment of each criterion can be tackled in a global or local way. The global strategy consists to consider together all the segments selected around an estimated position when affecting the masses. The local strategy separately treats each segment with respect to the criterion considered. In this way and if necessary, it is realistic to consider that the information sources are independent from each other.

3.1 Proximity criterion

The proximity criterion is based primarily on the measurement of the Euclidean distance between the estimated position and each segment taken in the road data base.

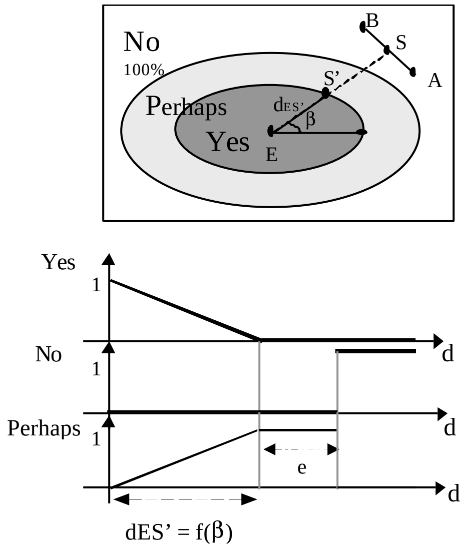

Fig. 6. Mass assignment of the proximity criterion.

The estimated error of the position is quantified by an ellipse of 99% equi-probability produced by the EKF (drawn in dark gray in Fig. 6). The estimated position E is at the center of the ellipse.

To allot a mass to a candidate segment [AB], we proceed in the following way. Let us note d the distance between the segment and point E:

$$d = \left\| \overrightarrow{SE} \right\| = d_{ES} \; .$$

The point S' falls at the intersection between the segment [ES] and the ellipse. The distance $d_{ES'}$ depends on the angle b which forms the segment [ES'] in the ellipse co-ordinates system. In the zone $d < d_{ES'}$, with a fuzzy modeling obtained by a probability-possibility transformation (Dubois and Prade (1993), Lassere (1997, 1998), Zadeh (1965, 1978, 1986)), the degree of membership is quantified. The first curve presented in Fig. 6 assigns a mass to the assumption *Yes*. In complementing the mass of *Yes*, the mass to the assumption *Perhaps* is allotted. Then, the mass of *Perhaps* remains constant (=1) for $d_{ES'} < d < d_{ES}+e$, in order to consider the projection error and the errors on the co-ordinates of the segments of the data base. Finally the mass of assumption *No*, is a step function starting from the distance $d = d_{ES'}+e$.

In conclusion, the mass assignment of the proximity criterion depends on two variables:

- the distance *d* between the center of the ellipse and the segment,
- the angle *b* between the distance support and the major axis of the ellipse.

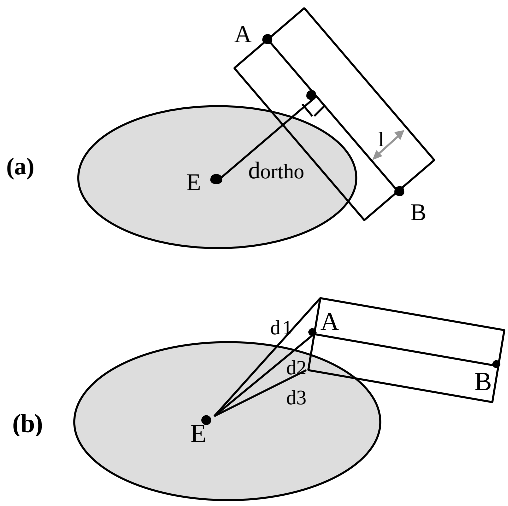

Fig. 7. Computation of the distance *d* with the road-box.

The problem becomes more complicated when considering the width of the road. We propose to model the road by a box centered on the segment, the length of which is equal to the one of the segment. The exact influence of the width of the road *l* is difficult to take into account in the computations of the criterion because *l* modifies the values of *b* and *d*. To simplify, we have chosen the following strategy:

1) if the orthogonal projection of *E* exists inside segment [*AB*], *d = dortho–l* (Fig. 7a).

2) if the orthogonal projection of *E* does not exist inside segment [*AB*], *d = min(d1,d2,d3)* (Fig. 7b).

