# Cayley Graphs and Interconnection Networks

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En raison de l'essor récent du calcul parallèle et distribué, la conception et l'analyse de divers réseaux d'interconnexion a été un sujet important de recherche ces dernières années. Les nouvelles technologies des réseaux de communications, telles que les fibres optiques, ont encore stimulé les recherches. Il y a de nombreux avantages à utiliser les graphes de Cayley, orientés ou non, comme modèle de réseau d'interconnexion. Ce rapport propose une synthèse des caractéristiques des réseaux d'interconnexions basés sur les graphes de Cayley. On présente d'abord quelques classes de graphes de Cayley, qui sont beaucoup étudiées comme modèles de réseau d'interconnexion. Des résultats et problèmes concernant le routage dans les réseaux sont ensuite abordés. On insistera sur ceux concernant la charge des nœuds et des liens.

Due to recent developments of parallel and distributed computing, the design and analysis of various interconnection networks has been a main topic of research for the past few years and is still stimulated by the new technologies of communication networks such as optic fibers. There are many advantages in using Cayley (di)graphs as models for interconnection networks. This work first surveys some classes of Cayley graphs which are well studied as models of interconnection networks. Results and problems related to routings in networks are then presented, with emphasis on loads of nodes and links in routings.

# Chapter 1

# Introduction

As defined by D.F. Hsu in the introduction of a special issue of the journal *Networks* ([97]), "an interconnection network consists of hardware and software entities that are interconnected to facilitate efficient computation and communication; these entities can be in the form of processors, processes, memory modules or computer systems". Due to recent developments in parallel and distributed computing, the design and analysis of various interconnection networks has been a main topic of research for the past few years. An essential component of a supercomputer based on large-scale parallel processing is the interconnection network. It provides communication among the processors and the memories. Interconnection networks also play a key role in the design and implementation of communication networks and the recent advent of optic technology adds more design problems. This explains that a growing number of research articles and congresses (such as IWIN every two years since 1991) are devoted to interconnection networks, and that special issues of journals have already been published on the subject (for example [97], [17] and [107]).

Interconnection networks are often modeled by finite graphs (or digraphs). The vertices of the graph represent the nodes of the network, that is, processing elements, memory modules or switches, and the edges correspond to communication lines. What qualities are required in order for a graph to be a good model for interconnection networks ? A basic constraint in many network design problems is that a bound on the maximum node degree is imposed by cost and fundamental engineering limitations. That is, the nodes of the network can be connected by at most a fixed number of communication lines to other nodes. At the same time other properties are crucial for many applications which need efficient network communications. This leads to some well studied design problems.

We can distinguish at least three categories of interconnection network problems which lead to well studied problems in graph theory:

(a) network topologies and analysis: find graphs or digraphs which model good interconnection networks with regard to quality measures such as diameter and connectivity. For example, consider the diameter (i.e. the maximum of the distance between two vertices in the graph) as a measure of the maximum communication delay. The *degree/diameter problem* is then to construct the largest possible graphs satisfying bounds on the maximum vertex degree and diameter (see Section 4).

(b) communication: efficiently implement distributed algorithms such as those for broadcasting, gossiping or accumulation on a specific network (a graph with a communication hypothesis).

(c) mapping and simulation: in order to efficiently emulate a network A on a network

B, study mappings from a graph into another one (in particular homomorphisms which are arc-preserving mappings). This problem has received increasing attention from researchers in the last years, but the results concerning symmetric graphs are not as developped as for the first two problems.

Symmetry is important for all of these problems since it simplifies their study and allows the use of algebra, which is an efficient tool in proofs. In particular, there are many advantages in using Cayley (di)graphs. They have the property of node symmetry (vertex transitivity) which makes it possible to implement the same routing and communication schemes at each node of the network they model. Furthermore, some of them have other advantages such as line symmetry (edge transitivity), hierarchical structure (allowing recursive constructions), high fault tolerance and so on, as we will see in the following. This explains why the topology chosen for some parallel machines is or was a hypercube (for example the nCUBE2 is based on a hypercube of dimension 13) or a torus (like the parallel machine Fujitsu AP1000) despite their drawbacks, the degree for the former, the diameter for the latter. It also explains why hypercubes have probably been the most studied graphs for interconnection networks in the past decade, and why star graphs, toruses and other graphs generalizing hypercubes, are much studied in recent publications.

Most of the time, researchers first study some problems in a particular Cayley graph, for example one kind of communication problem in the hypercube (like [95] or [73]) or the star graph (like [72]). Then the same problem is studied in other Cayley graphs and, finally, the result is obtained in a more general class of Cayley graphs which includes the graphs studied to begin with (like [21] or [74]). Therefore, it seems important to specify and study large classes of Cayley graphs having good properties for given interconnection problems.

We cannot survey here all the problems of interconnection networks modeled by Cayley graphs. Good references on this subject are the first two published surveys [3] and [103]. In this report, we will focus on routing problems including connectivity, diameter and loads of routings. We place the emphasis on load problems, since new classes of vertex-transitive graphs (see quasi-Cayley graphs in Section 5.3) or edge-transitive graphs (see regular orbital graphs in Section 5.6) were studied in order to enlarge the class of graphs behaving well for loads of routings. More precisely, after recalling some basic definitions and properties in Chapter 2, we survey particular classes of Cayley graphs which are often studied as models of interconnection networks. First, we study Cayley graphs generated by transpositions (Section 3.1) and those with a special automorphism which rotates the generators and is called a complete rotation (Section 3.2). In Section 3.3, we present hierarchical Cayley digraphs and results about fault tolerance. Finally, multistage Cayley digraphs and their quotients are studied in Section 3.4.

In a second part of this course (Section 4), we consider diameter and path problems. The last part is devoted to the specific problem of routing. The notion of load of routings was first studied in graph theory in [35] which introduced the parameter called (vertex-)forwarding index. This notion becomes more and more important, and Chapter 5 introduces the subject. One of the reasons why Cayley graphs behave well as models for networks is that they allow routings of shortest paths with uniform loads on the nodes. Generalizations of Cayley graphs with the same behaviour are then presented. A similar notion for load of edges instead of vertices is considered in Section 5.5 with the edge-forwarding index. Multicommodity flow theory can be useful here as shown in Section 5.7. As a conclusion, Chapter 6.11 gives some trends of research in this rich field.

# Chapter 2

# Definitions and basic properties

In this chapter, some basic definitions are given, and notation is introduced. We also recall some well-known properties of groups and Cayley graphs that we will use later. For more definitions and basic results on groups, we refer the reader for example to [117], [7], [32], for graph theory to [16], [28], [27], and for interconnection networks to [106], [47].

## 2.1 Groups and permutations

We only consider finite groups. Therefore the inverse of any element is a power of this element. The identity (unit element) of a group $G$ will be denoted by $I$ and the operation will be written as multiplication, except for abelian groups, where the operation is denoted additively and the unit element is 0. By abuse of notation, we use the same letter to denote a group and the subset of its elements and specify the law of the group only when a confusion can arise. Throughout $\mathbb{Z}_d$ will denote the group of integers modulo $d$. The elements of a subset $S$ of a group $G$ are called *generators of* $G$, and $S$ is said to be a *generating set*, if every element of $G$ can be expressed as a finite product of their powers. We will also say that $G$ is generated by $S$. We will denote by $Gr(S)$[1] the subgroup of a group $G$ generated by a subset $S$ of $G$. A subset $S$ of $G$ is said to be *unit free* if $I \notin S$. It is said to be *symmetric* (or closed under inverses) if $s \in S$ implies $s^{-1} \in S$.

The direct product of two groups $G$ and $G'$ is denoted by $G \times G'$.

A *permutation* $\sigma$ on the set $X = \{1, \ldots, n\}$ is a bijective mapping (i.e. one-to-one and surjective) from $X$ to $X$. As usual, it is denoted by the images of the elements $(\sigma(1), \ldots, \sigma(n))$. In a product of permutations, $\sigma\tau$ signifies that we apply first mapping $\tau$ on the set $\{1, \ldots, n\}$ and then the mapping $\sigma$, i.e. $\sigma\tau = (\sigma(\tau(1)), \ldots, \sigma(\tau(n)))$. We denote by $\mathfrak{S}_X$ the group of all permutations on a set $X$, and by $\mathfrak{S}_n$ the group of all permutations on $\{1, \ldots, n\}$, also called the *symmetric group*. A *permutation group* on $\{1, \ldots, n\}$ (we will also say a group of permutations) is some subgroup $G$ of $\mathfrak{S}_n$.

For a permutation $\sigma$ on a set $X$, $Supp(\sigma)$ is the set of elements $i$ of $X$ such that $\sigma(i) \neq i$. The *degree of a permutation* $\sigma$ is the cardinality of $Supp(\sigma)$. If the permutation $\sigma$ is a *cycle* of length $\ell \geq 2$, we denote it by $\langle \sigma(a), \sigma^2(a), \ldots, \sigma^\ell(a) \rangle$, where $a$ is any element of $X$ such that $\sigma(a) \neq a$. For example, $\langle 3, 5 \rangle$ denotes the transposition of the elements 3 and 5.

The *stabilizer* of an element $x \in X$ in a permutation group $G$ on $X$ is the subgroup of $G$, defined

[1] We do not use the classical notation $\langle S \rangle$ since we need it for transpositions and cyclic permutations.

by $Stab_G(x)$ $=\{g \in G : g(x) = x\}$.

Let $G$ be any subgroup of $\mathfrak{S}_X$. The group $G$ *acts transitively* on $X$ if for any $u, v \in X$, there exists $f \in G$ such that $f(u) = v$. The permutation group $G$ is said to be *semi-regular* if the stabilizer of every element of $X$ in $G$ is reduced to the identity. Finally, $G$ is said to be *regular* (or *acts regularly* on $V$) if $G$ acts transitively and is semi-regular, or equivalently, if for any two elements $u$ and $v$ of $V$, there exists a unique permutation $f \in G$ such that $f(u) = v$.

The following well-known theorem of Cayley allows one to consider only groups of permutations.

**Theorem 1** *Every finite group is isomorphic to a group of permutations.*

Thus, except for commutative groups, groups are mainly defined as groups of permutations in articles on interconnection networks as in [3], [103]. But it should be noticed that it can also be convenient to use a definition by generators and relators. This is the case in recent articles like [69], [53], [92].

## 2.2 Graphs and Cayley graphs

A *digraph* $\Gamma = (V, E)$ is defined by a set of *vertices* $V$ and a set $E$ of *arcs* or *oriented edges* which is a subset of elements $(u, v)$ of $V \times V$. If the subset $E$ is symmetric ($(u, v) \in E$ implies $(v, u) \in E$), we can identify two opposite arcs $(u, v)$ and $(v, u)$ to a single edge denoted by $uv$, or $[u, v]$ if $u$ and $v$ are composite. We then obtain a *graph*. The *out-degree* (respectively *in-degree*) of a vertex $u$ of a digraph $\Gamma$ is the number of arcs $(u, v)$ (respectively $(v, u)$) of $\Gamma$ and is denoted by $d^+(u)$ (respectively $d^-(u)$). A digraph $\Gamma$ is said to be *regular of out-degree* $d$ if $d^+(u) = d^-(u) = d$ for every vertex $u$ of $\Gamma$.

A *directed path* or *dipath* from vertex $u$ to vertex $v$ in a digraph $\Gamma$ is a sequence of vertices $x_0, x_1, \ldots, x_l$ with $x_0 = u$, $x_l = v$, such that $(x_i, x_{i+1})$ is an arc of the digraph for any $i$, $0 \leq i \leq l - 1$. In such a case, $l$ is the *length* of the dipath, that is, the number of arcs it contains. The *distance* from vertex $u$ to $v$, denoted by $d(u, v)$, is the length of a shortest dipath from $u$ to $v$. A digraph is *strongly connected* if there is a dipath from $u$ to $v$ for all $u \neq v$ in $\Gamma$. The *diameter* of $\Gamma$ is $Diam(\Gamma) = \max_{(u,v)\in V\times V} d(u, v)$.

Let $G$ be a group and let $S$ be a subset of $G$. Then $Cay(G, S)$ denotes the Cayley[2] digraph of the group $G$ and the subset $S$. The vertices of $Cay(G, S)$ are the elements of $G$ and the arcs are all ordered pairs $(g, gs)$ for $g \in G, s \in S$. The arc $(g, gs)$ is said to be *labeled by* $s$. If $S$ is a generating set of $G$, then $Cay(G, S)$ is strongly connected. In this case, we will say that $Cay(G, S)$ is the *Cayley digraph of $G$ generated by $S$*. If $S$ is unit free and closed under inverses, then $Cay(G, S)$ is a simple graph. The Appendix in Chapter 6 gives examples of well-known Cayley graphs. Notice that the definitions of a Cayley digraph given in the literature can differ. Many authors consider only the case where $S$ is a symmetric generating set of $G$. Furthermore, the terminology can also differ. For example, [27] speaks of *Cayley diagrams* instead of Cayley digraphs.

[2]Arthur Cayley (1821-1895) lived in his youth in St-Petersburg, studied in Cambridge, and became professor of Pure Mathematics at Cambridge in 1863 after he had been a lawyer during 14 years. He published over 900 papers and notes covering nearly every aspect of modern mathematics. The most important of his work is in developing the algebra of matrices. He also worked on non-euclidean geometry and n-dimensional geometry. He introduced the drawing in groups in 1878. At this time groups were considered mainly related to geometry. Its definition used in 1878 the left product by a generator, but it used the right product in 1889... The first drawing of a Cayley graph concerned the group of the isometries of the tetrahedron. The usual form of Cayley digraphs is more that of Max Dehn (1878-1951) than that of Arthur Cayley.

## 2.3 Homomorphisms

An *isomorphism* of a digraph from $\Gamma = (V, E)$ onto $\Gamma'(V', G')$ is a bijective mapping $f$ from $V$ to $V'$ such that $(u, v)$ is an arc of $\Gamma$ if and only if $(f(u), f(v))$ is an arc of $\Gamma'$. An automorphism of a digraph $\Gamma$ is an isomorphism from $\Gamma$ onto $\Gamma$. The automorphisms of $\Gamma$ form a subgroup of $\mathfrak{S}_V$ denoted by $Aut(\Gamma)$.

If $Aut(\Gamma)$ acts transitively on $V$, the digraph $\Gamma$ is said to be *vertex transitive*. In a Cayley digraph, the automorphisms $f_a : x \mapsto ax$, $a \in G$ (also called *translations*), form a subgroup of $Aut(\Gamma)$ which is isomorphic to the group $G$. This subgroup acts regularly on $V (= G)$ since for any two vertices $u$ and $v$ of $V$, there exists a unique element $a$ of $G$, $a = vu^{-1}$, such that $f_a(u) = v$. Therefore a Cayley digraph $\Gamma(V, E) = Cay(G, S)$ is vertex transitive. Furthermore, the group of translations allows a characterization of Cayley digraphs.

**Theorem 2 [124]** *Let $\Gamma$ be a strongly connected digraph. Then $\Gamma$ is a Cayley digraph if and only if $Aut(\Gamma)$ contains a subgroup which acts regularly on $V$.*

The following results on automorphisms of Cayley graphs will be used in the next section.

**Proposition 3** *Let $G = Gr(S)$ and $G' = Gr(S')$ be two groups generated by $S$ and $S'$ respectively, and with the same cardinality $|G| = |G'|$. Let $h$ be a group homomorphism from $G$ into $G'$ such that $h(S) = S'$. Then $h$ is an isomorphism from the digraph $Cay(G, S)$ onto the digraph $Cay(G', S')$.*

**Proof.** Let $x'$ be any element of $G'$. Then $x'$ can be written as a product of elements of $S'$, $x' = s'_1 s'_2 \cdots s'_k$, and since $h(S) = S'$, every element $s'_j$, $1 \le j \le k$, can be written as $s'_j = h(s_j)$, so that $x' = h(s_1)h(s_2)\cdots h(s_k) = h(x)$ with $x \in G$. Thus, the mapping $h$ is surjective. Since $|G| = |G'|$, $h$ is bijective, whence it follows from $h(S) = S'$ that $|S| = |S'|$, showing that the Cayley graphs have the same degree and consequently the same number of edges. Thus, it is sufficient to prove that $h$ is edge-preserving to finish the proof. For this, consider any two adjacent vertices $x$, $y$ of $Cay(G, S)$, with $y = xs$ for some element $s$ of $S$. By $h(y) = h(xs) = h(x)h(s)$, with $h(s) \in S'$, $h(y)$ and $h(x)$ are adjacent. □

By applying Proposition 3 to the case $S = S'$, we get the following well-known result.

**Corollary 4 [146, 24]** *Let $h$ be a homomorphism of the group $G$ generated by $S$ into itself such that $h(S) = S$. Then $h$ is a graph automorphism of $Cay(G, S)$.*

A graph $\Gamma$ is said to be *edge transitive* if for any given pair of edges $uv$, $u'v'$, there exists an automorphism $f \in Aut(\Gamma)$ such that $f(\{u, v\}) = \{u', v'\}$. A digraph (symmetric or not) is *arc transistive* (1-*arc-transitive* for some authors, *symmetric* in [24]) if for any given pair of arcs $(u, v)$, $(u', v')$ there exists an automorphism $f \in Aut(\Gamma)$ such that $f(u) = u'$ and $f(v) = v'$. Moreover, a digraph $\Gamma$ is said to be *distance transitive* if for all vertices $u, v, u', v'$ of $V$ such that $d(u, v) = d(u', v')$, there exists an automorphism $f$ of $\Gamma$ such that $f(u) = u'$ and $f(v) = v'$. Figure 2.1 shows the implications between these definitions in the case of graphs. An arrow from one property to another one indicates that the former implies the latter. If an implication is not true, the name of a graph which is a counterexample is given besides a crossed arrow.

We now recall a condition which is sufficient for a Cayley digraph to be arc transitive.

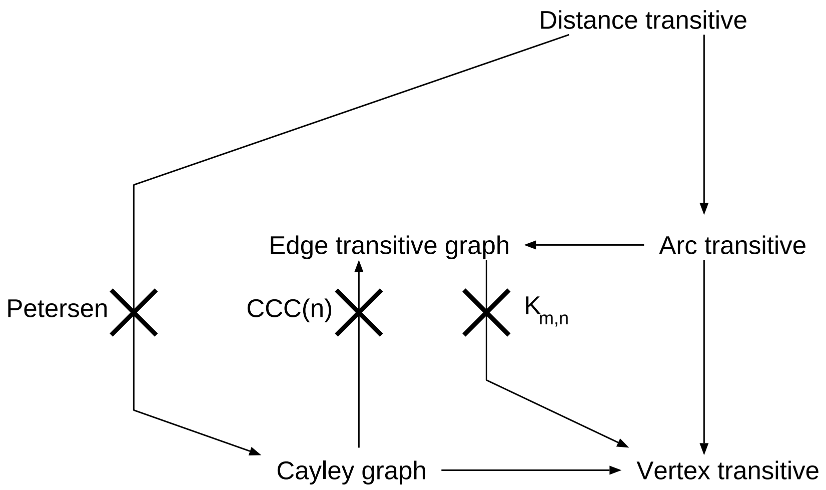


Figure 2.1: Relations between definitions.

**Proposition 5** **[3]** *A Cayley digraph $Cay(Gr(S), S)$ such that, for every pair $s, s'$ of $S$, there exists a group homomorphism $h$ of $Gr(S)$ satisfying $h(S) = S$ and $h(s) = s'$, is arc transitive.*

**Proof.** Consider two arcs $(u, v)$ and $(u', v')$ of $Cay(Gr(S), S)$ with $v = us$, $s \in S$, and $v' = u's'$, $s' \in S$. Let $h$ be a group homomorphism satisfying $h(S) = S$ and $h(s) = s'$. By Corollary 4, $h$ is a graph automorphism. Consider $\phi$ defined by $\phi(x) = u'h(u^{-1}x)$. Then $\phi$ is a graph automorphism since it is the composition of the three automorphisms $x \mapsto u^{-1}x$, $h$ and $y \mapsto u'y$. Furthermore, $\phi(u) = u'$, $\phi(v) = u'h(u^{-1}v) = u'h(s) = u's' = v'$. $\square$

**Remark 6** *The converse of Proposition* 5 *is false.*

**Proof.** (An error in [3]) Consider the complete graph $K_n$ which is arc transitive as the Cayley graph on the additive group $\mathbb{Z}_n$ generated by $S = \mathbb{Z}_n \setminus \{0\}$. Notice that any group homomorphism $h$ on $\mathbb{Z}_n$ satisfies $h(x) = h(\underbrace{1 + \cdots + 1}_{x}) = xh(1)$. If $h(S) = S$, then $h(1) \neq 0$ and $h(x) = ax$ with $a = h(1) \neq 0$. If for every pair $s, s'$ of $\mathbb{Z}_n \setminus \{0\}$, there exists $a \neq 0$, and $s' = as$, then in particular $1 = as$ for some $a \neq 0$. But if $n$ is not prime, not every element has a multiplicative inverse in $\mathbb{Z}_n \setminus \{0\}$, and therefore the condition of Proposition 5 cannot be fulfilled. This proves that the converse of Proposition 5 is not true for $K_n$ if $n$ is not prime. $\square$

## 2.4 Products

We recall the definitions and notation of the products of graphs we will use later.

- The *cartesian product* of two digraphs $\Gamma = (V, E)$ and $\Gamma' = (V', E')$, denoted by $\Gamma \Box \Gamma'$, is the digraph with vertex set $V \times V'$ and arc set

$$\{((x, x'), (y, x')) : x' \in V', \ xy \in E\} \cup \{((x, x'), (x, y')) : x \in X, \ x'y' \in E'\}.$$

  For example, the hypercube $H(n)$ and the $n$-dimensional torus $MT_{n,k}$ (see definitions in Chapter 6) are cartesian products of $n$ complete graphs $K_2$ and of $n$ cycles $C_k$, respectively.

- The *weak product* (also known as the *categorical* or *cardinal* product) of two digraphs $\Gamma = (V, E)$ and $\Gamma' = (V', E')$, denoted by $\Gamma \times \Gamma'$, is the digraph with vertex set $V \times V'$ and arc set

$$\{((x, x'), (y, y')) : \; xy \in E, \; x'y' \in E'\}.$$

- The *strong product* of two digraphs $\Gamma = (V, E)$ and $\Gamma' = (V', E')$, denoted by $\Gamma \boxtimes \Gamma'$, is the digraph with vertex set $V \times V'$ and arc set the union of the arc sets of $\Gamma \Box \Gamma'$ and of $\Gamma \times \Gamma'$.

- The *lexicographic product* of two digraphs $\Gamma = (V, E)$ and $\Gamma' = (V', E')$, denoted by $\Gamma[\Gamma']$, is the digraph with vertex set $V \times V'$ and arc set

$$\{((x, x'), (x, y')) : x \in V, \; x'y' \in E'\} \cup \{((x, x'), (y, y')) : \; x' \in V', y' \in V', \; xy \in E\}$$

A digraph produced by any of the three products: cartesian product, weak product, lexicographic product, is vertex transitive if and only if each factor is vertex transitive [125], [126]. For factors which are Cayley graphs, the following results are easy to obtain.

**Proposition 7** *Let $\Gamma = Cay(G, S)$ and $\Gamma' = Cay(G', S')$ be two Cayley digraphs. Then*

- *the cartesian product $\Gamma \Box \Gamma'$ is the Cayley digraph of the direct product $G \times G'$ and the subset $(S \times I) \cup (I \times S')$;*
- *the weak product $\Gamma \times \Gamma'$ is the Cayley digraph of the direct product $G \times G'$ and the subset $S \times S'$;*
- *the strong product $\Gamma \boxtimes \Gamma'$ is the Cayley digraph of the direct product $G \times G'$ and the subset $(S \times I) \cup (I \times S') \cup (S \times S')$;*
- *the lexicographic product $\Gamma[\Gamma']$ is the Cayley digraph of the direct product $G \times G'$ and the subset $(S \times G') \cup (I \times S')$.*

In Section 3.4.2 we will consider some Cayley digraphs of wreath products of groups.

## 2.5 Quotients

We recall here some classical definitions and fix our notation.

For any group $G$ and any subgroup $H$ of $G$, the *left cosets of* $H$ in $G$ are the subsets $gH$ of $G$ defined by $gH = \{gh : h \in H\}$. We denote by $(G/H)_l$ the set of the left cosets of $H$ in $G$. Similarly, the *right cosets of* $H$ in $G$ are the subsets $Hg$ of $G$ defined by $Hg = \{hg : h \in H\}$. We denote by $(G/H)_r$ the set of the right cosets of $H$ in $G$.

A *set of left coset representatives* of $H$ in $G$, also called *left transversal*, is a set of elements $\{g_i : 1 \leq i \leq r\}$ such that $G$ is the disjoint union of the left cosets $g_iH$, $1 \leq i \leq r$. A set of *right coset representatives* or *right transversal* is defined similarly. Given a Cayley digraph $Cay(G, S)$ and a subgroup $H$ of $G$, it is possible to construct two natural quotient graphs of $Cay(G, S)$.

The *left coset digraph of* $G$ with respect to a subgroup $H$ and to a subset $S$, denoted by $Cos_l(G, H, S)$, is the digraph with vertex set $(G/H)_l$, such that there exists an arc $(gH, g'H)$ if and only if there exists an element $s \in S$ and $ghs = g'h'$ for some $h, h' \in H$. This digraph can be seen as obtained from $Cay(G, S)$ by contracting the vertices of each left coset and deleting multiple arcs. For $H = \{I\}$, $Cos_l(G, H, S) = Cay(G, S)$.

The Sabidussi representation theorem [127] asserts that a (di)graph is vertex transitive if and only if it is a left coset (di)graph.

The *right coset digraph of* $G$ with respect to a subgroup $H$ and to a subset $S$, denoted by $Cos_r(G, H, S)$, is the digraph with vertex set $(G/H)_r$, such that there exists an arc $(Hg, Hg')$ if and only if there exists $s \in S$ and $Hgs = Hg'$. This digraph is also called a *Schreier coset graph* or a *Schreier diagram of* $G(mod(H))$ [27]. For $H = \{I\}$, $Cos_r(G, H, S) = Cay(G, S)$. As we will see in Section 3.4.3 a right coset digraph may not be vertex transitive.

# Chapter 3

# Special classes of Cayley graphs

## 3.1 Cayley graphs generated by transpositions

In this part we only consider Cayley digraphs $Cay(G, S)$, where $S$ is a set of transpositions on $\{1, 2, \ldots, n\}$. This case is simpler than the general case since $s = s^{-1}$ for any $s \in S$. Thus $Cay(G, S)$ is a graph. It is called a *Cayley graph generated by transpositions.* This includes many of the frequently studied interconnection Cayley graphs such as hypercubes, (generalized) star graphs, complete transposition graphs, (modified) bubble sort graphs (see the definitions in Chapter 6 and other examples in [103]). Notice that a Cayley graph generated by transpositions is bipartite, each class of the bipartition being defined by the parity of the permutations.

Following [103], we define the transposition graph associated to $S$, and hence to the Cayley graph $Cay(Gr(S), S)$, as follows.

**Definition 8** *Let $S$ be a set of transpositions of $\{1, 2, \ldots, n\}$. The* transposition graph *of $S$, denoted by $T(S)$, is the graph with vertex set $\{1, 2, \ldots, n\}$ and edges $ij$ for all $\langle i, j\rangle \in S$.*

For example, the transposition graph associated to the star graph $ST(n)$ is the star $K_{1,n-1}$ (see Figures 6.6 and 6.7 which display the star graph $ST(4)$ and its associated transposition graph). The transposition graph associated to the bubble sort graph is a hamiltonian path on $n$ vertices. Other examples of transpositions graphs are given in the Appendix.

If $T(S)$ is a tree, it is called *a transposition tree.* This case is studied in [3] (see Theorem 33 in Section 3.3). For this case a useful result is the following lemma ([15], p. 118).

**Lemma 9** *If the set of transpositions $S$ has cardinality $n - 1$, then $Gr(S) = \mathfrak{S}_n$ if and only if $T(S)$ is a tree.*

Lemma 9 implies that if $T(S)$ contains a spanning tree, in other words, if $T(S)$ is connected, then $Gr(S) = \mathfrak{S}_n$. This is in particular the case for the generalized star graph defined in Section 3.2.

The problem is to get some properties of the Cayley graphs $Cay(G, S)$ generated by transpositions, using the characteristics of the associated transposition graphs $T(S)$. Following the results of [103], we will see below that in fact $T(S)$ characterizes $Cay(G, S)$, and that if $T(S)$ is edge transitive, then $Cay(G, S)$ is arc transitive. Our proofs differ from the proofs of [103].

We will use the following lemma to prove that $T(S)$ characterizes $Cay(G, S)$.

**Lemma 10** *Let $S$ and $S'$ be two sets of transpositions of $\{1, 2, \ldots, n\}$, with $s = \langle i, j\rangle \in S$ and $s' = \langle i', j'\rangle \in S'$. Assume that $f$ is an isomorphism from $T(S)$ onto $T(S')$ such that $\{f(i), f(j)\} = \{i', j'\}$. Then there exists a group automorphism $h$ of $\mathfrak{S}_n$ mapping $Gr(S)$ onto $Gr(S')$ such that $h(s) = s'$ and $h(S) = S'$.*

**Proof.** Let $f$ be an isomorphism from $T(S)$ onto $T(S')$ satisfying the hypothesis. We can consider $f$ as a permutation of $\{1, 2, \ldots, n\}$ and define a mapping $h$ by $h(g) = fgf^{-1}$, for any permutation $g$ of $\mathfrak{S}_n$. Then $h$ is a group automorphism of $\mathfrak{S}_n$. Notice that for $t = \langle k, l\rangle$, we have $h(t) = \langle f(k), f(l)\rangle$. In particular, $h(s) = s'$. Since $f$ preserves edges, if $\langle k, l\rangle$ belongs to $S$, then $\langle f(k), f(l)\rangle$ belongs to $S'$. Thus $h(S)$ is a subset of $S'$ and $h$ is a group homomorphism from $Gr(S)$ into $Gr(S')$. Furthermore $h(S) = S'$, since $f$ is an isomorphism from $T(S)$ onto $T(S')$. □

**Corollary 11 [103]** *Let $S$ and $S'$ be two sets of transpositions of $\{1, 2, \ldots, n\}$ such that the two graphs $T(S)$ and $T(S')$ are isomorphic. Then the Cayley graphs $Cay(Gr(S), S)$ and $Cay(Gr(S'), S')$ are also isomorphic.*

**Proof.** Since the two graphs $T(S)$ and $T(S')$ are isomorphic, the groups $Gr(S)$ and $Gr(S')$ are also isomorphic. By Proposition 3, the group homomorphism $h$ defined in Lemma 10 is an isomorphism from $Cay(Gr(S), S)$ onto $Cay(Gr(S'), S')$. □

Recently C. Delorme [52] and J. Fournier [70] have proved the converse of Corollary 11.

**Theorem 12 [52, 70]** *Let $S$, $S'$ be two sets of transpositions of $\{1, \ldots, n\}$. Then the graphs $T(S)$ and $T(S')$ are isomorphic if and only if the corresponding Cayley graphs $Cay(Gr(S), S)$ and $Cay(Gr(S'), S')$ are isomorphic.*

**Corollary 13 [103]** *If the transposition graph $T(S)$ is edge transitive, then the Cayley graph $Cay(G, S)$ is arc transitive.*

**Proof.** Apply Lemma 10 with $S = S'$ and then Proposition 5. □

Finally, let us recall that the conjecture affirming that any Cayley graph $Cay(G, S)$ is hamiltonian is proved if $G = \mathfrak{S}_n$ and $S$ is a set of transpositions, but open if $G = \mathfrak{S}_n$ and every element of $S$ is of order 2 (see the survey [42]).

## 3.2 Cayley digraphs with the rotation property

In order to get more properties of symmetry on Cayley graphs and easier communication schemes in networks, a class of graphs with special graph automorphisms called *rotations* have been considered independently by P. Fragopoulou and S. Akl in [71], [74] and by J.-C. Bermond, T. Kodate and S. Perennes in [21]. The notion of rotation in graph theory was first used in the context of imbeddings (see for example [25], [147]). In this context, a *rotation* of a graph $\Gamma = (V, E)$ is a cyclic permutation of all the vertices adjacent to a vertex, and a *rotation scheme* is a collection $\{r_i, i \in V\}$, where $r_i$ is a rotation at the vertex $i$. It is used to imbed the graph $\Gamma$ into a surface. For a Cayley graph, any cyclic permutation $r$ of the generators allows one to define a rotation scheme by $r_i(j) = ir(i^{-1}j)$ for any edge $ij$ (see [25], page 117).

The notion of *complete rotation* we now define is related, but different.

**Definition 14 [21]** *Let $S = \{s_i, 0 \le i \le d-1\}$ be a subset of a group $G$ with $G = Gr(S)$. A mapping $\omega : G \to G$ is a* complete rotation *of $(G, S)$ if it is bijective and satisfies, for any $x \in G$ and any $i \in \mathbb{Z}_d$, the following two properties:*

$$\omega(I) = I, \tag{3.1}$$
$$\omega(xs_i) = \omega(x)s_{i+1}. \tag{3.2}$$

Some properties of complete rotations are listed here. Some of them are used in [21], [71] and [74]. A study of complete rotations can also be found in [92].

**Lemma 15** *A mapping $\omega$ is a complete rotation of $(G, S)$ if and only if it is a group automorphism of $Gr(S)$ which cyclically permutes the elements of $S$.*

**Proof.** Let $\omega$ be a complete rotation of $(G, S)$. By taking $x = I$ in the equation (3.2) of Definition 14 and using the equation (3.1), we get $\omega(s_i) = s_{i+1}$. By induction on the number of terms in the product of generators which gives an element of $G$, we get from condition (3.2) of Definition 14 that $\omega(xy) = \omega(x)\omega(y)$. Thus $\omega$ is a bijective group homomorphism. Conversely, it is obvious that a group automorphism of $Gr(S)$ which cyclically permutes the elements of $S$ is a complete rotation of $(G, S)$. □

By applying Corollary 4, we get that a complete rotation $\omega$ of $(G, S)$ induces a graph homomorphism and therefore a graph automorphism of $Cay(Gr(S), S)$. In the following, we will also say that the graph $Cay(Gr(S), S)$ *has a complete rotation.*

A digraph $\Gamma$ such that there exists a group $G$ and a subset $S$ with $\Gamma = Cay(G, S)$ and such that $(G, S)$ has a complete rotation is said to be *rotational.* Most popular Cayley digraphs used for interconnection networks such as the hypercube, the generalized hypercube, the star graph, the modified bubble sort graph, the $n$-dimensional torus with the same number of vertices for each dimension (see definitions given in Chapter 6) are rotational. In fact, for these Cayley digraphs, the set of generators $S = \{s_i, i \in \mathbb{Z}_d\}$ of the group $G$ (considered as a permutation group on a set $X$) satisfies $s_{i+1} = \sigma s_i \sigma^{-1}$ for some permutation $\sigma$ on $X$ (depending on the digraph) and any $i \in \mathbb{Z}_d$, or equivalently, $s_i = \sigma^i s_0 \sigma^{-i}$. In such a case $\omega(x) = \sigma x \sigma^{-1}$ is a complete rotation as noticed in [21] and [74].

We use this property to define a complete rotation for a new class of Cayley graphs, the generalized star graphs. Other examples can be found in the Appendix and in [71] and [74].

**Definition 16** *Let $k$ and $n$ be two integers such that $1 \le k < n$. The* generalized star graph $GST(n, k)$ *is the Cayley graph $Cay(Gr(S), S)$, where $S$ is the set of the transpositions $\langle i, j\rangle, i \in \{1, \ldots, k\}, j \in \{k+1, \ldots, n\}$.*

The transposition graph $T(S)$ associated to $GST(n, k)$ is the complete bipartite graph $K_{k,n-k}$. In the case $k = 1$, $GST(n, 1)$ is isomorphic to the star graph $ST(n)$ and $GST(2, 1)$ to the graph $K_2$. Notice that $GST(4, 2)$, which is depicted in Figure 6.8, is isomorphic to the modified bubble sort graph $MBS(4)$, since in both cases $T(S)$ is a cycle of length 4.

The graph $GST(n,k)$ is related to the *arrangement graph* $A(n,k)$ studied in [44]. The graph $A(n,k)$ is the graph with vertex set the set of arrangements of $k$ elements chosen out of $n$ elements in which an edge connects two vertices if and only if they correspond to arrangements differing in exactly one position. Thus $A(n,k)$ is obtained from $GST(n,k)$ by contracting into one vertex all the vertices of $GST(n,k)$ which are permutations of $\mathfrak{S}_n$ which fix $i$, $1 \leq i \leq k$ (and deleting loops and multiple edges). In other words, $A(n,k)$ is the left coset graph of $\mathfrak{S}_n$ with respect to the subgroup $H$ and the subset $S$, where $H = \{\sigma \in \mathfrak{S}_n : Supp(\sigma) = \{k+1,\ldots,n\}\}$.

**Proposition 17 [92]** *If $k$ and $n-k$ are relatively prime, the generalized star graph $GST(n,k)$ is rotational.*

**Proof.** Consider the permutation $\sigma$ defined on $\{1,\ldots,n\}$ by $\sigma = \langle 1,2,\ldots,k\rangle\langle k+1,k+2,\ldots,n\rangle$ Then $\sigma^{-1} = \langle k,k-1,\ldots,1\rangle\ \langle n,n-1,\ldots,k+1\rangle$. For $1 \leq i \leq k$, $k+1 \leq j \leq n$, $\sigma\langle i,j\rangle\sigma^{-1} = \langle 1+imod(k),\ k+1+(j-k)mod(n-k)\rangle$. So let $\omega$ be the group automorphism defined by $\omega(x) = \sigma x \sigma^{-1}$. Since $n-k$ and $k$ are relatively prime, the order of $\sigma$ is the least common multiple of $k$ and $n-k$, that is $k(n-k)$, and condition (2) of Definition 14 is fulfilled. Thus $\omega$ is a complete rotation. □

Not all pairs $(Gr(S),S)$ have a complete rotation, as can be seen in the following example.

**Example 18** *Let $(G,S)$ be $(\mathbb{Z}_n,\mathbb{Z}_n\backslash\{0\})$. Then $(G,S)$ has a complete rotation if and only if $n$ is prime.*

**Proof.** A nonzero group homomorphism $\omega$ of the additive group $\mathbb{Z}_n$ satisfies $\omega(x) = x\omega(1)$, so that if $\omega(1) = a$, then by induction $\omega(x) = ax$, and the set of generators $\mathbb{Z}_n \setminus \{0\}$ is equal to $\{1,a,a^2,\ldots,a^{n-2}\}$. But this is possible if and only if $n$ is prime. □

Note that this example does not give all the values of $n$ for which the complete graph $K_n$ is rotational. For this, we have to consider $K_n = Cay(G, G\setminus I)$, where $G$ is any group.

**Proposition 19 [112]** *The complete graph $K_n$ is rotational if and only if $n$ is a power of a prime.*

This result shows that a graph can have a complete rotation for some, but not for every structure as a Cayley graph.

By applying Proposition 5, we get the following corollary.

**Corollary 20** *Any rotational Cayley digraph is arc transitive.*

**Remark 21** *Example 18 shows that there exist arc transitive Cayley digraphs which are not rotational.*

If $G$ and $H$ are two rotational graphs, is the cartesian product $G\times H$ also rotational? The following proposition was found independently by several people, in particular [11] and [92].