3.2 Adaptable angular criterion

In this section, a mass assignment function is proposed to express the fact that the most possible segments are those which have an angle close to the heading of the vehicle. Figure 5 represents the computation of *DHeading* :

$$DHeading = min(|a\text{-}q|\,,\,|a\text{-}q+p|) \quad \text{with } q \in [0,\pi] \qquad (4)$$

Figure 8 presents the fuzzy modeling of the absolute value of the difference between the heading of the vehicle and the heading of the candidate segment. In fact, This curve is an adaptive one according to two parameters:

1- the speed of the vehicle. The scalar value *B* fixes the angular limit tolerated at a given velocity *V*. *B(V) = 90°-kV*, with *k = (90-10)/Vmax.*

2- the standard deviation of the estimation error of the heading angle $q \in \left[\hat{q} - 3s_q, \hat{q} + 3s_q\right]$. *m* represents the maximum belief which can be assigned to the hypothesis *YES*. Therefore, *m* varies according to s_q :

$$m(s_q) = 1 - \frac{6}{p} s_q \qquad (5)$$

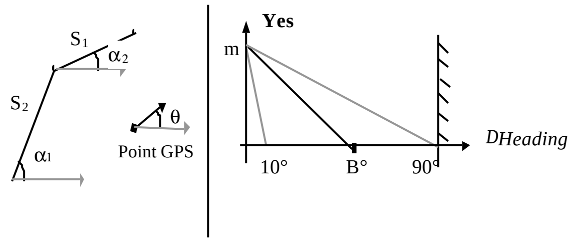

Fig. 8. Mass assignment of *Yes* for the adapted angular criterion

The *Perhaps* mass assignment is done by computing the complement of the mass of *Yes*. The mass of *No* starts from the limit angle tolerated for a given speed (B) and reaches one when the angle is equal to 90 degrees (fig.6).

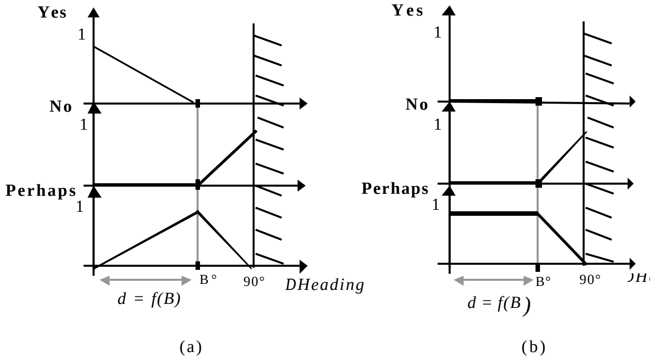

Fig. 9. Exemples of the mass assignment at a given velocity (a): $s_q = 0$. (b): $s_q = \frac{p}{2}$

3.3 Criteria Fusion

Theory of evidence is a mathematical theory which allows one to reason with uncertainty and which suggests a way for combining uncertain data. This is the reason why it is used as a basic tool for multi-sensor data fusion in situation assessment process.

This theory was introduced by Dempster (1968, 1976) and mathematically formalized by Shafer in 1976 (Shafer 1976). It is the generalization of Bayes Theory in the treatment of the notion of uncertainty. It allows to take into account the uncertainty of partial knowledge. Generally, this theory is used in a multisensor context to fuse heterogeneous information in order to obtain the best decision.

The basic entity, in Dempster-Shafer Theory, is a set of all possible answers (also called hypotheses) to a specific question. This set is called the *frame of discernment* and is denoted *Q*. All the hypotheses must be exclusive and exhaustive and, each subset of the frame of discernment can be a possible answer to the question. The degree of belief of each hypothesis is represented by a real number in [0,1] called a mass function *m(.)*. It satisfies the following rules:

$$m(f) = 0 \qquad \sum_{A \subseteq \Theta} m(A) = 1 \qquad (6)$$

A mass function is defined for all the different evidences. Each evidence *A*, for which *m(A) ¹ 0*, is called a focal element.

Associated with each basic assignment m, the belief (Bel) and the plausibility (Pl) are defined by:

$$Bel(A) = \sum_{B \subseteq A} m(B)$$
$$Pl(A) = \sum_{B \cap A \neq f} m(B) \qquad (7)$$

The belief and plausibility are interrelated by the relationship:

$$Pl(A) = 1 - Bel(\overline{A}) \qquad (8)$$

where $\overline{A}$ denotes the complement of A.