**Proposition 22** *Let $\Gamma$ be a rotational Cayley graph. Then, for any integer $n \geq 1$, the cartesian product of $n$ graphs isomorphic to $\Gamma$, $\Gamma_n = \underbrace{\Gamma\Box\cdots\Box\Gamma}_{n}$, is also rotational.*

**Proof.** Assume $\omega$ is a complete rotation of $(G,S)$, with $S=\{s_i, i\in\mathbb{Z}_d\}$ and $\Gamma = Cay(G,S)$. We denote the vertices of $\Gamma_n$ by $(x_0,x_1,\ldots,x_{n-1})$. The $n\times d$ generators of $\Gamma_n$ are:

$$\begin{array}{rclrclcrcl}
t_0 &=& (s_0,I,\ldots,I) & t_1 &=& (I,s_0,\ldots,I) & \ldots & t_{n-1} &=& (I,\ldots,I,s_0)\\
t_n &=& (s_1,I,\ldots,I) & t_{n+1} &=& (I,s_1,\ldots,I) & \ldots & t_{2n-1} &=& (I,\ldots,I,s_1)\\
\vdots &&& \vdots &&&& \vdots &&\\
t_{(d-1)n} &=& (s_{d-1},I,\ldots,I) & & \ldots & & \ldots & t_{dn-1} &=& (I,\ldots,I,s_{d-1})
\end{array}$$

The rotation on $\Gamma_n$ is given by $\rho(x_0,x_1,\ldots,x_{n-1}) = (\omega(x_{n-1}),x_0,x_1\ldots,x_{n-2})$.

The mapping $\rho$ is a group homomorphism since

$$\begin{aligned}
\rho((x_0,x_1,\ldots,x_{n-1})(y_0,y_1,\ldots,y_{n-1})) &= \rho(x_0y_0,x_1y_1,\ldots,x_{n-1}y_{n-1})\\
&= (\omega(x_{n-1}y_{n-1}),x_0y_0,\ldots,x_{n-2}y_{n-2})\\
&= (\omega(x_{n-1})\omega(y_{n-1}),x_0y_0,\ldots,x_{n-2}y_{n-2})\\
&= \rho(x_0,x_1,\ldots,x_{n-1})\rho(y_0,y_1,\ldots,y_{n-1}).
\end{aligned}$$

Furthermore, $\rho(t_i) = t_{i+1}$ for $0\le i\le dn-1$. □

Proposition 22 implies that cartesian products of isomorphic modified bubblesort graphs and cartesian products of isomorphic generalized star graphs are rotational. Recall that the hypercube is a particular case of a cartesian product of isomorphic (generalized) star graphs $GST(2,1)$, so that Proposition 22 implies also that the hypercube is rotational.

The following results characterizes Cayley graphs generated by a set of transpositions which have a complete rotation.

**Theorem 23 [92]** *The only Cayley digraphs $Cay(G,S)$, with $G$ a permutation group generated by a set $S$ of transpositions, such that $(G,S)$ has a complete rotation are cartesian products of isomorphic modified bubblesort graphs and cartesian products of isomorphic generalized star graphs $GST(n,k)$ with $k$ and $n-k$ relatively prime.*

The proof is based on the following proposition.

**Proposition 24 [92]** *Consider a set $S$ of $d$ transpositions and let $G$ be the permutation group generated by $S$. Then $(G,S)$ has a complete rotation if and only if the transposition graph $T(S)$ is the vertex disjoint union of isomorphic cycles $C_q$ or the vertex disjoint union of isomorphic complete bipartite graphs $K_{t,q}$ with $t$ and $q$ relatively prime.*

## 3.3 Hierarchical Cayley digraphs and fault tolerance

One important problem in parallel computers, and more generally in distributed systems, is the problem of fault tolerance. We only consider here the basic notion corresponding to connectivity of the graph or the digraph (see the survey of R. J. Faudree [67] and references in [58] for advanced concepts). Other notions of fault tolerance will be considered in Section 4.5.

The *vertex connectivity* of a non-complete (di)graph is the minimal number of vertices that must be removed in order to disconnect the (di)graph, while the *edge connectivity* of a (di)graph is the minimal number of edges (arcs) needed to break connectivity. A vertex transitive graph is maximally edge connected ([145], [110]), that is, its edge connectivity equals its degree. Similarly, every strongly connected vertex transitive digraph of out-degree $d$ is $d$-arc connected ([85]).

The *fault tolerance* of a (strongly) connected (di)graph $\Gamma$ is the maximum number $k$ such that, if any $k$ vertices are removed from $\Gamma$, the resulting subgraph is still (strongly) connected. The fault tolerance is equal to one less than the vertex connectivity of the (di)graph unless the (di)graph is complete, in which case it is equal to the vertex connectivity. A regular (di)graph $\Gamma$ of out-degree $d$ and fault tolerance $d-1$ is said to be *optimally fault tolerant.* We only present here results about undirected graphs.

**Proposition 25** [4] *Every connected vertex transitive graph of degree* 2*,* 3*,* 4 *or* 6 *is optimally fault tolerant.*

The following example gives a Cayley graph of degree 5 having vertex connectivity equal to one less than the degree, i.e. fault tolerance equal to two less than the degree.

**Example 26** [2] *Consider the additive commutative group* $\mathbb{Z}_4 \times \mathbb{Z}_2$ *and the ordered set of generators* $S = \{(2,0),(1,0),(3,0),(1,1),(3,1)\}$ *(see Figure 3.1). The corresponding graph* $Cay(G,S)$ *has 8 vertices and is of degree 5. It is composed of two complete graphs* $K_4$*, one with vertices* $(x,0)$*, the other with vertices* $(x,1)$*, joined by edges all incident to one vertex of the set* $\{(1,0),(1,1),(3,0),(3,1)\}$*, which is therefore a cutset.*

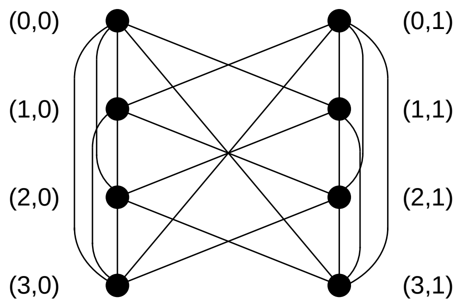


Figure 3.1: Example of graph having fault tolerance equal to two less than the degree.

This example was generalized by B. Alspach ([4], see Theorem 34) to a family of graphs that was more precisely studied in [114]. In order to introduce a class of Cayley graphs which are optimally fault tolerant, we need the following definition.

**Definition 27** *A* minimal generating set *$S$ of a group $G = Gr(S)$ is a unit free generating set such that $Gr(S \setminus \{s\})$ is a proper subgroup*[1] *of $G$ for any $s \in S$.*

**Definition 28** *A* minimal symmetric generating set *$S$ of a group $G = Gr(S)$ is a unit free symmetric generating set such that $Gr(S \setminus \{s, s^{-1}\})$ is a proper subgroup of $G$ for any $s \in S$.*

**Definition 29** *A Cayley digraph (resp. Cayley graph) $Cay(G,S)$ is said to be* strong hierarchical *if $S$ is a minimal generating set (resp. minimal symmetric generating set).*

[1] A proper subgroup is a subgroup different from the group itself.

For example, the hypercube and the star graph are strong hierarchical Cayley graphs. C. Godsil proved that if $S$ is a minimal generating set, then the graph $Cay(G, S)$ is optimally fault tolerant [81]. The result of Godsil was extended to a larger class of Cayley graphs as we explain now.

**Definition 30** *An ordered subset $S = \{s_1, s_2, \ldots, s_d\}$ of $G$ is said to be* quasiminimal *if $s_i \notin Gr(s_1, \ldots, s_{i-1})$, $2 \leq i \leq d$.*

In other words, a minimal generating set is a quasiminimal generating set for any order of the generators.

We give the definition used in [4].

**Definition 31** *A* quasiminimal symmetric generating set *is a unit free and symmetric generating set such that there exists a numbering of the generators in $S$, $s_1, s_2, \ldots, s_d$, such that*

- *if the order of $s_i$ is greater than 2, then $s_i^{-1}$ is either $s_{i-1}$ or $s_{i+1}$;*
- *if the order of $s_i$ is 2, then $Gr(s_1, s_2, \ldots, s_{i-1})$ is a proper subgroup of $Gr(s_1, s_2, \ldots, s_i)$;*
- *if the order of $s_i$ is greater than 2 and $s_i^{-1} = s_{i-1}$, then $Gr(s_1, s_2, \ldots, s_{i-2})$ is a proper subgroup of $Gr(s_1, s_2, \ldots, s_i)$.*

For example, the generating set $S = \{(2, 0), (1, 0), (3, 0), (1, 1), (3, 1)\}$ with the given order is a quasiminimal symmetric generating set of the group of Example 26. But it is not a minimal symmetric generating set since $Gr(S) = Gr(S \setminus (2, 0))$.

This notion is related to the notion of *hierarchical* Cayley graph introduced in [1].

**Definition 32 [1]** *A Cayley graph $Cay(G, S)$ is said to be* hierarchical *if $S$ is a quasiminimal symmetric generating set of $G$.*

By definition, a strong hierarchical Cayley graph is hierarchical, but the converse is false. For example, the pancake graph $PR(n)$ (defined in Section 6.9 and depicted in Figure 6.11) is a hierarchical Cayley graph, but not a strong hierarchical Cayley graph. Other examples of hierarchical Cayley graphs are given by Cayley graphs associated to a transposition tree, as shows the following theorem.

**Theorem 33 [3]** *If the transposition graph $T(S)$ is a tree, then $\Gamma = Cay(G, S)$ satisfies the following properties.*

- *$\Gamma$ has $n!$ vertices and is bipartite;*
- *the connectivity of $\Gamma$ is equal to its degree;*
- *the edges of $T(S)$ can be ordered to have $\Gamma$ as a hierarchical Cayley graph. In this case, $\Gamma$ is obtained by interconnecting $n$ copies of a Cayley graph defined by a transposition tree of order $n - 1$.*

Theorem 33 applies in particular to the star graph and the bubble sort graph.

Hierarchical Cayley (di)graphs are fault tolerant since their connectivity is equal to their degree, except for a particular family. Many authors have studied their connectivity. The strongest result is the following theorem found independently (and by different methods) by B. Alspach and M. Baumslag. It generalizes results of [81] on minimal generating sets, and of [1] on some hierarchical Cayley graphs.

**Theorem 34 [4, 14]** *If $S$ is a quasiminimal symmetric generating set of the group $G$, then the graph $Cay(G,S)$ has connectivity $|S|$, unless $Cay(G,S)$ belongs to a particular family of graphs of connectivity $|S|-1$.*

This particular family contains graphs for which the quasiminimal generating set $S$ satisfies the following conditions:

- $S = \{s_1, s_2, s_2^{-1}, \ldots, s_k, s_k^{-1}\}$ (thus the degree is $2k+1$) is quasiminimal;
- $s_1$ is of order 2;
- for $2 \leq i \leq k$, $s_i$ is of order 4 with $s_i^2 = s_1$ (and therefore $s_i s_1 = s_1 s_i$).

These graphs are studied in [114].

The analogue of Theorem 34 for digraphs has been proved by Y. Hamidoune, A. Llado and O. Serra [87].

Notice that the connectivity of circulant digraphs has been studied by Van Doorn [144]. Maximally connected circulant graphs have been characterized by F. Boesch and R. Tindell [26]. This article considered furthermore superconnected circulant graphs in the following sense. A *superconnected digraph* is a digraph in which every disconnecting set has cardinality at least $d+1$ (where $d = \min_{u \in V}\{d^+(u), d^-(u)\}$), unless it is the set of out-neighbours or the set of in-neighbours of a vertex. Y. Hamidoune, A. Llado and O. Serra have also studied the superconnectivity of abelian Cayley digraphs [86] and the superconnectivity of hierarchical Cayley graphs [88]. In particular, they have proved the following theorem.

**Theorem 35 [88]** *Let $G$ be a group containing a quasiminimal generating set $M$ such that $|M| \geq 4$ and, for $S = M \cup M^{-1}$, the connectivity of the Cayley digraph $Cay(G,S)$ is $|S|$. Then either $Cay(G,S)$ is superconnected or there exists a hierarchical ordering of $M$ with a minimal element $u \in M$ and an element $v \in M$ such that $u^2 = v^2 = I$ and $s^2 = u$ for any $s \in S, s \neq u, s \neq v$.*

Other results on hierarchical Cayley graphs improving the preceding ones can be found in [89] and [90].

The interest of hierarchical Cayley graphs is that they often allow inductive proofs by decomposing (stripping) the graph into smaller members of the same family. These graphs are *scalable* in the sense that they recursively consist of copies of smaller Cayley graphs of the same variety. For example, the pancake graph of dimension 4, $PR(4)$, with generators $S = \{(2,1,3,4),(3,2,1,4),(4,3,2,1)\}$ (see Section 6.9), is composed of 4 pancake graphs of dimension 3 connected together by edges labeled by the generator $(4,3,2,1)$.

Numerous results about communications (as in [83], [22], [38]) and embeddings (as in [29]) in the star graph $ST(n)$ or the pancake graph, $PR(n)$ have been proved by induction on $n$ using the recursive construction of these graphs. In fact, the proofs work for a class of hierarchical Cayley graphs which generalizes both the star graph and the pancake graph, and is defined as follows.

**Definition 36 [83]** *A Cayley graph on the symmetric group, $Cay(\mathfrak{S}_n, S)$, is* recursively decomposable *if $S$ is composed of $n-1$ permutations each of order 2, which can be arranged in a sequence $s_2, s_3, \ldots, s_n$, such that for every $s_i \in S$, $s_i(1) = i, s_i(i) = 1$, $s_i(j) = j$ for $i < j \leq n$.*

To emphasize the importance of hierarchical Cayley graphs, we recall that J. Liu proves in [108] that there exists a hamiltonian decomposition of strong hierarchical Cayley graphs on abelian groups of odd order.

## 3.4 Multistage Cayley digraphs and quotients

### 3.4.1 Motivation

The hypercube is a model of interconnection networks with many good properties which are most of the time a consequence of the fact that it is a Cayley graph and a cartesian product. But it has a major drawback: its degree is $\log_2 N$, where $N$ is the number of vertices. So other graphs with the same number of vertices and the same diameter, but a constant degree, have been studied, such as the binary de Bruijn graph. In fact, a family of graphs, called *hypercubic graphs*, has been intensively studied in the literature with respect to the relations between them. Some of these graphs are called multistage graphs since they model dynamic networks in which nodes are commutators displayed in different levels (see [47] for a short introduction to multistage interconnection networks). We present here some of these hypercubic graphs and their relations, following [13] and [6].

### 3.4.2 Cube-connected cycles and butterfly graphs

Binary strings are used as vertex labels or components of vertex labels for all of the classes of graphs studied in this part. We use $x(i)$ or $x_0x_1\dots\overline{x_i}\dots x_{n-1}$ to denote the binary string obtained from $x = x_0x_1\dots x_{n-1}$ by complementing the bit in position $i$. In this section, all arithmetic on indices and levels is assumed to be $mod(n)$, and all arithmetic on bits is assumed to be $mod(2)$. For example, the hypercube $H(n)$ is here considered as a Cayley graph on the additive group $(\mathbb{Z}_2)^n$ with vertices denoted by $x = x_0x_1\dots x_{n-1}$ with $x_i \in \mathbb{Z}_2, 0 \le i \le n-1$.

**Definition 37** *The* cube-connected cycles graph *of dimension $n$, denoted by $CCC(n)$, is the graph with vertex set $\mathbb{Z}_n \times (\mathbb{Z}_2)^n$ and with edges defined as follows. Any vertex $(l, x)$, where $l \in \{0, \dots, n-1\}$, $x = x_0x_1\dots x_{n-1}$, $x_i \in \mathbb{Z}_2, 0 \le i \le n-1$, is adjacent to the vertex $(l+1, x)$ and to the vertex $(l, x(l))$, where $x(l) = x_0x_1\dots x_{l-1}\overline{x_l}x_{l+1}\dots x_{n-1}$.*

For $n \ge 3$, the graph $CCC(n)$ is derived from $H(n)$ by replacing each vertex $x$ of $H(n)$ by a cycle of length $n$ (see $CCC(3)$ in Figure 6.12). Such a cycle is composed of the vertices labelled $(l, x)$, $0 \le l \le n-1$, where $l$ is called the *level* of the vertex. Each of the $n$ vertices $(l, x)$ of $CCC(n)$ that replaces the vertex $x$ of $H(n)$ inherits one of the $n$ edges that were incident to $x$ in $H(n)$, namely the edge $[(l, x), (l, x(l))]$, which is called an $H$-edge.

This graph has been introduced by Preparata and Vuillemin [116] as a model of very efficient network for algorithms of the "Ascend-Descend" class, which contains classical algorithms (sorting algorithms, Fast-Fourier-Transform algorithms, permutations, convolutions, matrix operations, ...). Furthermore, it has a good VLSI layout area. It also contains cycles of almost every length [80].

A directed version of $CCC(n)$, called in what follows the *digraph* $CCC(n)$, can be considered by replacing each $H$-edge by two opposite arcs and each edge $[(l, x), (l+1, x)]$ by the arc $((l, x), (l+1, x))$. The resulting digraph is regular of out- and in-degree 2.

The *butterfly* graph of dimension $n$ (also called *wrapped butterfly* by some authors), $BF(n)$, is derived from the hypercube in a similar way to $CCC(n)$. The difference is that $BF(n)$ has two $H$-edges corresponding to each former hypercube edge instead of one (Figure 6.14).

**Definition 38** *The* butterfly graph *of dimension $n$, denoted by $BF(n)$, is the graph with vertex set $\mathbb{Z}_n \times (\mathbb{Z}_2)^n$ and with edges defined as follows. Any vertex $(l, x)$, where $l \in \{0, \dots, n-1\}$, $x = x_0 x_1 \dots x_{n-1}$, $x_i \in \mathbb{Z}_2, 0 \le i \le n-1$, is adjacent to the vertex $(l+1, x)$ and to the vertex $(l+1, x(l))$, where $x(l) = x_0 x_1 \dots x_{l-1} \overline{x_l} x_{l+1} \dots x_{n-1}$.*

The edges incident with the vertex $(l, x)$ in $BF(n)$ and inherited from hypercube edges are $[(l, x), (l+1, x(l))]$ and $[(l, x), (l-1, x(l-1))]$. Similarly as for $CCC(n)$, the graph $BF(n)$ can be oriented as a regular digraph of out- and in-degree 2. Each edge $[(l, x), (l+1, y)]$ is replaced by an arc $((l, x), (l+1, y))$.

Both oriented versions of $BF(n)$ and $CCC(n)$ are Cayley digraphs on the group which is the *wreath product* $\mathbb{Z}_2 \wr \mathbb{Z}_n$, i.e. a particular semi-direct product of $\mathbb{Z}_n$ by $\mathbb{Z}_2^n$. The elements of this group $G$ are $(l, x)$, where $l \in \mathbb{Z}_n$ and $x = x_0 x_1 \dots x_{n-1}$ with $x_i \in \mathbb{Z}_2$ for $0 \le i \le n-1$. The identity of the group is $(0, 0^n)$. The homomorphism $h$ from the additive group $\mathbb{Z}_n$ into the automorphisms of the additive group $\mathbb{Z}_2^n$ used in the semi-direct product is defined by $l \mapsto \{y \mapsto \sigma^l(y)\}$, where $\sigma(y_0 y_1 \dots y_{n-1}) = (y_{n-1} y_0 \dots y_{n-2})$, and therefore the product is given by:

$$(l, x)(l', x') = \left(l + l', x + \sigma^l(x')\right) = \left(l + l', (x_0 + x'_{n-l})(x_1 + x'_{n-l+1}) \dots (x_{n-1} + x'_{n-l-1})\right).$$

It is easy to prove the following result.

**Lemma 39** [6] *The digraph $BF(n)$ is a Cayley digraph $Cay(G, S)$ for $S = \{(1, \underbrace{0 \dots 0}_{n}), (1, 1\underbrace{0 \dots 0}_{n-1})\}$. The digraph $CCC(n)$ is a Cayley digraph $Cay(G, S')$ for $S' = \{(1, \underbrace{0 \dots 0}_{n}), (0, 1\underbrace{0 \dots 0}_{n-1})\}$.*

By definition, the hypercube $H(n)$, considered as a symmetric digraph, is a quotient of both graphs $BF(n)$ and $CCC(n)$, obtained by contraction using the mapping $(l, x) \mapsto x$ and deleting loops. $H(n)$ can also be viewed as the left coset digraph of $G$ with respect to the subgroup $H = \{(l, 0^n) : l \in \mathbb{Z}_n\}$ and to $S$ or $S'$, $H(n) = Cos_l(G, H, S) = Cos_l(G, H, S')$.

We will see in the next section what is the right coset graph of $G$ with respect to $H$ and $S$. We finish this presentation by a few remarks on these digraphs.

Despite the similarity of the digraphs $CCC(n)$ and $BF(n)$, it has been proved only recently that $CCC(n)$ is a subgraph of $BF(n)$ [68].

A generalization of these Cayley digraphs is defined similarly on the wreath product $\mathbb{Z}_d \wr \mathbb{Z}_n$ (see [6] and [13]).

There exist other multistage graphs defined in the literature, like the *baseline graph* and the *omega graph*, but they are isomorphic to the butterfly graph (see [47] and the references given there).

### 3.4.3 Group action graphs

We present here mainly the results of [6].

**Definition 40 [6]** *Given a set $V$ and a set $\Pi$ of permutations on $V$, the* group action graph *(*GAG *for short) defined by $V$ and $\Pi$, and denoted by $GAG(V, \Pi)$, is the digraph with vertices set $V$ and arcs $(x, \pi(x)), x \in V, \pi \in \Pi$.*

In such a case, we will say that the arc $(x, \pi(x))$ is labelled by $\pi$. To illustrate this definition, we consider the de Bruijn digraph.

**Definition 41** *The* de Bruijn digraph *$B(d, n)$ is the digraph with vertices set $V = \mathbb{Z}_d^n$ and arcs set given by $E(B(d, n)) = \{(x_0x_1 \ldots x_{n-1}, x_1 \ldots x_{n-1}x_n) : x_i \in \mathbb{Z}_d, 0 \leq i \leq n\}$.*

This digraph is regular of out-degree $d$, but not vertex transitive (only $d$ vertices, $i^n, 0 \leq i \leq d-1$, are incident with loops). It has been intensively studied and proposed as a model of parallel architecture. Many of its good properties come from its being an iterated line-digraph (see [47] and references cited here).

For the sake of simplicity here we only consider the binary de Bruijn digraph $B(2, n)$ (see Figure 3.2).

**Example 42** *The* binary de Bruijn digraph *of order $n$, $B(2, n)$, is a GAG, by taking $V = \mathbb{Z}_2^n$ and $\Pi = \{\pi_0, \pi_1\}$ with*

$$\pi_0(x_0x_1 \ldots x_{n-1}) = x_1 \ldots x_{n-1}x_0 \text{ and } \pi_1(x_0x_1 \ldots x_{n-1}) = x_1 \ldots x_{n-1}\overline{x_0}.$$

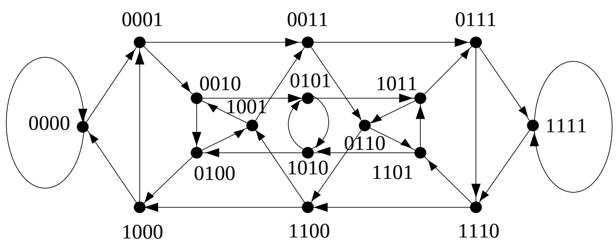


Figure 3.2: Binary de Bruijn digraph: an example of a GAG.

**Lemma 43 [6]**

*(i) Every GAG is regular of out-degree $|\Pi|$.*

*(ii) Every regular digraph is a GAG.*

*(iii) Every connected GAG is strongly connected.*

**Proof.** (i) is evident by definition. (ii) results from Proposition 76 in Section 5.3. (iii) can be proved by using the fact that a connected GAG has an eulerian circuit. □

Let $\Pi$ be a set of permutations on $V$. We now consider $Gr(\Pi)$, the group of permutations on $V$ generated by $\Pi$, and the Cayley digraph $Cay(Gr(\Pi), \Pi)$. The digraph $Cay(Gr(\Pi), \Pi)$ is said to be the *Cayley digraph associated to* $GAG(V, \Pi)$.

The following theorem shows how GAGs and Cayley digraphs are related. Recall that the stabilizer of a vertex $v \in V$, $Stab_{Gr(\Pi)}(v) = \{\sigma \in Gr(\Pi) : \sigma(v) = v\}$, is a subgroup of $Gr(\Pi)$.

**Theorem 44 [6]** *Every connected $GAG(V, \Pi)$ is isomorphic to the right coset digraph of $G = Gr(\Pi)$ with respect to $Stab_{Gr(\Pi)}(v_0)$ and $\Pi$, where $v_0$ is any vertex of $V$.*

**Proof.** Following [6], for convenience of notation, we adopt here a different sense for products: $\pi_1 \cdot \pi_2$ is the permutation such that $\pi_1 \cdot \pi_2(u) = \pi_2(\pi_1(u))$.

If $GAG(V, \Pi)$ is connected, then $GAG(V, \Pi)$ is strongly connected by Lemma 43. This implies that $Gr(\Pi)$ acts transitively on $V$ since for any vertices $u$, $v$ of $V$ there exists a path from $u$ to $v$. If the arcs of this path are labelled from $u$ to $v$ successively by $\pi_1, \pi_2, \ldots, \pi_l$, where $\pi_i \in \Pi$, $1 \leq i \leq l$, then $v = \pi_l(\pi_{l-1}(\cdots(\pi_1(u))\cdots))$. Therefore, $v = g(u)$ with $g = \pi_1 \cdot \pi_2 \cdot \ldots \cdot \pi_l$, $g \in Gr(\Pi)$ (notice here the order of the product).

Let $v_0$ be any vertex of $V$ and $H = Stab_{Gr(\Pi)}(v_0)$. An isomorphism from $(G/H)_r$ to $GAG(V, \Pi)$ results from the mapping $\phi$ defined as follows:

$$\begin{array}{ccl} G & \longrightarrow & V \\ g & \longmapsto & v_0 g = g(v_0). \end{array}$$

In fact, $\phi(g) = \phi(g')$ if and only if $v_0 g = v_0 g'$, or $gg'^{-1} \in H$, i.e. $Hg = Hg'$. Furthermore, $\phi$ is surjective since $Gr(\Pi)$ acts transitively on $V$. Thus the mapping $Hg \in Cos_r(G, H, \Pi) \mapsto g(v_0) \in GAG(V, \Pi)$ is bijective. It is arc preserving since if $Hg' = Hg\pi$, then $\pi(g(h(v_0))) = \pi(g(v_0)) = g'(v_0)$ and $(g(v_0), g'(v_0))$ is an arc in $GAG(V, \Pi)$. Conversely we have that any arc $(g(v_0), \pi(g(v_0)))$ is the image of an arc $(Hg, Hg \cdot \pi)$ of $Cos_r(G, H, \Pi)$. □

**Example 45 [6]** *The binary butterfly digraph $BF(n)$ of dimension $n$ is the Cayley digraph associated to the de Bruijn digraph $B(2, n)$ considered as the GAG of Example 42.*

Taking $v_0 = 0^n$, the mapping $\phi$ of the proof of Theorem 44 is defined in the case of $BF(n)$ by

$$(l, x_0 x_1 \cdots x_{n-1}) \mapsto x_l x_{l+1} \cdots x_{l-1}.$$

Thus, keeping the notation of Lemma 39, the right coset digraph $Cos_r(G, H, S)$, which is a quotient of $BF(n)$, is $B(2, n)$ which is not vertex transitive. Similarly, $Cos_r(G, H, S')$ is a non-transitive digraph, different from $B(2,n)$, and called the *shuffle-exchange digraph* of dimension $n$ (see [6]). This digraph is a subgraph of $B(2,n)$ [68].

To illustrate the interest of the relation between GAGs and Cayley digraphs and the power of algebraic tools, we give an application of the results cited in this section.

Notice that, since $Gr(\Pi)$ acts transitively on $V$, all the stabilizers of elements of $V$ are conjugate subgroups of $Gr(\Pi)$ and therefore are all isomorphic.

**Theorem 46 [6]** *If the group action graph $GAG(V,\Pi)$ has a subgraph $T$ which is a directed tree, then the Cayley digraph $Cay(Gr(\Pi),\Pi)$ contains $|Stab_{Gr(\Pi)}(v_0)|$ vertex-disjoint directed trees isomorphic to $T$, where $v_0$ is any vertex of $GAG(V,\Pi)$.*

**Corollary 47 [6]** *The butterfly digraph $BF(n)$ contains $n$ vertex disjoint copies of a complete directed binary tree of height $n-1$.*

**Proof.** This is an easy consequence of Theorem 46, Theorem 44, and Example 42, since the de Bruijn graph $B(2,n)$ contains a complete binary tree of height $n-1$. This tree can be easily constructed taking as the root $0^{n-1}1$, and adding, for each vertex $x_0x_1\dots x_{n-1}$ already put in the tree, the two children $x_1\dots x_{n-1}0$ and $x_1\dots x_{n-1}1$. □

Corollary 47 was first proved by a complicated purely combinatorial proof (see [6]). We conclude this part with an open problem on group action graphs.

**Definition 48 [100]** *The* Kautz digraph $K(d,n)$ *is the digraph with vertex set*

$$V = \{x_0x_1\dots x_{n-1} \in \mathbb{Z}_{d+1}^n : x_i \neq x_{i+1}, 0 \leq i \leq n-2\}$$

*and arc set given by*

$$\{(x_0x_1\dots x_{n-1}, x_1\dots x_{n-1}x_n) : x_0x_1\dots x_{n-1} \in V, x_n \in \mathbb{Z}_{d+1}, x_{n-1} \neq x_n\}.$$

The Kautz digraph $K(d,n)$ is a Cayley digraph if and only if $n=1$ or $n=2$ and $d+1$ is a prime power [30]. For any integers $d,n$, $K(d,n)$ is regular of out- and in-degree $d$ and strongly $d$-connected, thus it is a GAG and a coset digraph.

**Problem 49** *Give an explicit construction of the Cayley digraph $Cay(Gr(\Pi),\Pi)$ associated to $K(d,n)$ considered as $GAG(V,\Pi)$.*

# Chapter 4

# Routing and diameter problems

## 4.1 Paths and factoring

Let $Cay(G, S)$ be a Cayley digraph and $g, g' \in G$ two of its vertices. If $g' = gs_1s_2 \ldots s_t$ with $s_i \in S, 1 \leq i \leq t$, then $s_1, s_2, \ldots, s_t$ defines a path from vertex $g$ to vertex $g'$ with arcs labelled by $s_1, s_2, \ldots, s_t$. It is equivalent to finding a path from $g^{-1}g' = s_1s_2 \cdots s_t$ to $I$. Thus, as noticed in [132], to represent an element $g$ in the group $Gr(S)$ as a word written with generators of $S$, is equivalent to finding a path in the Cayley digraph $Cay(G, S)$ from the vertex $I$ to the vertex $g$.

Rewriting techniques can be used to find a shortest path between two vertices if the presentation of the group is known [69]. Anyway, a minimal presentation of a finite group can be constructed starting from fundamental cycles in the Cayley graph (see [34] or [140]). Notice that the minimal word problem in groups (for a given element $g$, find $s_1, s_2, \ldots, s_t$ with $s_i \in S, 1 \leq i \leq t$, and $t$ minimal) is known to be NP-hard [64] if the set of generators is not fixed in advance (more precisely, this problem is PSPACE-complete [99]).

In the case of permutation groups, the problem of factoring has been studied extensively. Algorithms that have been developed by group theorists for permutation groups can be used. One tool is the *Sim's factorisation* ([31], [132]). Let $G$ be a group of permutations on $X$. A *base* $B$ is a subset of $X$ such that if $g \in G$ satisfies $g(\beta) = \beta$ for every $\beta \in B$, then $g = I$. Thus any permutation of $G$ is defined by its action on $B$. Assume $B = \{\beta_1, \beta_2, \cdots, \beta_b\}$ and consider the subgroups $G_i$ of $G$,

$$G_i = \{g \in G : g(\beta_j) = \beta_j, 1 \leq j \leq i+1\}, i = 0, \ldots, b-1.$$

$G_i$ is a subgroup of $G_{i-1}$. $S$ is said to be a *strong generating set* relative to the base $B$ if $S \cap G_i$ generates $G_i$ for any $0 \leq i \leq b-1$.

**Example 50** **[31]** *Take for $G$ the group of the symmetries of the square generated by $S = \{s_1, s_2\}$ with $s_1 = \langle 1, 2, 3, 4\rangle$ and $s_2 = \langle 2, 4\rangle$. The order of $G$ is 8. A base is $B = \{1, 2\}$, $G_0 = Stab_G(1) = Gr(s_2)$, $G_1 = Stab_G(1) \cap Stab_G(2) = I$, and $S$ is a strong generating set relative to $B$.*

**Proposition 51** **[132]** *With the notation given above, let $U_i$ be a set of right coset representatives of $G_i$ in $G$. Then every element of $G$ has a unique representation as a product $u_1u_2 \cdots u_b$, with $u_i \in U_i, 1 \leq i \leq b$.*

Sim's algorithm gives a factorisation of each group element as a unique product of coset representatives. This property is used in [132] to find efficient routing algorithms for Cayley graphs $Cay(G, S)$ such that $S$ is a strong generating set, in particular, for the star graph and the pancake graph.

## 4.2 Diameter calculation

Let $\Gamma = Cay(G, S)$ be a Cayley digraph of order $n$ and out-degree $d$. Since $\Gamma$ is vertex transitive its diameter can be obtained by constructing a breadth-first search tree from any vertex of the graph, in particular $I$. Thus, the diameter is the maximum distance $d(I, g)$ from $I$ to a vertex $g$ of the graph. It can be computed in time in $O(dn)$ instead of $O(dn^2)$ for a general digraph.

Let $D = Diam(Cay(G, S))$ be the diameter of $Cay(G, S)$. Then we can express $D$ as follows:

$$D = \max_{g \in G} \{\min \{t : \exists s_1, s_2, \ldots, s_t, g = s_1 s_2 \ldots s_t, s_i \in S, \forall i,\ 1 \leq i \leq t\}\}.$$

In the case of a Cayley digraph $Cay(G, S)$, where $G$ is a group of permutations, the calculation of $d(\sigma, I)$ is equivalent to sorting the permutation $\sigma$ of $G$ using the generators in the minimum number of operations.

According to Cooperman *et al.* [40] it is important to observe that the computation of the diameter for large groups is more often limited by memory resources rather than by the CPU time. They give the example of the calculation of the diameter of the Cayley graph of the cube $2 \times 2 \times 2$, with 9 generators, which needs $3,674,160$ nodes to be stored in semiconductor memory. They succeed with 8 megabytes using techniques from [39] against 80 megabytes in a method using simple breadth-first search to construct a spanning tree for the graph.

## 4.3 Diameter of particular graphs

For transposition graphs we have the following theorem.

**Theorem 52 [3]** *If the transposition graph $T(S)$ is a tree, then $\Gamma = Cay(Gr(S), S)$ satisfies*

$$Diam(()\Gamma) \leq \max_{\pi \in Gr(S)} \{c(\pi) - n + \sum_{i=1}^{n} \delta(i, \pi(i))\},$$

*where $\delta$ is the distance in the tree $T(S)$, and $c(\pi)$ is the number of cycles in the permutation $\pi$ on $\{1, 2, \ldots, n\}$.*

This can be made precise for the star graph and the bubble sort graph.

**Proposition 53 [1] [103]** *The diameter of the star graph $ST(n)$ is $\lceil 3(n-1)/2 \rceil$ for $n \geq 3$. The diameter of the bubble sort graph $BS(n)$ is $n(n-1)/2$.*

Despite the similarity between the star and pancake graphs, the diameter of the pancake graph $PR(n)$ is unknown for $n > 13$ (for example, the diameter of $PR(13)$ is 15). This problem is known as the *pancake flipping problem*. There exist bounds given by Gates and Papadimitriou.

**Proposition 54 [76]** *The diameter of the n-pancake graph $PR(n)$ satisfies*

$$Diam(PR(n)) \leq \frac{5n+5}{3}$$

*for all $n$, and*

$$Diam(PR(n)) \geq \frac{17n}{16}$$

*for infinitely many $n$.*

A generalization of this problem to *burnt pancakes* has been studied in [91]. A vertex in the burnt pancake graph is a signed permutation $(a_1, a_2, \ldots, a_n)$, where $a_i \in X$ or $-a_i \in X$. It is conjectured that the furthest vertex from $I$ (or equivalently the hardest to sort permutation) is $-I$ with $-I = (-1, -2, \ldots, -n)$. If this is true, results of [91] give an upper bound equal to $3(n+1)/2$ for all $n$ congruent to $3 mod(12)$ which is better than the one of proposition 54. D. Cohen and M. Blum have shown that $-I(n)$ can be sorted in $\frac{23}{14}n = 1.63n$ steps. The problem is also to find the furthest vertex from $I$ in the $n$-pancake graph. M. Heydari and H. Sudborough also showed that the problem of deciding, given two vertices $x$ and $y$ in the pancake graph and an integer $k$, whether there is a path of length $k$ connecting $x$ and $y$, is NP-complete [91].

It should be noted that it can even be difficult to compute the diameter of Cayley (di )graphs on additive groups, as proves the study of *circulant (di )graphs* (also called *multiple loop networks*), which are Cayley (di)graphs defined on $\mathbb{Z}_n$ (see the survey [18] and references given here).

In [13] and [5], relationships are established between the diameter of a Cayley graph and the diameter of a subgraph and of a coset graph.

## 4.4 $(\Delta, D)$-graphs

In this part we only consider graphs. The design of interconnection networks comes up against two fundamental constraints: the number of connections for every node is limited, and the number of intermediate nodes on a communication path must be small, in order for the time to broadcast information to be acceptable. This is translated in graph theory as follows.

**Problem 55 ($(\Delta, D)$-problem) [61]** *Find graphs with the maximal number of vertices with given constraints on the maximum degree $\Delta$ and the diameter $D$.*

This problem has been intensively studied in the last fifteen years (see, for example, [19], [56] and the references given there).

A $(\Delta, D)$-graph is defined as a graph of maximum degree $\Delta$ and diameter $D$. Following [47], we denote by $N(\Delta, D)$ the maximum order of a $(\Delta, D)$-graph. The only known bound on $N(\Delta, D)$ is the Moore bound. This bound is obtained by calculating the maximum number of vertices at distance $0, 1, \ldots, D$ from one vertex.

**Proposition 56 [113]** $N(\Delta, D) \leq 1 + \Delta + \Delta(\Delta-1) + \cdots + \Delta(\Delta-1)^{D-1}$, *thus* $N(\Delta, D) = (\Delta(\Delta-1)^D - 2)/(\Delta-2)$ *for* $\Delta \geq 3$ *and* $N(2, D) \leq 1 + 2D$.