To obtain a better information from two different single sources S1 and S2, a combination of their mass function is performed according to the Demspter-Shafer rule:

$$m_\Theta(A) = \sum_{A_i \cap B_j = A} m_\Theta^{S_1}(A_i).m_\Theta^{S_2}(B_j) \qquad (9)$$

If there are some conjunctions that are empty of focal elements, a step of re-normalization is necessary to fulfil the rule m(f)=0. The coefficient of re-normalization is called k_q and is defined as:

$$k_q = \sum_{A_i \cap B_j = f} m_\Theta^{S_1}(A_i)m_\Theta^{S_2}(B_j) \qquad (10)$$

It represents the incoherence between the different sources. If we set $K_q = \frac{1}{1-K_q}$, we obtain the following normalized expression of the combination:

$$m_\Theta(A) = k_q \cdot \sum_{A_i \cap B_j = A} m_\Theta^{S_1}(A_i).m_\Theta^{S_2}(B_j) \qquad (11)$$

This combination rule is independent to the order in which evidences are combined when more than two evidences are involved.

After the combination step, several decision rules can be used to obtain the final result. It is then possible to adjust the wanted behavior for the decision step. If one wants to have an optimistic decision, the maximum of plausibility must be used and for a pessimistic decision one can use the maximum of belief. Many other decision rules exist in the Belief theory, especially for non-exhaustive frames of discernment. More information about them can be found in (Janez 1996, Nifle 1998 and Fabiani 1996).

In the decision-making, the strategy adopted to keep segments among the candidates, is to keep the most credible segments according the law of *ideal* decision. The likelihood of singleton assumption is characterized by two quantities (credibility and plausibility) which are calculated using the set of masses. These quantities respectively correspond to the minimal probability and the maximum probability of that assumption to be true. Consequently, a law of decision without ambiguity is where an assumption have a credibility higher than the plausibility of any other assumption (Kim 1990, Zadeh 1978).

The conflict computed in the Dempster-Shafer fusion rule is large if the two criteria are in total confusion. Therefore, we eliminate the segments which present an important conflict. Experimentally, we have taken a threshold equal to 0.5.

4 Experimental results

The algorithm works in real time conditions (with a frequency of 1Hz) under WIN NT (Pentium III 700 MHz) Fig. 10 presents the DGPS receiver used (a Trimble AgGPS132).

Fig. 10. The experimental vehicle "STRADA".

The following figure presents a top view of an experimental test performed in Compiegne. The map data-base are managed and interfaced by the software "Geoconcept".

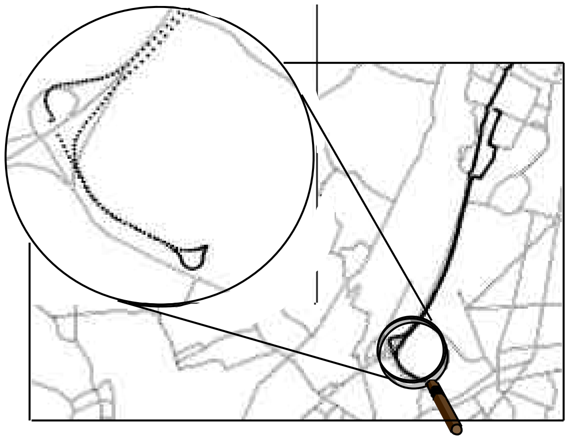

Fig. 11. Experimental situation on the "IGN Géoroute" data-base (The DGPS points are dotted).

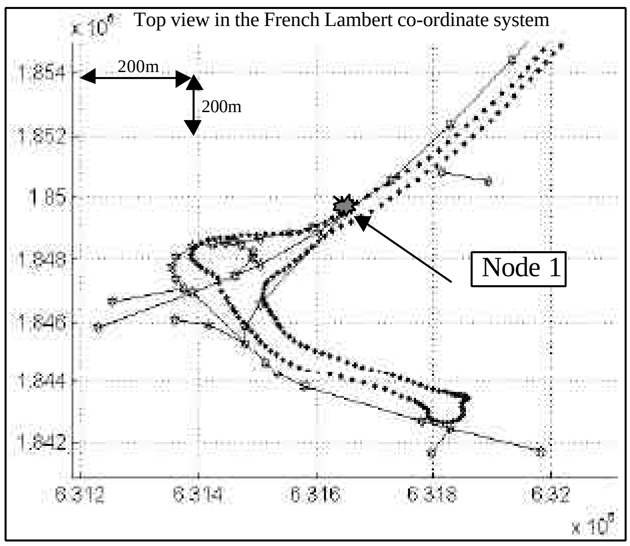

Fig. 12. Candidate segments extracted from the "IGN Géoroute" data-base by Matlab (The DGPS points are dotted).