Graphs which attain the bound are called *Moore graphs.*

**Theorem 57 [10, 43, 96]** *Moore graphs exist only for:*

$\Delta = 2$*: cycles* $C_{2D+1}$*;*

$D = 1$*: complete graphs* $K_{\Delta+1}$*;*

$D = 2$ *and* $\Delta = 3$*: Petersen graph;*

$D = 2$ *and* $\Delta = 7$*: Hoffman-Singleton graph (a vertex-transitive graph with* 50 *vertices);*

*and, perhaps, for* $D = 2$ *and* $\Delta = 57$ *(if such a graph exists it has 3250 vertices and is not vertex transitive).*

Furthermore, apart from Moore graphs, the only graphs known to be optimal are $(3,3)$ , $(4,2)$- and $(5,2)$-graphs with 20, 15, and 24 vertices, respectively. Except for the cycle of length 4, $N(\Delta, D)$ is not equal to the Moore bound minus 1. In the absence of good bounds, researchers are running a race to construct ever "better" $(\Delta, D)$-graphs. A table of large $(\Delta, D)$ graphs is regularly updated by C. Delorme ([20], see Table given in [56]). The constructions are mostly algebraic ones, using Cayley graphs, coset graphs and compounds (see [51]). For some values of $(\Delta, D)$, the best graphs presently known are Cayley graphs. In this case the problem splits into two parts: choice of the group and then choice of the generating set.

With experimental computing the nine authors of [33] have found small diameter Cayley graphs constructed from linear groups. Their experiment gives no method except the fact that abelian groups are to be avoided in this kind of problem. They use a particular kind of matrix groups and fix the cardinality of the generating set. The diameter is then computed by repeated random choices of the generating set using the expression for the diameter given in Section 4.2. The bounds given in [33] have been improved by constructions of M. Dinneen and P. R. Hafner [56] which use semidirect products of cyclic groups $\mathbb{Z}_n$. For example, they obtain more than 30 entries of Delorme's table by taking a semidirect product of $\mathbb{Z}_m$ with $\mathbb{Z}_n$ of the type

$$[x, y] \times_a [u, v] = [(x+u) mod(m), (ya^u + v) mod(n)].$$

In particular, with $m = 15$, $n = 77$, $a = 4$ and $S = \{[6,2],[9,5],[10,9],[5,24]\}$, they construct a $(4,7)$-graph with 1155 vertices. Recently, M. Sampels obtained better constructions by using abstract presentations of groups. In the article [131], the use of abstract presentations of groups is explained, and presentations of (new) graphs are given with the following values of $(\Delta, D, N)$: $(4,10,17316)$, $(6,5,1344)$ and $(4,8,3080)$. The value $(4,10,17316)$ is now updated to $(4,10,17525)$ by O. Delmas [48] ; F. Comellas and M. Mitjana have also a construction for the value $(8,3,253)$ [37].

The results of a simulation study about the practical use of degree/diameter-graphs as communication networks are presented in [129]. Since then, using the technique above and genetic algorithms on metacyclic groups, the following values of $(\Delta, D, N)$ were obtained by M. Sampels [130]: $(8,3,253)$, $(5,4,210)$, $(7,4,672)$, $(5,5,546)$, $(6,5,1395)$, $(5,8,15990)$, $(4,10,17535)$.

A table on this problem with details on all the graphs can be found on the WWW at

`http://maite71.upc.es/grup_de_grafs/table_g.html`

Let us note that similar work exists for bipartite graphs and digraphs. The Moore bounds for these classes of graphs are only reached for small degrees or small diameters (see [47]); F. Comellas and M.A. Fiol found some vertex-symmetric digraphs with small diameter (see [36]). Also, some good non-symmetric $(\Delta, D)$ digraphs are known, like Kautz digraphs (see Definition 48 and results cited in [47]) or Cyclic prefix digraphs which are based on coset digraphs [65].

Cayley graphs of non-solvable groups, in particular non-abelian simple groups (groups without non-trivial normal subgroup) are considered by many authors as producing good interconnection networks. This is a field which is of interest also to group theorists.

**Theorem 58** [8] *There exists a constant $c$ such that every non-abelian simple group has a set $S$ of at most seven generators for which the diameter of the Cayley graph $Cay(G, S)$ is at most $c \log_2(|G|)$.*

It is proved in [5] that this property is not true for Cayley graphs of abelian groups. On the other hand, it is conjectured that a general bound exists in the non-abelian case:

**Conjecture 59** [9] *There exists a constant $c$ such that for every non-abelian finite simple group $G$ and any generating set $S$,*

$$Diam(Cay(G, S)) \leq (\log_2(|G|))^c.$$

## 4.5 Diameter vulnerability

It is of practical importance to know what is the diameter of the resulting digraph after faulty vertices (or edges) have been deleted from a digraph. A first result of this kind concerns the hypercube.

**Theorem 60** [47] *Every pair of vertices of the hypercube $H(n)$ is connected by $n-1$ paths of length at most $n$ and a path of length at most $n+1$, all these paths being internally vertex disjoint.*

Thus after deleting at most $n-1$ vertices, the diameter of the resulting graph is at most $n+1$, i.e. increases by at most 1. But Theorem 60 gives more than the value of the diameter of the graph obtained after deleting faulty vertices, since it bounds the length of parallel paths between any two vertices.

Different generalizations of the diameter can be found in the literature in order to measure the diameter after deleting vertices, such as the notions of *fault-diameter* and *wide-diameter* (see [101] and the survey of Hsu [98]). We give here some definitions and results from [98].

**Definition 61** *Let $k$ and $\kappa$ be two integers, $k \leq \kappa$. Let $\Gamma$ be a $\kappa$-connected graph, and $x$ and $y$ two distinct vertices.*

- *The $k$-*fault diameter* of $\Gamma$, denoted by $D_k(\Gamma)$, is the maximum of the diameters of the graphs obtained from $\Gamma$ after deleting any set of $k-1$ vertices.*
- *The $k$-*distance* $d_k(x, y)$ between $x$ and $y$ is equal to the least number $l$ such that there exist $k$ vertex-disjoint paths between $x$ and $y$ whose lengths are at most $l$.*
- *The $k$-*diameter* of $\Gamma$, $Diam(k)\Gamma$, is the maximum of the $k$-distances $d_k(x, y)$ taken over all distinct pairs $x$, $y$.*

Notice that $D_1(\Gamma) = Diam(1)\Gamma = Diam(\Gamma)$. By Menger's theorem, if a graph $\Gamma$ is $\kappa$-connected then, for $k \leq \kappa$, its $k$-diameter and its $k$-fault diameter are well defined and finite.

If $\Gamma$ is a $k$-connected graph, its $k$-diameter is also called its *wide-diameter* and is denoted by $wdiam(\Gamma)$, and its $k$-fault diameter is simply called *fault diameter*. It is easy to see that

$$Diam(\Gamma) \leq D_k(\Gamma) \leq wdiam(\Gamma).$$

Graphs with small wide-diameter and large connectivity are desirable because they enable fast communication using parallel paths between two vertices. A graph with connectivity $k$ is said to be *strongly resilient* if $wdiam(\Gamma) = Diam(\Gamma) + O(1)$. For a $k$-regular, $k$-connected graph $\Gamma$ it is known that

$$wdiam(\Gamma) \geq Diam(\Gamma) + 1.$$

In this case $wdiam(\Gamma) = Diam(\Gamma) + 1$ is optimal. Thus by Theorem 60, the hypercube $H(n)$ which is $n$-regular, $n$-connected and has wide-diameter equal to $n+1$, is optimal in this sense. Notice that this is not the case for every Cayley graph, since, for the cycle $C_n$, which is 2-regular and 2-connected, the wide-diameter is equal to $n-1$ and the fault diameter is equal to $n-2$. Thus it is interesting to know which graphs, and in particular which Cayley graphs, are strongly resilient.

The star graph is strongly resilient like the hypercube. The diameter of the resulting graph after deleting at most $n-1$ vertices from the star graph $ST(n)$ increases by at most one for $n \geq 7$ (see [121] and [104]). Furthermore, between two vertices $x$ and $y$ of $ST(n)$, there exist $n-1$ paths of length $d(x,y)$ or $d(x,y)+2$ or $d(x,y)+4$ [45].

The generalized hypercubes (defined as cartesian products of complete graphs) are also strongly resilient [60].

The same notion of wide-diameter can be defined for digraphs and some results have been obtained for particular Cayley digraphs, proving that there are optimal. For example, Hsu proved that if $\Gamma$ is the circulant digraph $Cay(\mathbb{Z}_{a^n}, \{1, a, a^2, \ldots, a^{n-1}\})$, then $wdiam(\Gamma) \leq Diam(\Gamma)+1$. It can be noticed that in the case of the multidimensional torus, the undirected case is more complicated than the directed one [98].

To finish this section we give some related questions and results.

- The results on the wide-diameter of de Bruijn and Kautz digraphs (which are quotients of Cayley digraphs, see Section 3.4.3) extend to iterated line digraphs (see [115] and references given there).

- It can be interesting to know the connectivity under forbidden faulty sets (for example, the neighborhood of any vertex) so that more faults are supported without increasing the diameter too much. In [120] it is proved that any set of $2n-5$ vertices not containing all the neighbors of some vertex can be deleted from the star graph $ST(n)$, and the resulting diameter is at most increased by 2. The same property is true for hypercubes [123].

- One can relax the condition on the diameter in order to find better algorithms. For example, Q. Gu et S. Peng [84] give linear time algorithms for finding paths in the faulty hypercube $H(n)$ obtained by deleting $(n-1)$ arbitrary vertices, and in the faulty star graph $ST(n)$ after deleting $(n-2)$ arbitrary vertices. They obtain in particular a path of length at most $Diam(ST(n))+3$ between any two non-faulty vertices of $ST(n)$ in $O(n)$ time.

- Given a strongly connected digraph $\Gamma$, a routing $R$ (see definition in Chapter 5) and a set $F$ of faults (arcs and vertices), the *surviving routing digraph* $S(\Gamma, R, F)$ is defined as the digraph with vertex set the non-faulty vertices of $\Gamma$, and in which $(x,y)$ is an arc if and only if the path $R(x,y)$ does not contain any element from $F$ [57]. The diameter of $S(\Gamma, R, F)$ gives the maximum number of retransmissions needed to insure the communication between two nodes of the network after failure. It is interesting to find routings $R$ such that $Diam(S(\Gamma), R, F)$ is small. Some results are known for particular Cayley graphs, like hypercubes and some circulant digraphs [63].

# Chapter 5

# Indices of routings

In this section, we study loads on vertices and arcs induced by routings in digraphs. Following [35], we define a routing as follows.

For a given (strongly) connected (di)graph $\Gamma$ of order $n$, a *routing* $R$ is defined as a set of $n(n-1)$ simple (di)paths $R(x,y)$, specified for every ordered pair $(x,y)$ of vertices in $\Gamma$, such that $R(x,y)$ joins $x$ to $y$. A *partial routing* is any subset of a routing.

## 5.1 Types of routings

Different types of routings can be considered. We define some of them below. These types of routings have interesting properties and are often used in practice.

A routing $R$ is said to be:

(a) *of shortest paths* or *minimal* if, for any two vertices $x$ and $y$, the path $R(x,y)$ is a path of shortest length between $x$ and $y$;

(b) *symmetric* if, for any two vertices $x$ and $y$, the path $R(y,x)$ is the reverse of the path $R(x,y)$;

(c) *consistent* if, for any two vertices $x$ and $y$ and for each vertex $z$ belonging to the dipath $R(x,y)$, the dipath $R(x,y)$ is the concatenation of the dipaths $R(x,z)$ and $R(z,y)$.

**Example 62** *Let $\Gamma$ be the cycle $C_4$ with vertices labeled by $\mathbb{Z}_4$. We illustrate the above definitions by considering some routings on $\Gamma$.*

*A routing $R$ such that $R(0,3)=(0,1,2,3)$ and $R(0,2)=(0,3,2)$ is not consistent. Notice that since the diameter is equal to 2, any routing of shortest paths is consistent.*

*Consider $R_1$ such that $R_1(i,j)$ is the path from $i$ to $j$ following a chosen direction of the cycle. More precisely, $R_1(0,1)=(0,1)$, $R_1(0,2)=(0,1,2)$, $R_1(0,3)=(0,1,2,3)$. $R(i,j)$ is obtained from $R(0,j-i)$ by applying the translation $x \mapsto x+i mod(4)$. $R_1$ is not a routing of shortest paths and is not symmetric, but it is consistent.*

*We define $R_2$ such that $R_2(0,2)=(0,1,2)$, $R_2(2,0)=(2,1,0)$, $R_2(1,3)=(1,2,3)$, $R_2(3,1)=(3,2,1)$, and if $i$ and $j$ are adjacent, $R_2(i,j)=(i,j)$. $R_2$ is a symmetric routing of shortest paths.*

*We define* $R_3$ *such that* $R_3(0,2) = (0,1,2)$, $R_3(2,0) = (2,3,0)$, $R_3(1,3) = (1,2,3)$, $R_3(3,1) = (3,0,1)$, *and if* $i$ *and* $j$ *are adjacent,* $R_3(i,j) = (i,j)$. $R_3$ *is a routing of shortest paths but it is not symmetric.*

## 5.2 Vertex-forwarding index

For a given digraph $\Gamma$, a vertex $x$ of $\Gamma$ and a specified routing $R$, the *load* of $x$, denoted by $\xi(\Gamma, R, x)$, is the number of dipaths of $R$ going through $x$. The *vertex-forwarding index* $\xi(\Gamma, R)$ is the maximum number of dipaths of $R$ going through a vertex of $\Gamma$. The minimum of $\xi(\Gamma, R)$ over all possible routings $R$ of $\Gamma$ is denoted by $\xi(\Gamma)$ and is called the *vertex-forwarding index of* $\Gamma$ or for short the *forwarding index of* $\Gamma$:

$$\xi(\Gamma) = \min_R \xi(\Gamma, R).$$

The minimum of $\xi(\Gamma, R)$ over all possible routings of *shortest paths* $R$ of a strongly connected digraph $\Gamma$ is called the *minimal forwarding index* of $\Gamma$ and is denoted by $\xi_m(\Gamma)$.

**Example 63** *Let us consider the routings defined in Example 62 on the graph* $C_4$. *We have:*

$\xi(\Gamma, R_1, x) = 3$ *for any vertex* $x$, *and* $\xi(\Gamma, R_1) = 3$;

$\xi(\Gamma, R_2, 1) = \xi(\Gamma, R_2, 2) = 2$, $\xi(\Gamma, R_2, 0) = \xi(\Gamma, R_2, 3) = 0$, $\xi(\Gamma, R_2) = 2$;

$\xi(\Gamma, R_3, x) = 1$ *for any vertex* $x$, *and* $\xi(\Gamma, R_3) = 1$;

*thus* $\xi(\Gamma) = \xi_m(\Gamma) = \xi(\Gamma, R_3) = 1$.

Since forwarding indices were first defined in graphs [35], most of the results in the literature are given for graphs instead of digraphs, but they can often be easily extended to digraphs [111]. Nevertheless, we give here most of the results for graphs as they appear in the literature.

To compute $\xi(G)$ for an arbitrary graph is difficult as shows the complexity of the following problem.

**Problem 64 (FI (Forwarding Index problem))** Instance: A graph $\Gamma$ and an integer $k$. Question: $\xi(\Gamma) \leq k$ ?

**Theorem 65 [122]** *FI is NP-complete.*

It has been shown that there are graphs for which the forwarding index is obtained neither for a minimal nor for a symmetric routing. For example, the forwarding index of wheels cannot be obtained for minimal routings [35], and the forwarding index of cycles of even length cannot be obtained for symmetric routings [94].

Similarly, we define the problems **SFI** (symmetric FI), **MFI** (minimal FI), as the above problems when the considered routings are restricted to symmetric routings, and routings of shortest paths, respectively.

**Theorem 66 [122]** *SFI is NP-complete.*

The problem FI is still NP-complete for graphs of diameter 2. By contrast:

**Proposition 67 [93]** *MFI is polynomial for graphs of diameter* 2.

The preceding results show how important it is to find classes of graphs for which the forwarding index can be easily calculated. The following proposition gives bounds on $\xi(\Gamma)$ and characterizes the routings for which the lower bound is attained.

**Proposition 68 [35]** *Let* $\Gamma$ *be a simple connected graph of order* $n$*. Then*

*(i)* $\frac{1}{n}\sum_u \sum_{v\neq u}(d(u,v)-1) \ \leq \xi(\Gamma) \leq \xi_m(\Gamma) \leq (n-1)(n-2)$.

*(ii) The equalities* $\xi(\Gamma) = \xi_m(\Gamma) = \frac{1}{n}\sum_u \sum_{v\neq u}(d(u,v)-1)$ *hold if and only if there exists a routing of shortest paths in* $\Gamma$ *for which the load of all vertices is the same.*

The proof given in [35] works also for digraphs. This proposition shows how interesting are the graphs in which there exists a routing of shortest paths which loads all vertices of $\Gamma$ uniformly, in which case $\xi(\Gamma)$ has the lowest possible value, given by $\frac{1}{n}\sum_u \sum_{v\neq u}(d(u,v)-1)$. The next theorem proves that Cayley digraphs form such a class of digraphs.

**Theorem 69 [93]** *In any Cayley digraph* $\Gamma = (V, E)$*, there exists a consistent routing of shortest paths in which the load of every vertex* $u$*, and therefore also the vertex forwarding index, is equal to*

$$\sum_{v\in V, v\neq u} d(u,v) - (n-1).$$

**Proof.** Let $\Gamma = Cay(G, S)$ be a Cayley digraph and consider an order of $S$, $S = s_1, s_2, \ldots, s_d$, with $s_1 < s_2 < \ldots < s_d$. This induces a lexicographic order on the dipaths of shortest length from $x$ to $y$, for any two different vertices $x$ and $y$ of $\Gamma$. The dipath $\rho = x_0x_1\ldots x_l$ is smaller than the dipath $\rho' = y_0y_1\ldots y_l$ ($x_0 = y_0 = x$ and $x_l = y_l = y$) if there exists $i$, $1 \leq i < l$, such that

$$x_k = y_k \text{ for } 0 \leq k < i, \text{ and } x_{i-1}^{-1}x_i < y_{i-1}^{-1}y_i.$$

For any two vertices $x$ and $y$ of $\Gamma$, define $R(x,y)$ to be the smallest dipath (for the induced order) of shortest length among all dipaths of shortest length from $x$ to $y$.

It is easy to verify that the routing $R$ is consistent. If $R(x,y) = xx_1\ldots x_{l-1}y$, the path $xx_1\ldots x_i$, $1 \leq i < l$, has length $d(x, x_i)$ since $R(x,y)$ is of shortest length. Moreover, it is a smallest path for the order, otherwise $x_0x_1\ldots x_l$ would not be a smallest path. Thus $R(x, x_i) = x_0x_1\ldots x_i$. Similarly, $R(x_i, y) = x_i\ldots x_{l-1}y$ and $R(x,y)$ is the concatenation of $R(x, x_i)$ and $R(x_i, y)$.

Let us now prove that the number of paths of the routing going through any vertex is a constant. Consider the regular subgroup of translations of $Aut(\Gamma)$, $F = \{f_a : a \in G\}$, as defined before Theorem 2. For any two vertices $x$ and $y$ of $\Gamma$, there exists a unique $a \in G$ such that $f_a(x) = y$. Notice that an arc labeled by $s \in S$ is transformed by $f_a$ into an arc with the same label, since $xs = y$ implies $axs = ay$ (the associativity of the law is here important). Thus any path $R(u,v)$ going through $x$ is transformed by $f_a$ into a path of shortest length from $au$ to $av$ which is, moreover, a smallest path, i.e. the path $R(au, av)$, which goes through $y$. Therefore, by symmetry, $\xi(\Gamma, R, x) = \xi(\Gamma, R, y)$. We can then finish by Proposition 68. □

In order to appreciate the strength of this property of Cayley graphs, observe that a class of graphs is constructed in [93] for which the vertex forwarding index is not given by a consistent routing of shortest paths. Thus it is worth considering the following problems.