To illustrate the road reduction method, we will discuss how it processes certain ambiguous situations. In the test shown on figure 11, the vehicle exits a motorway. This situation is very ambiguous because the angles of three segments (motorway, exit ramp, entrance ramp) are close to the car heading. Moreover, they have a common point very close to the estimated position (see Fig.13).

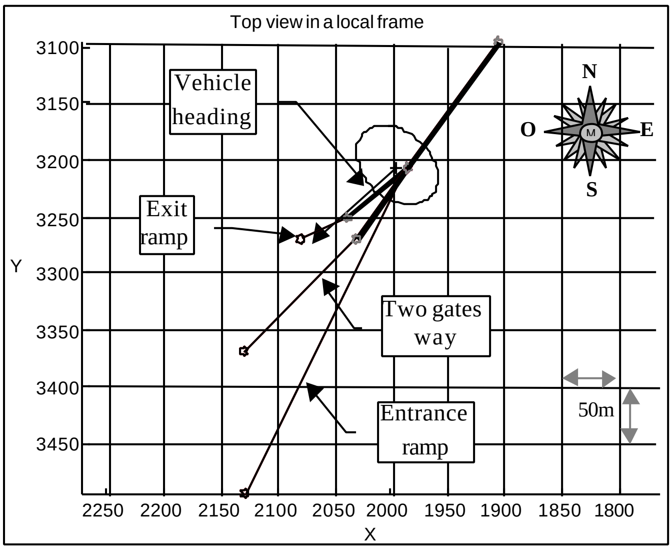

Fig. 13. The car exiting the motorway.

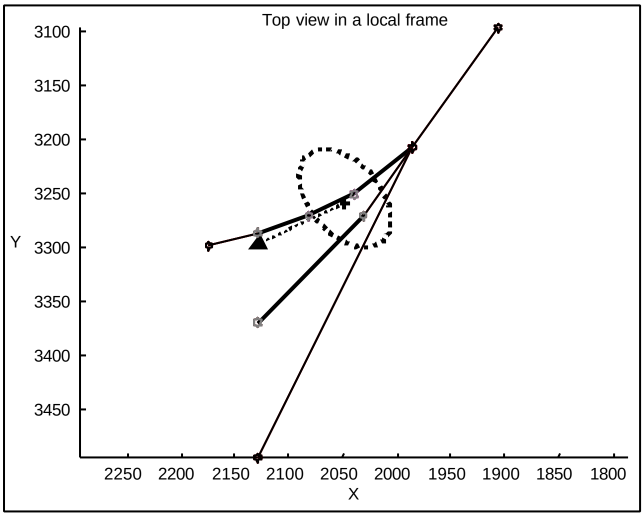

Fig. 14. The car in the exit ramp (a).

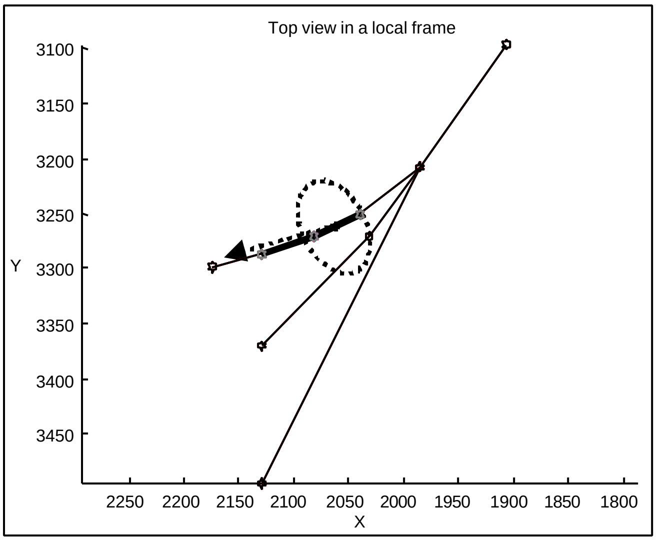

Fig. 15. The car in the exit ramp (b).