**Problem 70 [93]** *For which digraphs $\Gamma$ does there exist a consistent routing $R$ such that $\xi(\Gamma) = \xi(\Gamma, R)$ ?*

**Problem 71 [93]** *For which digraphs $\Gamma$ does there exist a consistent routing of shortest paths $R$ such that $\xi_m(\Gamma) = \xi(\Gamma, R)$ ?*

The following conjecture has attracted many researchers for the last ten years without a complete success.

**Conjecture 72 [94]** *In any vertex-transitive graph $\Gamma = (V, E)$, there exists a routing of shortest paths in which the load of every vertex $u$, and therefore also the vertex-forwarding index, is equal to*

$$\sum_{v \neq u} d(u, v) - (n - 1).$$

The next two sections present the major results obtained so far on this conjecture.

## 5.3 Quasi-Cayley digraphs

We now present the theory of quasi-Cayley (di)graphs as given by G. Gauyacq in [79] and [78]. The class of quasi-Cayley graphs is a new class of vertex-transitive graphs which contains the class of Cayley graphs, and which satisfies Conjecture 72.

**Definition 73** *A* quasigroup *$Q$ is a set of elements and a binary operation denoted as a product such that*

*(i) for every ordered pair $(a, b)$ of elements of $Q$ the product $a \cdot b$ exists and belongs to $Q$;*

*(ii) for every ordered pair $(a, b)$ of $Q$ there exists a unique element $x \in Q$ such that $a \cdot x = b$;*

*(iii) for every ordered pair $(a, b)$ of $Q$ there exists a unique element $y \in Q$ such that $y \cdot a = b$.*

By definition, the multiplication table of the binary operation of a quasigroup is a latin square. Notice that an associative quasigroup is a group [54].

**Example 74 [79]** *Take $\mathbb{Z}_2 \times \mathbb{Z}_5$ and the binary operation defined for any $a, c \in \mathbb{Z}_2$ and $b, d \in \mathbb{Z}_5$ by $(a, b) \cdot (c, d) = (a + c, b + 2^a d)$, where $2^0 = 1$ and $2^1 = 2$. Then $(\mathbb{Z}_2 \times \mathbb{Z}_5, \cdot)$ is a quasigroup.*

In the following, we denote for short by $Q$ the quasigroup $(Q,.)$ and omit the dot $\cdot$ in the products if no confusion can arise.

**Definition 75** *Let $Q$ be a quasigroup and let $S$ be a subset of $Q$. The* quasigroup digraph $QD(Q,S)$ *is defined as the digraph with vertex set $Q$ and arcs $(g, g \cdot s)$, $g \in Q$, $s \in S$.*

The digraph $QD(Q,S)$ is regular since for every vertex $g \in Q$, $d^+(g) = |S|$, and by property (iii) of the definition $d^-(g) = |S|$. By property (ii) of the definition, a quasigroup digraph has no multiple arc.

**Proposition 76 [54]** *A digraph $\Gamma = (V,E)$ is regular if and only if it is a quasigroup digraph.*

This notion is similar to the notion of group action graph (GAG) defined in Section 3.4.3. Every quasigroup digraph $QD(Q,S)$ is a $GAG(Q,\Pi)$ with $\Pi = \{x \mapsto x \cdot s, s \in S\}$. Conversely, every GAG without multiple arc is a quasigroup digraph [79].

Quasi-Cayley digraphs are particular quasigroup digraphs defined as follows.

**Definition 77** *A subset $S$ of $Q$ is said to be* right associative *if for any two elements $a$ and $b$ of $Q$, $a \cdot (b \cdot S) = (a \cdot b) \cdot S$.*

**Definition 78** *A subset $S$ of $Q$ is said to* generate $Q$ *if, for any element $g$ of $Q$, there exist elements $s_1, s_2, \ldots, s_t$ of $S$ such that $g = (\ldots((s_1 \cdot s_2) \cdot s_3)\ldots) \cdot s_t$.*

**Definition 79** *An element $I$ of $Q$ is a* right unit *if $a \cdot I = a$ for any $a \in Q$.*

**Definition 80** *Given a quasigroup $Q$ with a right unit and a subset $S$ of $Q$ which is right associative, the quasigroup digraph $QD(Q,S)$ is called a* quasi-Cayley digraph.

We list below some easily obtained properties of any quasi-Cayley digraph $QD(Q,S)$.

• If $S$ generates $Q$, then $QD(Q,S)$ is strongly connected. For this, one can construct a path from $I$ to every element $g$, using the facts that $g$ is a product of elements of $S$ and that $S$ is right associative. Similarly, a path from every element $g$ to $I$ can be obtained, using the existence of $g'$ such that $gg' = I$, with $g'$ a product of elements of $S$.

• If the right unit $I$ does not belong to $S$, $QD(Q,S)$ has no loop. If furthermore $Q$ is a group, then the quasi-Cayley digraph $QD(Q,S)$ is the same as the Cayley digraph $Cay(Q,S)$. Thus any Cayley digraph is a quasi-Cayley digraph.

• If $S$ has the property that for any $s \in S$, the element $s'$ which satisfies $ss' = I$ (see property (ii) of Definition 73) belongs to $S$, then $QD(Q,S)$ is a graph. For this, assume $(u,v)$ is an arc of $QD(Q,S)$, then $v = us$. Consider $s' \in S$ such that $ss' = I$. Since $S$ is right associative, there exists $s''$ such that $u(ss') = (us)s''$. This implies $u = uI = u(ss') = (us)s'' = vs''$, therefore $(v,u)$ is an arc of $QD(Q,S)$.

**Example 81 [79]** *The Petersen graph is a quasi-Cayley graph which is not a Cayley graph. Indeed, taking $Q$ as defined in Example 74 and $S = \{(0,1),\ (1,0),\ (0,4)\}$, $QD(Q,S)$ is the Petersen graph. Since it is well known that the Petersen graph is not a Cayley graph, this proves that not every quasi-Cayley digraph is a Cayley digraph.*

G. Gauyacq also proved that the Hoffman-Singleton graph and the metacirculant graphs are quasi-Cayley graphs, but that some Kneser graphs and some Johnson graphs, are not quasi-Cayley graphs. In particular the line-graph of $K_n$ for $n$ even and $n > 5$, is not a quasi-Cayley graph [79].

In the following we assume that $S$ generates $Q$, and we give some properties of quasi-Cayley digraphs which generalize properties of Cayley digraphs.

**Proposition 82 [79]** *Any quasi-Cayley digraph is vertex-transitive.*

**Proof.** Let $QD(Q,S)$ be a quasi-Cayley digraph. As for Cayley digraphs, every element $a \in Q$ defines an automorphism $f_a$ of $QD(Q,S)$ by $f_a(x) = ax$. This is a consequence of property (ii) of Definition 73 and the fact that $S$ is right associative. Furthermore, the set $F = \{f_a, a \in Q\}$ acts regularly on $Q$ by property (iii) since, for any two elements $x$ and $y$ of $Q$, there exists a unique element $a$ of $Q$ such that $ax = y$, i.e. $f_a(x) = y$. □

As for Cayley digraphs, regular automorphisms characterize quasi-Cayley digraphs as is shown by Theorem 84 which generalizes Theorem 2.

**Definition 83 [79]** *A* regular family *of a digraph $\Gamma = (V,E)$ is a subset $F$ of the automorphism group $Aut(\Gamma)$ which acts regularly on $V$, i.e. for any two vertices $x$ and $y$ of $\Gamma$ there exists a unique automorphism $f \in F$ such that $f(x) = y$.*

**Theorem 84 [79, 78]** *A strongly connected digraph $\Gamma = (V,E)$ is a quasi-Cayley digraph if and only if $Aut(\Gamma)$ contains a regular family on $V$.*

**Proof.** We only give a sketch of the *if* part of the proof. Assume $\Gamma = (V,E)$ is a strongly connected digraph and $F$ a regular family of $\Gamma$. Let $V = \{1,2,\ldots,n\}$ and define $f_i$ as the unique element of $F$ such that $f_i(1) = i$, $1 \leq i \leq n$. A binary operation $\cdot$ is defined on $V$ by

$$i \cdot j = f_i(j).$$

It can be easily verified that $(V,\cdot)$ is a quasigroup. Since by definition $f_i(1) = i \cdot 1 = i$, 1 is a right unit.

Let $S$ be the set of the out-neighbours of 1, i.e. the vertices $u$ of $V$ such that $(1,u) \in E$. One can verify that $S$ is right associative and generates $V$ (using the existence of a dipath from 1 to every element of $V$). Finally,

$$\begin{aligned}(u,v) \in E \quad &\Leftrightarrow \quad (f_u^{-1}(u), f_u^{-1}(v)) \in E \\ &\Leftrightarrow \quad (1, f_u^{-1}(v)) \in E \Leftrightarrow f_u^{-1}(v) \in S \Leftrightarrow v = f_u(s) = u \cdot s, s \in S.\end{aligned}$$

□

As for Cayley graphs, products of quasi-Cayley graphs are quasi-Cayley graphs.

**Proposition 85 [79]** *If $\Gamma_1$ and $\Gamma_2$ are two quasi-Cayley digraphs, then the cartesian product, the weak product and the lexicographic product of $\Gamma_1$ and $\Gamma_2$ are also quasi-Cayley digraphs.*

**Proof.** Consider a regular family $F_i \subseteq Aut(\Gamma)_i$ of $\Gamma_i$, $i = 1,2$. Given $f_i \in F_i$, $i = 1,2$, define the mapping $f$ on $V_1 \times V_2$ by $f(x_1,x_2) = (f_1(x_1), f_2(x_2))$. It is easy to see that $f$ is an automorphism for any of the three products and that $\{f : f_1 \in F_1, f_2 \in F_2\}$ is a regular family of the product. □

In [79], G. Gauyacq characterizes the quasi-Cayley graphs with the notion of quasi-normal subgroup.

**Definition 86 [79]** *A subgroup $H$ of a group $G$ is said to be* quasi-normal *if there exists a common system of left coset representatives of $H$ and all its conjugate subgroups $gHg^{-1}$ for $g \in G$.*

**Theorem 87 [79]** *Let $\Gamma$ be a vertex transitive graph, and $H$ the subgroup of $Aut(\Gamma)$ which is the stabilizer $Stab_{Aut(\Gamma)}(u)$ of any fixed vertex $u$ of $\Gamma$. Then $\Gamma$ is a quasi-Cayley graph if and only if $H$ is a quasi-normal subgroup of $Aut(\Gamma)$.*

Other properties of quasi-normal subgroups can be found in [79]. Finally, we give the theorem which motivates all this study.

**Theorem 88 [79, 78]** *In any quasi-Cayley digraph there exists a routing of shortest paths in which the load on every vertex is the same.*

**Proof.** We only give the idea of the proof. It is similar to that of Theorem 69 except that we cannot define the routing so easily and insure that it is consistent.

Let $\Gamma = QD(Q, S)$ be a quasi-Cayley digraph. The idea is to consider a partial routing $R'$ consisting of paths of shortest length from the vertex $I$ (the right unit) to any vertex $u$ of $\Gamma$, $\{R'(I, u) : u \in Q, u \neq I\}$ . Consider the regular family $F$ of $Aut(\Gamma)$, $F = \{f_x, x \in Q\}$, as defined after Proposition 82. In particular, $f_x(I) = xI = x$ for any $x \in Q$. Since the image by $f_x$ of a dipath of shortest length $P$ is a dipath of shortest length, denoted by $f_x(P)$, a routing $R$ can be defined by

$$R(x, y) = f_x(R'(I, f_x^{-1}(y)))$$

for any two vertices $x$ and $y$ (notice that $R(I, y) = f_I(R'(I, f_I^{-1}(y)))$ can be different from $R'(I, y)$ since $I$ is not assumed to be a left unit). Then $R = \{f_x(R'(I, u)) : x \in Q, u \in Q, u \neq I\}$. It can then be proved that the number of paths of the routing $R$ going through any vertex is a constant. □

## 5.4 Digraphs with regular orbits

G. Gauyacq has defined another class of vertex transitive graphs which strictly contains quasi-Cayley graphs and satisfies Conjecture 72. Our presentation differs a little from [79] (stating results for digraphs), but all the results and ideas come from it. To understand the definition of this class, recall that in the proof of Theorem 88, the key point is the uniform load induced on a vertex by a well chosen subset of paths obtained by automorphisms. Thus the idea is to consider a subset of $R$ consisting of paths between vertices at the same distance and such that these paths can be obtained from one of them by applying automorphisms. This leads to the notion of $V \times V$ orbit.

**Definition 89** *Let $\Gamma = (V, E)$ be a strongly connected vertex-transitive digraph, and $k \leq Diam(\Gamma)$. A* $k$-orbit *$\Omega$ of $\Gamma$ is a subset of $V \times V$ such that*

*(i) there exists $(u, v) \in \Omega$ with $d(u, v) = k$;*

*(ii) for every $(u', v') \in V \times V$, $(u', v') \in \Omega$ if and only if there exists $g \in Aut(\Gamma)$ such that $g(u) = u'$ and $g(v) = v'$.*

Since the distance between vertices is unchanged by automorphisms, $d(u', v') = d(u, v) = k$ for every $(u', v')$ in a $k$-orbit. Thus a $k$-orbit is an orbit for the action of the group $Aut(\Gamma)$ on $V \times V$ consisting of ordered pairs of vertices $(u', v')$ at distance $k$. Therefore $V \times V$ is partitioned into $k$-orbits.

**Lemma 90** *Let $\Omega$ be a $k$-orbit. Then:*

*(i) For any vertex $u \in V$, there exists $v \in V$ such that $(u, v) \in \Omega$.*

*(ii) For any vertex $u \in V$, the number of vertices $v \in V$ such that $(u, v) \in \Omega$ is independent of the choice of $u$; let $\alpha(\Omega)$ be this number.*

*(iii) $|\Omega| = |V| \times \alpha(\Omega)$.*

**Proof.** (i) Since $\Gamma$ is vertex-transitive, for any $u' \in V$ there exists an automorphism $f$ and a vertex $v'$ such that $f(u) = u'$, $f(v) = v'$ and $(u', v') \in \Omega$.

(ii) Let $(u, x_1), (u, x_2), \ldots, (u, x_p)$ be the elements of $\Omega$ whose first coordinate is $u$. Given any $(u', v') \in \Omega$, there exists $g \in Aut(\Gamma)$ such that $g(u) = u'$ and $(u', g(x_1)), (u', g(x_2)), \ldots, (u', g(x_p))$ are all in $\Omega$. Thus $|\{v : (u, v) \in \Omega\}| \leq |\{v' : (u', v') \in \Omega\}|$ and by symmetry, $|\{v : (u, v) \in \Omega\}| = |\{v' : (u', v') \in \Omega\}|$.

(iii) The equality $|\Omega| = |V| \times \alpha(\Omega)$ comes from facts (i) and (ii). □

**Proposition 91 [79]** *Let $\Gamma$ be a vertex-transitive digraph and $\Omega$ a 2-orbit of $\Gamma$. Then there exists a partial routing $\{R(u, v) : (u, v) \in \Omega\}$, consisting of shortest paths between the couples of vertices of $\Omega$, which loads uniformly all vertices of $\Gamma$.*

**Proof.** Consider a bipartite graph $B$ in which one class consists of $\Omega$ and the other, denoted by $Y$, consists of the ordered pairs $(x, i)$, where $x \in V$ and $i$ is any integer $\in \{1, \ldots, \alpha(\Omega)\}$. Notice that $|\Omega| = |V| \times \alpha(\Omega) = |Y|$. An edge connects $(u, v) \in \Omega$ with $(x, i)$ if and only if $d(u, x) = d(x, v) = 1$. $B$ is regular, thus it contains a perfect matching. Let $R(u, v) = (u, s, v)$ if $(u, v)$ is matched with $(s, i)$. The load of any vertex $s$ is then equal to $\alpha(\Omega)$, since $s$ belongs to the paths connecting the $\alpha(\Omega)$ ordered pairs $(u, v)$ matched respectively with $(s, 1), (s, 2), \ldots, (s, \alpha(\Omega))$. □

The following corollary shows that Conjecture 72 is true for vertex-transitive digraphs of diameter 2.

**Corollary 92 [79]** *If $\Gamma$ is a vertex-transitive digraph of diameter 2, then there exists a routing of shortest paths which loads uniformly all vertices of $\Gamma$.*

We turn now to the general case.

**Definition 93 [79]** *A $k$-orbit $\Omega$ is said to be* regular *if, for every $(u, v) \in \Omega$, there exists a shortest path $u, s_1, s_2, \ldots, s_k = v$ and a subset $F$ of $Aut(\Gamma)$ such that*

*(i) for every $(u', v') \in \Omega$, there exists a unique $f \in F$ such that $f(u) = u'$ and $f(v) = v'$;*

*(ii) for any $i, 1 \leq i \leq k-1$, and for any $x \in V$, the number of $f \in F$ such that $f(s_i) = x$ depends only on $\Omega$.*

Let $\Omega$ be a regular $k$-orbit. By property (i) of Definition 93, $|\Omega| = |F|$. By the property (ii) of the definition, let $\beta(\Omega)$ be the number of $f \in F$ such that $f(s_i) = x$ for any $i, 1 \leq i \leq k-1$, and for any $x \in V$. Let $(u, x_1), (u, x_2), \ldots, (u, x_p)$, with $p = \alpha(\Omega)$, be the elements of $\Omega$ with first coordinate $u$. Consider $t$ such that $(s_1, t) \in \Omega$. There exist $\beta(\Omega)$ automorphisms $f \in F$ such that $f(s_1) = u$ so that $f(t) \in \{x_1, x_2, \ldots, x_p\}$ and by property (i), $\{f(t), f \in F\} = \{x_1, x_2, \ldots, x_p\}$. Thus $\beta(\Omega) = \alpha(\Omega)$, i.e. the number of $v \in V$ such that $(u, v) \in \Omega$ for any $u \in V$.

We first give the result which justifies Definition 93, before defining a new class of vertex-transitive digraphs.

**Lemma 94 [79]** *If $\Omega$ is a regular $k$-orbit, then there exists a partial routing $R_\Omega$ defined on $\Omega$ for which the load of all the vertices of the digraph $\Gamma$ is the same.*

**Proof.** Define $R_\Omega$ by $R_\Omega(u, v) = u, s_1, s_2, \ldots, s_k$ with $s_k = v$, and for any $(u', v') \in \Omega$ $R_\Omega(u', v') = f(R_\Omega(u, v))$, where $f$ is the unique automorphism of $F$ such that $f(u) = u'$ and $f(v) = v'$. Thus the partial routing $R_\Omega$ is the collection of paths $f(R_\Omega(u, v))$, $f \in F$. To calculate the load of any vertex $x$ by $R_\Omega$, consider the number of times $x$ is in position $i$ on a path $f(R_\Omega(u, v))$. By property (ii) of Definition 93, this is exactly $\alpha(\Omega)$. Thus the load of the vertex $x$ by $R_\Omega$ is equal to $(k-1)$ times $\alpha(\Omega)$ and is independent of $x$. □

From Lemma 94, it is easy to get:

**Corollary 95 [79]** *If $\Gamma$ is a vertex-transitive digraph in which all $k$-orbits are regular for $1 \leq k \leq Diam(\Gamma)$, then there exists a routing of shortest paths which loads uniformly all vertices of $\Gamma$.*

**Proposition 96 [79]** *In any quasi-Cayley digraph, all $k$-orbits are regular.*

G. Gauyacq gives some examples of vertex-transitive digraphs in which all $k$-orbits are regular and which are not quasi-Cayley digraphs, such as the odd graphs $O_4, O_5, O_6, O_8$ [79]. Notice that the proof of this involves far from evident combinatorics.

**Problem 97 [79]** *Find an example of vertex-transitive digraph in which not all $k$-orbits are regular.*

Similar to Figure 2.1, Figure 5.1 summarizes the implications between the different classes of vertex transitive graphs introduced in this part.

## 5.5 Edge-forwarding index

The vertex-forwarding index of a routing can be considered as a measure of the load of the vertices. Similarly, one can consider the *load of any arc $e$* in a given routing $R$ of a strongly connected digraph $\Gamma = (V, E)$, that is, the number of dipaths of $R$ going through the arc $e$. If $\Gamma$ is a graph, the original definition of [94] considers for each edge $uv$ the sum of the loads of the two arcs $(u, v)$ and $(v, u)$ of the equivalent symmetric digraph. In the following, to avoid confusion, we only consider graphs and give results as they appear in the literature. But the reader can keep in mind that the results can also be considered for digraphs.

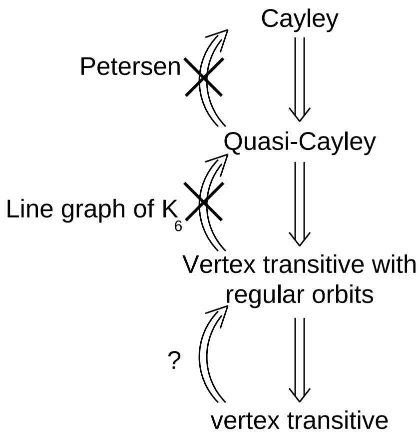


Figure 5.1: Relations between classes of vertex transitive graphs.

For a graph $\Gamma = (V, E)$ and a routing $R$, let $\pi(\Gamma, R, e)$ be the number of paths of $R$ going through the edge $e$. The *edge-forwarding index* of $(\Gamma, R)$, denoted by $\pi(\Gamma, R)$, is the maximum number of paths of $R$ going through any edge of $\Gamma$, that is:

$$\pi(\Gamma, R) = \max_{e \in E} \pi(\Gamma, R, e).$$

The *edge-forwarding index* of $\Gamma$ [94] is defined as

$$\pi(\Gamma) = \min_{R} \pi(\Gamma, R).$$

Similarly, the *minimal edge-forwarding index*, $\pi_m(\Gamma)$, is defined by replacing paths with paths of shortest length in the definitions.

As for the vertex-forwarding index, the following inequality is easily obtained from the definition.

**Proposition 98 [94]** *Let $\Gamma = (V, E)$ be a connected graph. Then*

$$\pi_m(\Gamma) \geq \pi(\Gamma) \geq \frac{1}{|E|} \sum_{(u,v) \in V \times V} d(u, v)$$

*and the equalities*

$$\pi_m(\Gamma) = \pi(\Gamma) = \frac{1}{|E|} \sum_{(u,v) \in V \times V} d(u, v)$$

*hold if and only if there exists a routing of dipaths of shortest length in $\Gamma$ for which the load of all the edges is the same.*

Despite the similarity of the two notions of forwarding indices, there does not exist an edge-uniform routing of shortest paths in every Cayley graph (consider for example the cartesian product of the two complete graphs $K_2$ and $K_p$ for $p \geq 3$). On the other hand, there exist infinite families of Cayley graphs in which such routings exist [50], [138]. In [77] it is proved that the edge-forwarding index of star graphs and complete transposition graphs achieves (within one) the lower bound given above.

We finish this section by giving two conjectures in order to find sufficient conditions on graphs insuring the existence of routings with uniform loads for the edges or almost uniform loads (only different by one) for the vertices and the edges.

**Conjecture 99** **[94]** *In any distance-transitive graph $\Gamma = (V, E)$, there exists a routing $R$ of shortest paths such that for any vertex $u$, the edge-forwarding index is equal to*

$$\pi(\Gamma) = \pi(\Gamma, R) = \left\lceil \frac{n}{|E|} \sum_{v \in V} d(u, v) \right\rceil .$$

**Conjecture 100** **[94]** *In any distance-transitive graph there exists a routing of shortest paths for which the load is the same for any vertex, and almost the same for any edge.*

No example is given in [94] of an arc-transitive graph which does not satisfy the conjectures. In view of the results obtained so far, the hypothesis of these conjectures can surely be reduced. One can also expect to prove the following conjecture.

**Conjecture 101** *In any arc transitive Cayley graph $G = (V, E)$, there exists a routing $R$ of shortest paths such that the load is the same for any vertex, and almost the same for any edge.*

## 5.6 Orbital regular graphs

P. Solé has introduced a new class of graphs which allows an easy calculation of the edge-forwarding index by averaging over the distances between pairs of distinct vertices [138]. We adopt here the presentation of [66] which is more precise than that of [138].

**Definition 102** *Let $\Gamma = (V, E)$ be a simple connected graph and $H$ a subgroup of $Aut(\Gamma)$. $\Gamma$ is said to be $H$-*orbital regular *if:*

*(i) $H$ acts regularly on each of its orbits in $\{(x, y) : x, y \in V, x \neq y\}$.*

*(ii) One of these orbits, say $O$, is such that $E = \{xy : (x, y) \in O\}$.*

*A graph $\Gamma$ is said to be* orbital regular *if it is $H$-orbital regular for some subgroup $H$ of $Aut(\Gamma)$.*

Note that an orbital regular graph is edge-transitive.

**Example 103** *(C. Delorme) The Cayley graph defined on the additive group $\mathbb{Z}_{13}$ with generators $S = \{1, -1, 5, -5\}$ is orbital regular. The orbits are*

$$\begin{aligned} O &= \{(x, y) : \ x - y = \pm 1, \pm 5\}, \\ \Omega_2 &= \{(x, y) : \ x - y = \pm 2, \pm 3\}, \\ \Omega_3 &= \{(x, y) : \ x - y = \pm 4, \pm 6\}. \end{aligned}$$

*The automorphisms of $H$ are given by $f(x) = ax + b$, for $x \in \mathbb{Z}_{13}$, where $a \in S$ and $b \in \mathbb{Z}_{13}$.*

In fact, almost all orbital regular graphs are Cayley graphs as follows from the characterization given in [66] using Frobenius groups. Let us recall that a *Frobenius group* is a group $G$ of permutations on a set $V$ such that the only element that fixes at least two elements of $V$ is the identity. For a Frobenius group $G$, a *$G$-Frobenius graph* is a graph $\Gamma$ with vertex set $V$ such that for some $G$-orbit $O$ in $\{(x, y) : x, y \in V, x \neq y\}$, the edge set of $\Gamma$ is $E = \{xy; (x, y) \in O\}$.

**Theorem 104 [66]** *Let $\Gamma = (V, E)$ be a connected graph with $n$ vertices, and let $H$ be a subgroup of $Aut(\Gamma)$. Then $\Gamma$ is $H$-orbital regular if and only if the following holds:*

*(i)* $\Gamma$ *is an $H$-Frobenius graph, or*

*(ii)* $\Gamma$ *is a cycle of length $n$ and $H = \mathbb{Z}_n$ with $n \geq 3$, or*

*(iii)* $\Gamma$ *is the star $K_{1,n-1}$.*

Properties of Frobenius groups imply that any Frobenius graph is a Cayley graph [66] so that we get:

**Corollary 105 [66]** *All orbital regular graphs except stars $K_{1,n-1}$ are Cayley graphs.*

We now return to the property which motivates the study of orbital regular graphs.

**Proposition 106 [138]** *In an orbital regular graph with $n$ vertices the edge-forwarding index is*

$$\pi(\Gamma) = \frac{1}{|E|} \sum_{(u,v) \in V \times V} d(u, v).$$

**Proof.** Let $\Gamma = (V, E)$ be an orbital regular graph. In each orbit $\Omega$ an ordered pair of vertices $(u, v)$ is selected and a shortest path $R(u, v)$ is chosen. For every other pair of vertices $(u', v')$ in the same orbit $\Omega$, put $R(u', v') = f(R(u, v))$, where $f$ is the unique automorphism of $H$ such that $f(u) = u'$ and $f(v) = v'$. Let $e$ and $e'$ be two different edges of $\Gamma$. Then by definition there exists a unique automorphism $h \in H$ such that $h(e) = e'$, so that the paths through $e'$ are the images by $h$ of the paths through $e$. □

Other results on the edge-forwarding index of orbital regular graphs can be found in [66].

## 5.7 Multicommodity flow problem

It is proved in [46] that the problem of determining the value of the vertex/edge-forwarding indices (respectively of the vertex/edge-forwarding indices in case of shortest paths) is an instance of the integral multicommodity flow problem (respectively integral flow with multipliers problem) for which good heuristics or approximation algorithms exist.

Another way to study forwarding indices is to enlarge the problem by considering it as an integer valued version of a real valued one. We adopt here the presentation of [136]. We first recall some definitions about multicommodity flows, and then present how multicommodity flows and forwarding indices are related.

Let $\Gamma$ be a connected graph considered as a symmetric digraph $\Gamma = (V, E)$ and $x, y$ two vertices of $\Gamma$. We denote by $\mathcal{P}_{x,y}$ the dipaths from $x$ to $y$ in $\Gamma$ and by $\mathcal{P}$ the set of all dipaths in $\Gamma$.

**Definition 107** *A* uniform concurrent multicommodity flow $f$ *on a graph* $\Gamma$ *is a function $f : \mathcal{P} \to \mathbb{R}^+$ such that, for any two vertices $x$ and $y$ of $\Gamma$, $\sum_{p \in \mathcal{P}_{x,y}} f(p) = 1$*

This definition uses an "edge-path" formulation which is equivalent to the more usual node-arc formulation (see for example [27]). This means that the flow $f$ associates to any dipath a real value in such a manner that, for any vertices $x, y$, the sum taken over all the dipaths from $x$ to $y$ is equal to 1. If we restrict the values of $f$, and take integral values ($f(p) = 0$ or $f(p) = 1$), the choice of $f$ is equivalent to the choice of a directed path between any pair of vertices.

The flow in an arc $e$ of $\Gamma$ is defined by

$$f(e) = \sum_{\{p: e \in p\}} f(p),$$

and the *edge congestion* of the flow $f$ is

$$\mu_f = \max_{e \in E} f(e).$$

Similarly, the flow in a vertex $x$ of $\Gamma$ is defined by

$$f(x) = \sum_{\{p: x \in p\}} f(p),$$

and the *vertex congestion* of the flow $f$ is

$$m_f = \max_{x \in X} f(x),$$

A flow that minimizes $\mu_f$ (resp. $m_f$) is said to be *edge* (resp. *vertex*) *optimal.*

Let $f$ be an integral flow, and $R$ the associated routing defined by $R(x, y) = p, p \in \mathcal{P}_{x,y}, f(p) = 1$. Then the following equalities link the forwarding indices of $R$ to the congestions of the flow $f$:

$$m_f = \xi(G, R) + 2(n-1), \qquad \mu_f = \pi(G, R).$$

(Notice that the added value $2(n-1)$ corresponding to the fact that a vertex is the extremity of $2(n-1)$ dipaths.)

Therefore the following problems, denoted by UCFP in [136], are generalizations of forwarding index problems.

**Problem 108 (Uniform concurrent multicommodity flow problem on the edges)** *Find a flow $f$ which minimizes the edge congestion $\mu_f$.*

**Problem 109 (Uniform concurrent multicommodity flow problem on the vertices)** *Find a flow $f$ which minimizes the vertex congestion $m_f$.*

The UCFP can be solved in polynomial time using polynomial time linear programming algorithms (see [133]). But the resulting algorithms are not considered efficient enough for practical applications. On the other hand, efficient approximations exist [134], [133].

The *integral uniform concurrent flow problem*(IUCFP) is to compute a flow $f$ such that $f(p) \in \{0, 1\}$, for any $p \in \mathcal{P}$, which is either edge or vertex optimal.

Let $G$ be a subgroup of $Aut(\Gamma)$. A uniform concurrent multicommodity flow $f$ on $\Gamma$ is said to be *$G$-invariant* if for any $g \in G$ and any $p \in \mathcal{P}$, we have $f(p) = f(g(p))$.

A dipath $p$ such that $f(p) = 1$ is said to be an *active path.* The following result gives bounds on the edge and vertex congestions of $G$-invariant flows.

**Theorem 110** **[136]** *Let $\Gamma$ be a graph, $G$ a subgroup of $Aut(\Gamma)$ and $f$ a uniform concurrent multicommodity $G$-invariant flow on $\Gamma$ such that any active path is a path of shortest length. Then*

$$\mu_f \leq \frac{1}{|O_E|} \sum_{(u,v)\ \in\ V\times V} d(u,v), \qquad m_f \leq \frac{1}{|O_V|}(|V|(|V-1|) + \sum_{(u,v)\ \in\ V\times V} d(u,v)),$$

*where $|O_E|$ and $|O_V|$ are the smallest edge and vertex orbits of $\Gamma$ under $G$.*

Translated in terms of forwarding indices, this theorem gives upper bounds for $\xi_m(\Gamma)$ and $\pi_m(\Gamma)$ which are to be compared to the lower bounds of Propositions 98 and 68. Applied to a Cayley graph and an optimal (for the vertex load) routing constructed by automorphisms in it, Theorem 110 proves that such a routing has an optimal edge-forwarding index up to a factor $d$.

**Theorem 111** **[136]** *Let $\Gamma = Cay(G,S)$ be a Cayley graph of degree $d$. There exists an integral uniform flow $f$ in $\Gamma$ such that*

*(i) $f$ is $G$-invariant*[1]*;*

*(ii) each active path is a path of shortest length;*

*(iii)* $\mu_f \leq \frac{d}{|E|} \sum_{(u,v)\ \in\ V\times V} d(u,v)$.

**Proof.** The proof of the first two assertions (i) and (ii) is similar to the proof of Theorem 69, since the construction of $f$ uses paths constructed as in this proof. Part (iii) is a corollary of Theorem 110. Notice that in a vertex transitive graph, the equalities $d|V|/2 = |E|$ and $d_1|V|/2 = |O_E|$ (where $d_1$ is the degree of the subgraph induced by $O_E$) imply $|E| \leq d|O_E|$. Thus, the inequality of Theorem 110 gives

$$\mu_f \leq \frac{d}{|E|} \sum_{(u,v)\ \in\ V\times V} d(u,v).$$

□

Using the equality $\pi(\Gamma) = \mu_f$ and the inequality of Proposition 98, we get:

**Corollary 112** *If $\Gamma$ is a Cayley graph, then*

$$\frac{1}{|E|} \sum_{(u,v)\in V\times V} d(u,v) \quad \leq \quad \pi(\Gamma) \quad \leq \quad \frac{d}{|E|} \sum_{(u,v)\in V\times V} d(u,v).$$

By using the inequality of Proposition 98, Theorem 110 shows also that, if there exists a subgroup $H$ of $Aut(\Gamma)$ which acts transitively on the edges of $\Gamma$ (in which case $|O_E| = |E|$) and such that there exists an $H$-invariant routing of shortest paths $R$ of $\Gamma$, then the edge-forwarding index satisfies

$$\pi(G) = 1/|E| \sum_{(u,v)\ \in\ V\times V} d(u,v),$$

and therefore is optimal. Orbital regular graphs are graphs for which this is true.

[1]The group $G$ is identified with the isomorphic subgroup of $Aut(\Gamma)$ which is composed of the translations on $\Gamma$.

In [137], F. Shahrokhi and L. Szekely construct integral flows corresponding to symmetric routings (they consider only one path instead of two dipaths between any pair of vertices) which are asymptotically vertex (resp. edge) optimal in vertex (resp. edge) transitive graphs. For this they construct first a *canonical* $Aut(\Gamma)$-invariant flow $\hat{f}$ in the graph as follows. For any pair of vertices $x$, $y$, and any (undirected) path $p$ between $x$ and $y$ they define $\hat{f}(p) = 1/2M_{x,y}$, where $M_{x,y}$ is the number of paths between $x$ and $y$. Then they apply to $\hat{f}$ a method called randomized rounding to obtain an integral flow $f$ such that the congestions of $f$ are close to the congestions of $\hat{f}$. In particular, if $|E| = o(n^2/\log n)$, where $n = |V|$, then putting $L = \frac{1}{2}\sum_{(u,v)\ \in\ V\times V}\ d(u,v)$, this flow satisfies:

- if the graph is edge transitive and $e$ is any edge,

$$\frac{1}{|E|}L(1-o(1)) \leq f(e) \leq \frac{1}{|E|}L(1+o(1))$$

- if the graph is vertex transitive and $x$ is any vertex of $V$;

$$(\frac{L}{|V|}+n-1)(1-o(1)) \leq f(x) \leq (\frac{L}{|V|}+n-1)(1+o(1)).$$

This proves Conjectures 99 and 72 asymptotically and for symmetric routings in graphs such that $|E| = o(n^2/\log n)$. The same authors prove also in [135] that the cube-connected cycles graph $CCC(n)$ and the butterfly graph $BF(n)$ have an integral flow which is asymptotically both edge and vertex optimal (recall that these Cayley graphs are not edge transitive). They introduce a class of graphs called *G-orbit proportional* which contains the class of the edge transitive graphs and for which they obtain some similar results.

Notice that [137] and some references cited therein contain applications of flows to parameters which are important for interconnection networks, such as the crossing number and the bisection width. Similarly, forwarding indices are related to expanding in [139].

Finally, the reader should be aware that some results exist for partial routings instead of routings. For example, flows are studied in [137] in a more general context of partial routings than what is presented here. A similar notion of *generalized forwarding index* is defined in [93] which considers the load of a partial routing on a given subset of vertices.

# Chapter 6

# Appendix: some families of Cayley digraphs

This chapter contains definitions and drawings of some Cayley graphs studied or used as examples in this report. More information on them can be found for example in [3], [103], [106] and [47].

## 6.1 Cycle

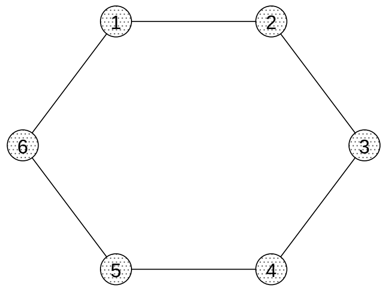


Figure 6.1: Cycle $C_6$.

The cycle $C_n$ is the connected graph with $n$ vertices and degree 2. It is the Cayley graph of the additive group $\mathbb{Z}_n$ generated by $\{-1,+1\}$. A complete rotation on $(\mathbb{Z}_n, \{-1,+1\})$ is defined by $(i \mapsto n-i)$.

It is also the Cayley graph on $\mathfrak{S}_n$ and the subset of the two cycles $\langle 1,2,\ldots,n\rangle$ and $\langle n,n-1,\ldots,1\rangle$. In this case a complete rotation $\omega$ is defined by $\omega(x) = \sigma x \sigma^{-1}$, where the permutation $\sigma$ is given by $(n, n-1, \ldots, 2, 1)$.

$Diam(C_n)= \lfloor \frac{n}{2} \rfloor$.

$C_n$ is a distance transitive graph.

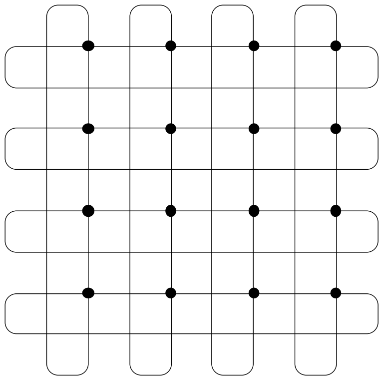

Figure 6.2: Torus $MT_{2,4}$.

## 6.2 Multidimensional torus

The multidimensional torus $MT_{n,k}$ is the cartesian product of $n$ cycles of length $k$. $MT_{n,k}$ has $k^n$ vertices and degree $2n$. It is the Cayley graph of the group $\mathbb{Z}_k^n$ generated by the subset of $2n$ elements $S = \{(\underbrace{0,\ldots,0}_{i},1,\underbrace{0,\ldots,0}_{n-i-1}),(\underbrace{0,\ldots,0}_{i},-1,\underbrace{0,\ldots,0}_{n-i-1}), 0 \le i \le n-1\}$.

$MT_{n,k}$ is rotational (see Proposition 22).