At the beginning, three segments are selected (those which are in bold in Fig.13). Two of them correspond to the motorway and one to the exit ramp. Logically, the entrance ramp (located on the opposite side of the road) is not selected, thanks to the angular criterion. Afterwards, the situation is still ambiguous (Fig. 14) until the difference between the car heading and the angles of the motorway segments becomes significant. Then, the system is able to assert that the car is on the exit ramp (Fig. . 15).

5 Conclusion

These experimental results show the aptitude of a multi-criterion fusion technique using Belief Theory to treat ambiguous situations frequently met by localization systems using maps. Moreover, it can detect situations where there's no credible segment, which means that the position of the vehicle does not correspond to any road on the map.

This methodology can be considered like an excellent tool to improve the positioning reliability and it makes possible to quantify the ambiguousness of a situation.

Other criteria using, for example, the local shape of the vehicle trajectory can be easily fused with the ones presented in this paper. The main problem is to develop judicious mass assignment functions.

Finally, future work will be dedicated to develop fusion techniques that use the selected segments if the situation is not ambiguous. This will improve the localization process.

6 References

Bernstein D and Kornhauser A. (1998). Map Matching for personal navigation assistants. 77th Annual meeting, The Transport Research Board, Jan 11-15, Washinton D.C.

Bétaille D. et Bonnifait Ph., Road Maintenance Vehicles Location using DGPS, Map-Matching and Dead-Reckoning: Experimental Results of a Smoothed EKF. IAIN World Congress in association with the US ION annual meeting. San Diego, pp. 409-416, June 26-28 2000

Bonnifait Ph., Bouron P., Crubille P., Meizel D. (2001). Data Fusion of Four ABS Sensors and GPS for an Enhanced Localization of Car-like Vehicles. *IEEE ICRA 2001 23-25, Séoul. pp. 1597-1602*.

Dempster A.P. (1976). Upper and Lower Probabilities Induced by a Multivalued Mapping. *Annals of Mathematical Statistics, vol.38*.

Dempster A.P. (1968). A Generalization of Bayesian Inference. *Journal of the Royal Statistical Society, vol.30, Série B.*

Dubois D. and Prade H. (1993). Fuzzy Sets and System Theory and Application. *Mathematics in science and engineering volume 144 Academic Press, Inc.*

Fabiani P. (1996). Représentation Dynamique de l'Incertain et Stratégie de Prise d'Information pour un Système Autonome en Environnement Evolutif. *PhD. Thesis Ecole Nationale Supérieure de l'Aéronautique et de l'Espace.*

Kim H. and Swain P.H. (1990). A Method Of classification for Multisource Data in Remote Sensing Based on Interval-Valued Probabilities. *Purdue University, West Lafayette, Indiana 47907, TR-EE 90- 48.*

Kim J.S., Lee, J.H., Kang, T.H., Lee, W.Y. and Kim, Y.G.,(1996), Node based map matching algorithm for car navigation system. Proceedings of the 29th ISATA Symposium, Florence, Vol. 10,pp121-126.

Janez F. (1996). Fusion de Sources d’Information Définies sur de Réferentiels Non-Exhaustifs Différents. *Ph.D. thesis Angers University*.

Lasserre V., Mauris G.and Foulloy L. (1997). A simple probability-possibility transformation for measurement error representation: a truncated triangular transformation. *Proc. of IFSA’97, Prague, Vol. 3, pp. 476-481*.

Lasserre V., Mauris G., Foulloy L. (1998). Fuzzy modeling of measurements acquired by an intelligent ultrasonic telemeter. *IEEE'IMTC'98, St. Paul, Minnesota, USA, 18-20 Mai, Vol. 2, pp. 837-842*.

Nifle A. and Renaud R. (1998). Fuzzy Temporal Multiple Hypothesis Testing for Behavior classification. *IPMU 98 Poster the Seventh Conference on Information Processing and Management of Uncertainty in Knowledge-Based Systems*.

Rogers S. (2000). Creating and Evaluating Highly Accurate Maps with Probes Vehicles. *IEEE Intelligent Transportation Systems, Dearborn (MI), USA.*

Shafer G. (1976) A. Mathematical Theory Of Evidence. *Princeton University Press, Princeton*.

Tanaka J. Hirano K., itoh, T., Nobuta H. and Tsunoda S., (1990). Navigation System with Map-Matching Method. Proceedings of the SAE International Congress and Exposition, pp 45-50.