$Diam(MT_{n,k}) = n\lceil k/2 \rceil$.

## 6.3 Hypercube

The hypercube $H(n)$ is the graph with vertex set $\{x_1x_2\ldots x_n : x_i \in \{0,1\}\}$, two vertices $x_1x_2\ldots x_n$ and $y_1y_2\ldots y_n$ being adjacent if and only if $x_i = y_i$ for all but one $i$ (see Figure 6.3).

$H(n)$ can be considered as a particular case of torus, namely $MT_{n,2}$, since it is the cartesian product of $n$ complete graphs $K_2$. $H(n)$ is the Cayley graph of the group $(\mathbb{Z}_2^n, +)$ generated by the $n$ generators $\underbrace{0\ldots0}_{i}1\underbrace{0\ldots0}_{n-i-1}$, $0 \le i \le n-1$.

$H(n)$ is also the Cayley graph of the permutation group $G$ generated by the $n$ transpositions $\langle 2i-1, 2i\rangle$, $1 \le i \le n$ defined on the set of $2n$ elements $X = \{1\ldots 2n\}$ ($H(4)$ is shown in Figure 6.4 and the associated transposition graph in Figure 6.5). Indeed, each vertex $x_1x_2\ldots x_n$, $x_i \in \{0,1\}$, can be renamed as the permutation $(a_1, a_2, \ldots, a_{2n})$ where $(a_{2i-1}, a_{2i}) = (2i-1, 2i)$ if $x_i = 0$ and $(a_{2i-1}, a_{2i}) = (2i, 2i-1)$ if $x_i = 1$.

$H(n)$ is rotational. A complete rotation $\omega$ is defined on $(G,S)$ by $\omega(x) = \sigma x \sigma^{-1}$, where $\sigma$ is the permutation given by $\sigma = (3,4,\ldots,2n-1,\ 2n,1,2) = \langle 1,3,\ldots,2n-1\rangle\langle 2,4,\ldots,2n\rangle$.

Thus, $\sigma^{-1} = (2n-1, 2n, 1, 2, \ldots, 2n-3, 2n-2)$ and $\sigma^i\langle 1,2\rangle\sigma^{-i} = \langle 2i+1, 2i+2\rangle$.

$Diam(H(n)) = n$.

$H(n)$ is a distance transitive graph.

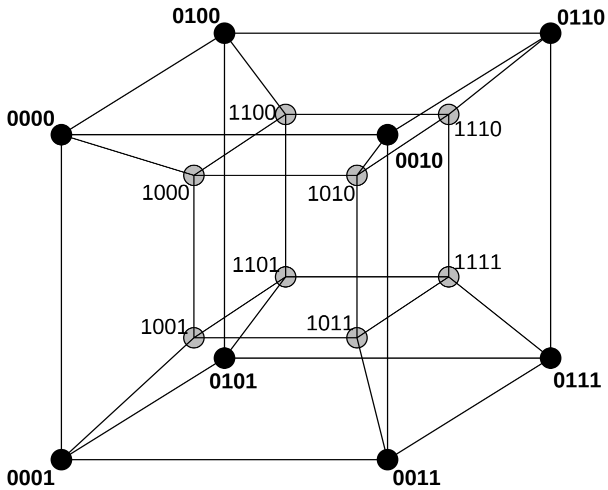


Figure 6.3: Hypercube graph $H(4)$.

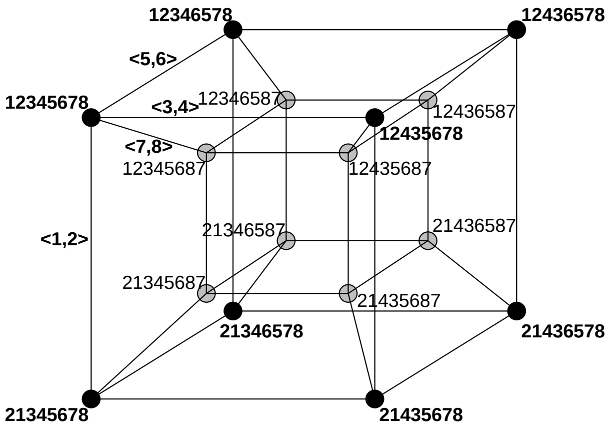


Figure 6.4: $H(4)$ is generated by transpositions.

## 6.4 Star graph

The star graph $ST(n)$ is defined as the Cayley graph of the group $\mathfrak{S}_n$ generated by the $n-1$ transpositions $S = \{\langle 1, i\rangle, 1 < i \leq n\}$. The associated transposition graph is the star $K_{1,n-1}$ (see $ST(4)$ depicted on Figure 6.6 and the associated transposition graph depicted on Figure 6.7). A complete rotation $\omega$ is defined on $(\mathfrak{S}_n, S)$ by $\omega(x) = \sigma x \sigma^{-1}$, where the permutation $\sigma$ is given by $\sigma = (1, 3, 4, \ldots, n, 2) = \langle 2, 3, \ldots, n\rangle$.

$Diam(ST(n)) = \lfloor \frac{3(n-1)}{2} \rfloor$ (see [3]).

## 6.5 Complete transposition graph

The complete transposition graph of dimension $n$, $CT_n$, is the Cayley graph of the group $\mathfrak{S}_n$ generated by all the transpositions $\{\langle i, j\rangle, 1 \leq i \leq n, 1 \leq j \leq n, i \neq j\}$. The associated transposition graph is the complete graph.

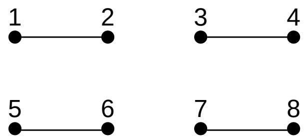


Figure 6.5: Transposition graph for $H(4)$

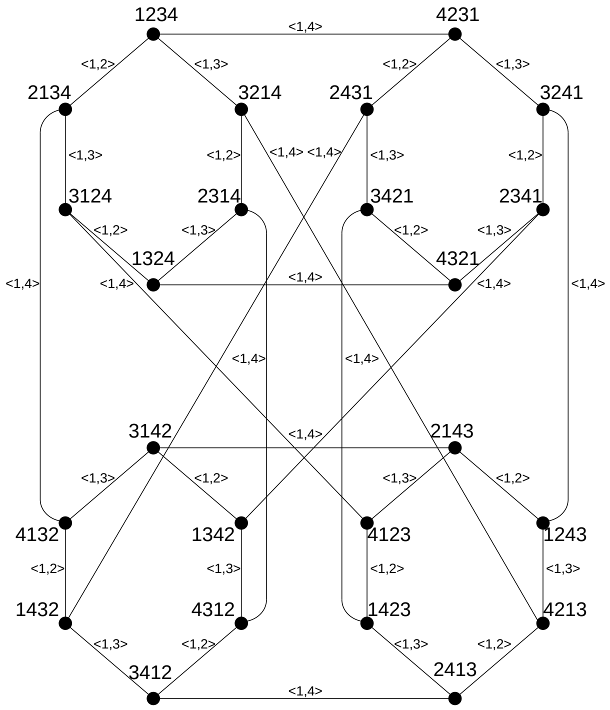


Figure 6.6: Star graph $ST(4)$.

$Diam(CT_n) = n - 1$. $CT_n$ is arc transitive but not distance transitive [103].

## 6.6 Generalized star graph

The generalized star graph $GST(n, k)$ is defined in section 3.2. $GST(n, k)$ is the Cayley graph of the group $\mathfrak{S}_n$ generated by the $k(n-k)$ transpositions $S = \{\langle i, j\rangle, 1 \le i \le k, k+1 \le j \le n\}$. The associated transposition graph is the bipartite complete graph $K_{k,n-k}$. $GST(4, 2)$ is depicted on Figure 6.8 and the associated graph on Figure 6.9. The graph $GST(n, k)$ is rotational if $n = 4$ and

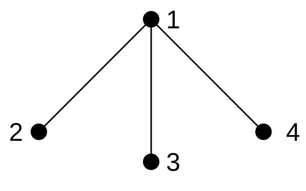


Figure 6.7: Transposition graph for $ST(4)$.

$k = 2$ or if $k$ and $n - k$ are relatively prime (see Proposition 17).

$Diam(GST(k,n)) = n - 1 + \max(\lfloor \frac{k}{2} \rfloor, \lfloor \frac{n-k}{2} \rfloor)$ [128].

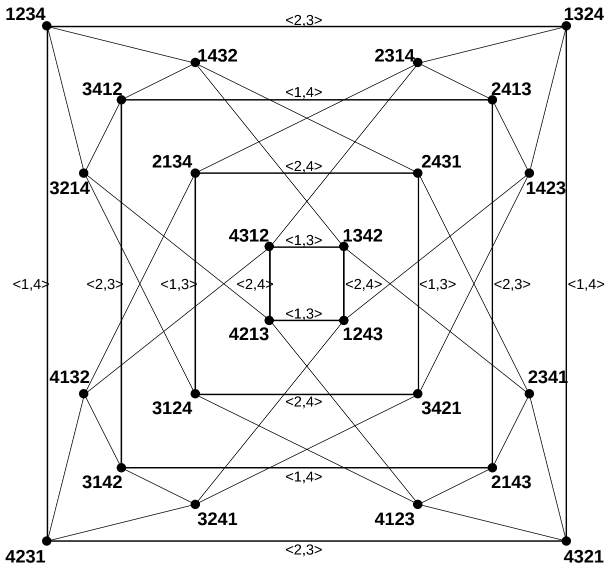


Figure 6.8: Generalized star graph $GST(4,2)$.

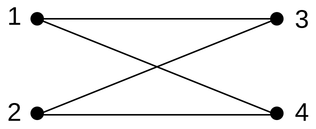


Figure 6.9: Transposition graph for $GST(4,2)$.

## 6.7 Bubble sort graph

The bubble sort graph of dimension $n$, $BS(n)$, is the Cayley graph of $\mathfrak{S}_n$ generated by the $n-1$ transpositions $\{\langle i, i+1 \rangle, 1 \leq i < n\}$[1] (see Figure 6.10). The associated transposition graph is a path of length $n - 1$.

$Diam(BS(n)) = \frac{n(n-1)}{2}$ [103].

## 6.8 Modified bubble sort graph

The modified bubble sort graph of dimension $n$, $MBS(n)$, is defined as the bubble sort graph, adding a $n^{th}$ transposition: $\langle n, 1 \rangle$. Thus the associated transposition graph is the cycle $C_n$. $MBS(n)$ has a complete rotation $\omega$ defined by $\omega(x) = \sigma x \sigma^{-1}$ where $\sigma$ is the cyclic permutation given by $\langle 1, 2, \ldots, n \rangle$.

$Diam(MBS(n)) = \lfloor \frac{n^2}{4} \rfloor$ [128].

[1] This Cayley graph is constructed on a Coxeter group [41].

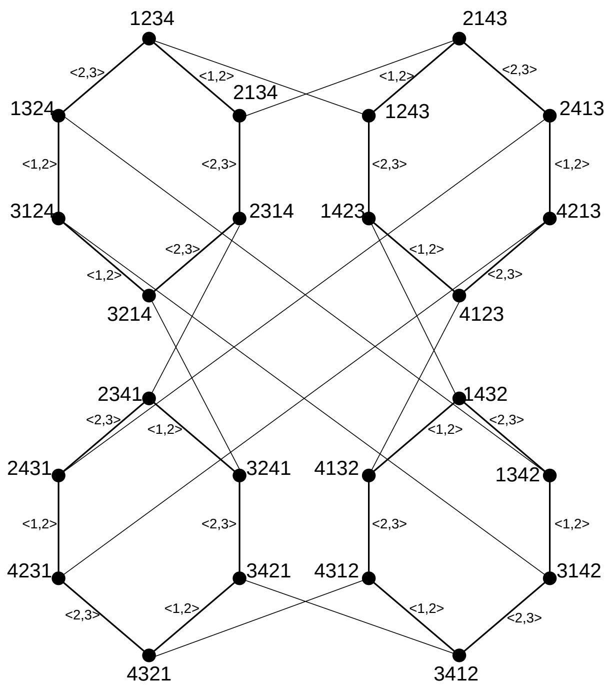


Figure 6.10: Bubble sort graph $BS(4)$.

## 6.9 Pancake

The pancake graph, $PR(n)$, also called *prefix reversal graph*, is the Cayley graph of the group $\mathfrak{S}_n$ generated by the $n-1$ permutations of order 2 $(\underbrace{i, i-1, \ldots 1}, i+1, \ldots, n)$.

$PR(2) = ST(2) = K_2$ and $PR(3) = ST(3) = C_6$. But $PR(n)$ is not edge transitive for $n \geq 4$ and therefore not rotational. $Diam(PR(n))$ is unknown for $n > 13$ (see Section 4).

## 6.10 Cube Connected Cycle

The cube-connected cycles graph $CCC(n)$ is defined in Section 3.4.2. $CCC(1) = K_2$, $CCC(2) = C_8$. For $n \geq 3$, $CCC(n)$ is obtained from $H(n)$ by replacing each vertex by a cycle of length $n$ such that the degree is 3 (see Figures 6.12 and 6.13). The number of vertices is $n \times 2^n$. For $n \geq 3$, $Diam(CCC(n)) = 2n - 1 + \max(1, \lfloor \frac{n-2}{2} \rfloor)$. This graph is not edge transitive and therefore not rotational for $n \geq 3$.

## 6.11 Butterfly

The butterfly graph $BF(n)$ is defined in Section 3.4.2. Its number of vertices is $n \times 2^n$. $BF(1) = K_2$, $BF(2) = H(3)$. For $n \geq 3$, the degree is 4 and $Diam(BF(n)) = n + \lfloor \frac{n}{2} \rfloor$. This graph is not edge transitive and therefore not rotational for $n \geq 3$. $BF(3)$ is depicted in Figure 6.14.

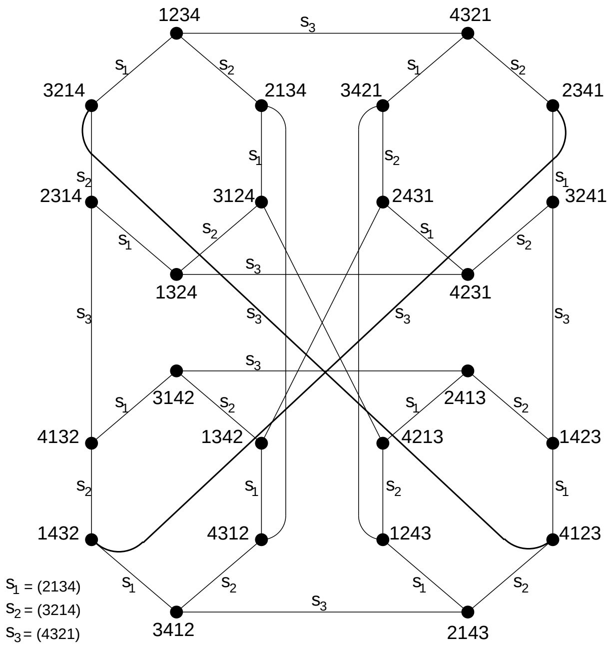


Figure 6.11: Pancake graph $PR(4)$.

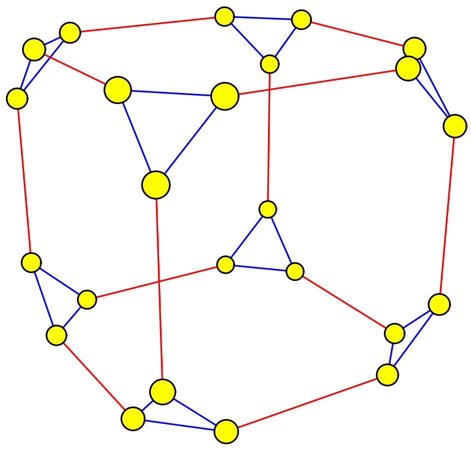

Figure 6.12: $CCC(3)$.

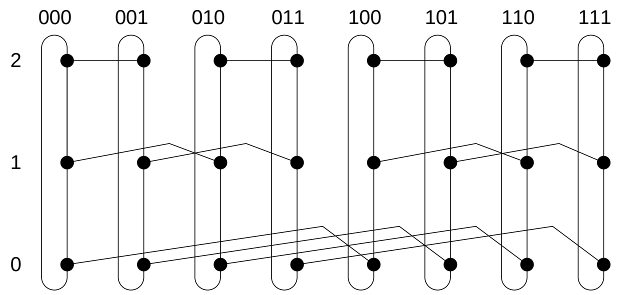


Figure 6.13: $CCC(3)$.

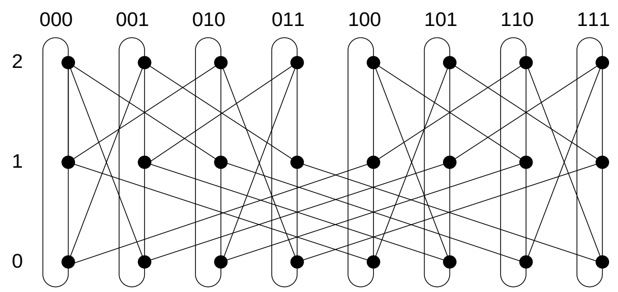


Figure 6.14: Butterfly graph $BF(3)$.

# Conclusion

In this report we have presented some Cayley digraphs which are studied as models of interconnection networks. We have surveyed neither all the possible graphs that have been studied as good architectures for interconnection networks, nor all aspects of those graphs that we have discussed. We have confined our attention to problems of structure, diameter, routing and loads of routings. We have paid much attention to Cayley graphs generated by transpositions, rotational Cayley graphs, hypercubic graphs and to new families of vertex transitive graphs which have good properties as far as routing is concerned, such as quasi-Cayley graphs and regular orbital graphs.

To conclude, we want to underline some trends of research in this rich field.

• The problem of finding new (di)graphs with good parameters improving the already known graphs is still being studied. Numerous articles have been devoted to the study of generalizations of the most famous graphs for interconnection networks, for example: the supertoroidal networks which are 4-regular Cayley graphs [59], the multiply-twisted hypercube (a Cayley graph with the same vertex set, degree and connectivity as the hypercube, but a diameter equal to half the diameter of the hypercube) [102], Cayley graphs with $n2^n$ vertices, degree 3 and 4, logarithmic diameter and constructed on signed permutations, [141], [143],[142]. For all these graphs, the authors justify their relevance for interconnection networks by studying their parameters but also algorithms on them (routing, embedding, communication).

• General theorems are missing which would extend to *classes* of Cayley graphs properties that have been proved for particular graphs (for example, the calculus of the wide-diameter in Section 4.5).

• Very few good algorithms are known to be efficient for all Cayley digraphs (as is the case in [55]). One usually starts with a given algorithm, and asks for which Cayley digraphs it is good, as in [82].

• More group theory has to be exploited to find properties of Cayley graphs. For example, in [5] the subgroup structure of a group reveals ways to decompose some Cayley graphs on that group.

• Bus interconnection networks generalize the notion of point to point interconnection networks. In a bus network, some sets of processors share a communication medium called a bus. In a natural way, they are modeled by hypergraphs in which the vertices represent the processors, and the hyperedges represent the buses that group the processors sharing the same communication link. An increasing number of articles consider problems related to bus interconnection networks (see for example [62]). Different ways of extension of Cayley graphs related to hypergraphs have already been considered. One is to consider the dual hypergraph of a Cayley graph (see for example [23]); a second one is to consider hypergraphs the line graphs of which are Cayley graphs (for example, [12]). A definition of Cayley hypergraphs is given in [49].

• We only gave an overview of some static problems. Some dynamic problems interesting for distributed computing are considered in [105]. Parallel models of computation include automata. Cayley automata are defined in [75]. Studies of cellular automata working on Cayley graphs can be found in [109], [118] and [119].

Are Cayley graphs the most appropriate models for interconnection networks ? We have seen that quotients of Cayley digraphs can also be very convenient, and some people have confidence in non-symmetric graphs. What is certain is that the relation between Cayley graphs and interconnection networks is fruitful for the research in both fields.

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