Zadeh L. A., (1965). Fuzzy Sets as a Basis for a Theory of Possibility. *Fuzzy Sets and Systems 1,pp.3-28*.

Zadeh L.A., (1978). Probability Measures of Fuzzy Events. *Journal of Mathematical Analysis and Applications 23 pp. 421-427*.

Zadeh L.A., (1986). A Simple View of Dempster-Shafer Theory of Evidence and its Implication for the Rule of Combination. *The AI Magazine*.

Zhao Y., (2000). Mobile Phone Location Determination and Its Impact on Intelligent Transportation Systems. *IEEE Transaction on Intelligent Transportation Systems vol. 1, No. 1, pp. 55-64.*

7 Appendix: computation of the covariance matrix of a DGPS measurement (*x*,*y*) using the NMEA GST sentence

GPS measurements are affected by many independent noise sources, so it is reasonable to assume that the position vector $X = \lfloor x_1, \ldots, x_n \rfloor$ has a Gaussian distribution :

$$P(X) = (2p)^{-n/2} [C]^{-1/2} \exp\left[-\frac{1}{2}(X-\bar{X})^T C^{-1}(X-\bar{X})\right] \qquad (2)$$

In the case of vehicle positioning, only two dimensions are of interest. The covariance matrix is therefore defined by:

$$Q_b = s_x s_y \begin{bmatrix} \frac{s_x}{s_y} & r \\ r & \frac{s_y}{s_x} \end{bmatrix} \qquad (12)$$

$$Q_b^{-1} = \frac{1}{s_x^2 s_Y^2 (1-r^2)} \begin{bmatrix} s_y^2 & -r s_x s_y \\ -r s_x s_y & s_x^2 \end{bmatrix}$$

where σ_x and σ_y are the standard deviations of the estimation error of *x* and *y* as observed in the xy-plane and

$r = \frac{s_{xy}^2}{s_x s_y}$ is the spatial correlation coefficient ($|r|$=1).

The NMEA "GST" sentence gives σ_x, σ_y and φ. Therefore, in order to compute Q_β, ρ has to be calculated.

The probabilistic distribution can be written :

$$P(x,y) = k_1.\exp\left\{-k_2\left[\frac{(x-\mu_x)^2}{s_x^2} - \frac{2r(x-\mu_x)(y-\mu_y)}{s_x s_y} + \frac{(y-\mu_y)^2}{s_y^2}\right]\right\} \qquad (13)$$

where $k_1 = \frac{1}{2p s_x s_y \sqrt{1-r^2}}$; $k_2 \frac{1}{2(1-r^2)}$

The constant probabilistic equation

$$\frac{(x-\mu_x)^2}{s_x^2} - \frac{2r(x-\mu_x)(y-\mu_y)}{s_x s_y} + \frac{(y-\mu_y)^2}{s_y^2} = (1-r^2)k^2 \qquad (14)$$

represents an ellipse.

This ellipse represents a contour of constant error probability. This contour varies according to k. Here are some particular values:

Proba = 40% → $k^2 \approx 1.0$
Proba = 50% → $k^2 \approx 1.386$
Proba = 90% → $k^2 \approx 4.605$

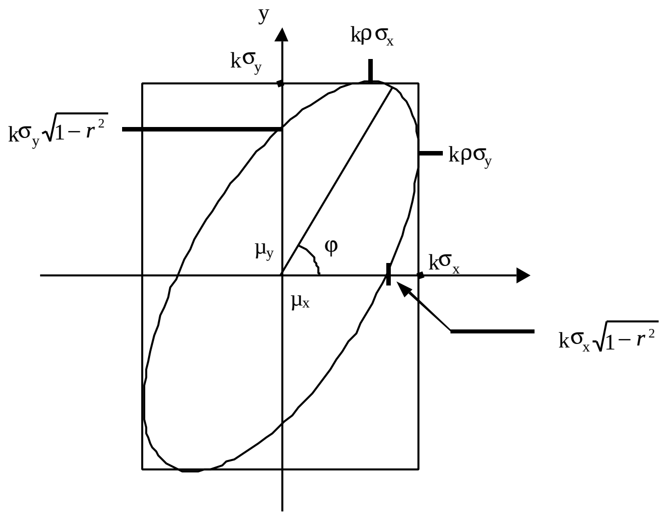

Fig. 16. Ellipse associated to μ_x, μ_y, σ_x, σ_y, ρ, φ.

Let us denote $M = \begin{bmatrix} A & B \\ B & C \end{bmatrix}$ with

$$A = \frac{1}{s_x^2(1-r^2)} \quad B = \frac{-r}{s_x s_y (1-r^2)} \quad C = \frac{1}{s_y^2(1-r^2)}$$

Eigen values of M : $l_{1,2} = \frac{A + C \mp \sqrt{(A-C)^2 + 4B^2}}{2}$

l_1 eigen vector is $u = \begin{bmatrix} u_x \\ u_y \end{bmatrix}$.

The non-normalized co-ordinate vector of u verifies the linear system:

$$\begin{cases} (A - l_1)\ u'_x + B\ u'_y = 0 \\ B\ u'_x + (C - l_1)\ u'_y = 0 \end{cases} \qquad (15)$$

u'_x can be fixed $u'_x = 1 \Rightarrow u'_y = \frac{-(A - l_1)}{B} = \frac{-B}{(C - l_1)}$

we then normalized

$$u = \frac{1}{\sqrt{u_x'^2 + u_y'^2}} \begin{bmatrix} u_x' \\ u_y' \end{bmatrix} \qquad (16)$$

Its trivial to proof that:

$$a = \sqrt{\frac{K^2}{l_1}}\ ,\ b = \sqrt{\frac{K^2}{l_2}} \text{ and } tg(j) = \frac{-(A - l_1)}{B} \qquad (17)$$

where *a* is the length of the semi-major axis of the ellipse, *b* is the length of the semi-minor axis and *j* is the orientation of the semi-major axis relatively to the East.

$$a = k\sqrt{\frac{2}{A + C - \sqrt{(A-C)^2 + 4B^2}}}$$
$$b = k\sqrt{\frac{2}{A + C + \sqrt{(A-C)^2 + 4B^2}}} \qquad (18)$$

<u>Theorem</u> :

The general equation of a conic is (B ≠ 0)

$$Ax^2 + 2Bxy + Cy^2 + Dx + Ey + F = 0$$

By rotating the coordinate axes through an angle φ, it can be rewritten as

$$A'u^2 + C'v^2 + D'u + E'y + F' = 0$$

where $j = \frac{1}{2} arctg\left(\frac{2B}{A - C}\right) \qquad 0 < j < \frac{p}{2}$

<u>Proof</u> :

To eliminate the xy-term, the x- and y-axes must rotate until they are parallel to the axes of the conic. Consider the rotation matrix:

$$x = u\cos j - v\sin j$$
$$y = u\sin j + v\cos j \qquad (19)$$

By substituting these value for *x* and *y* into the original equation and collecting terms, we obtain the following.

$$a = A\cos^2 j + 2B\sin j\cos j + C\sin^2 j$$
$$b = -2A\sin j\cos j + 2B\left(\cos^2 j - \sin^2 j\right)$$
$$+ 2C\sin j\cos j$$
$$c = A\sin^2 j + 2B\sin j\cos j + C\cos^2 j$$
$$d = D\cos j + E\sin j$$
$$e = -D\sin j + E\cos j$$
$$f = F$$

Now, in order to eliminate the uv-term, the value of φ must be selected such that *b=0*, as follows :

$$b = 2B\left(\cos^2 j - \sin^2 j\right) + (C - A)\sin(2j) \qquad (20)$$

If *B=0*, no rotation is necessary, because the xy-term is not present in the original equation. If *B¹0*, the only way to make *b=0* is to let:

$$tg(2j) = \frac{2B}{A - C} \qquad 0 < j < \frac{p}{2}. \qquad (21)$$

$$tg(2j) = \frac{2B}{C - A} \qquad -\frac{p}{2} < j < 0 \qquad (22)$$

By substituting these value for *A*, *B* and *C* into these equations and collecting terms, we obtain the following.

$$r = \frac{tg(2j)\left(s_x^2 - s_y^2\right)}{2s_x s_y} \qquad 0 < j < \frac{p}{2} \qquad (23)$$

$$r = \frac{tg(2j)\left(s_y^2 - s_x^2\right)}{2s_x s_y} \qquad -\frac{p}{2} < j < 0 \qquad (24)$$

Finally, Q_β is then given by (12).