Properties of the r-operators to insure termination of distributed computations

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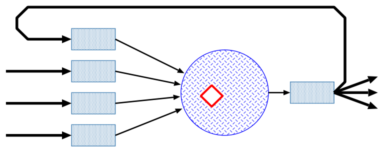
- Model
- Waves
- r-operators
- Applications
- Conclusion / Perspectives

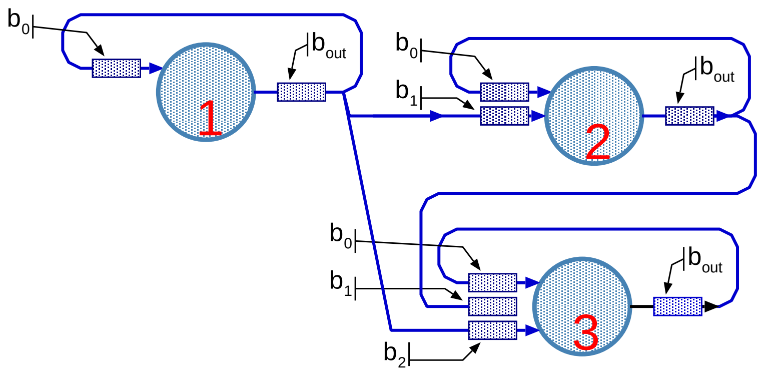
Computing entity, Network

Entity



$$b_{\text{out}} \leftarrow b_0 \;\diamond\; \cdots \;\diamond\; b_{\delta^-(v)} \qquad b_{\text{out}} \leftarrow \diamond\left(b_0, \ldots, b_{\delta^-(v)}\right)$$

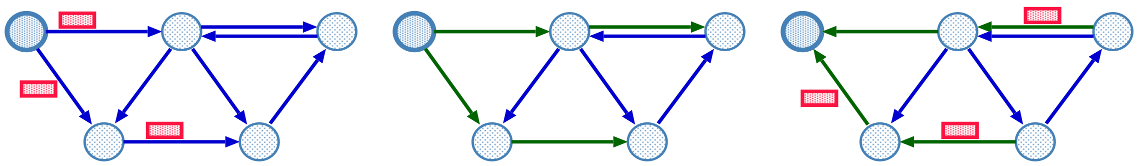
Network



- asynchronous system
- one-to-many communications
- weak safety hypotheses, depending on the operator
- termination detection display / algorithm

Distributed waves algorithms

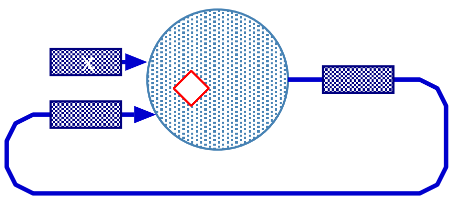
- Waves



  - diffusing computation (Dijkstra, Scholten, 1980)
  - echo algorithms (Chang, 1982)
  - PI / PIF (Segall, 1983)
  - extension to oriented networks (Tel, 1987)
  - waves on special topologies (Raynal and Hélary, 1988)
  - total, wave algorithms (Tel, 1991, 1994)

- $\rightsquigarrow$ unification : s-operator $\star$
  - $\star$ associative, commutative, idempotent
  - partial order $x \preceq_\star y \Leftrightarrow x \star y = x$ *sort*
  - $\wedge$, $\vee$, $\mathbf{min}$, $\mathbf{max}$, $\mathbf{gcd}$, $\mathbf{lcm}$, $\cap$, $\cup$, $\ldots$
  - Theorem $b_{\text{out}}(v) \leftarrow \star \left\{ b_0(u), u \in \widehat{\Gamma_G^{-1}}(v) \right\}$

- Necessary condition: idempotency of order k



Classical properties revisited: r-operators

- More general properties

r-associativity	$r : A \rightarrow A$
	$(x \diamond y) \diamond r(z) = x \diamond (y \diamond z)$
example:	*in* $\mathbb{Z}$, $x \oplus y \equiv x + 2y$ *with* $r(z) = 2z$
r-commutativity	$r : A \rightarrow A$
	$r(x) \diamond y = r(y) \diamond x$
example:	$\mathbf{op}(x) - y = -x - y = \mathbf{op}(y) - x$
r-idempotency	$r : A \rightarrow A$
	$r(x) \diamond x = r(x)$
example:	$x\mathbf{p}y = 0$ *if* x *and* y *have same parity,* $r(2k) = 0$, $r(2k+1) = 1$, $k \in \mathbb{Z}$

- r-operators

  $\triangleleft$ is an r-operator on A if there exists $r : A \rightarrow A$ bijective mapping such that $\triangleleft$ is:

r-associative	$(x \triangleleft y) \triangleleft r(z) = x \triangleleft (y \triangleleft z)$
r-commutative	$r(x) \triangleleft y = r(y) \triangleleft x$
r-idempotent	$r(x) \triangleleft x = r(x)$

Applications (1). New possibilities

- s-operators

$$b_{\text{out}}(v) \leftarrow \star \left\{ b_0(u), u \in \widehat{\Gamma_G^{-1}}(v) \right\}$$

- binary r-operators

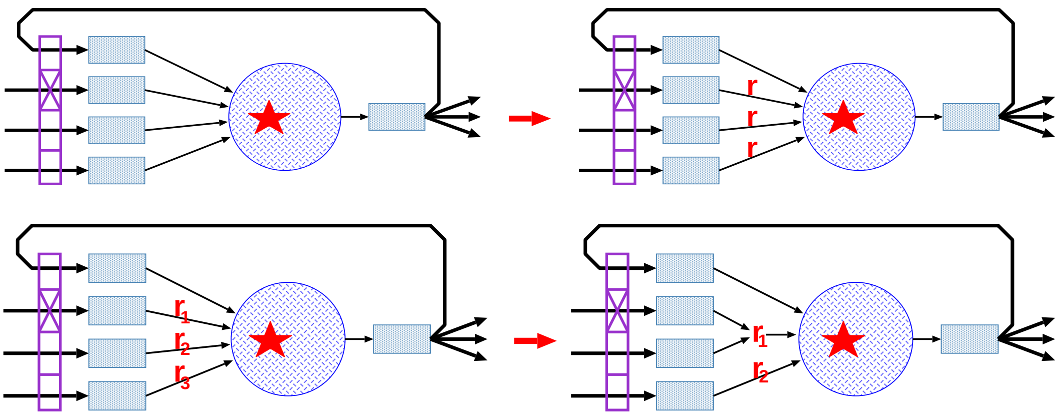
$$b_{\text{out}}(v) = \star \left\{ r^{\mathrm{d}_G(u,v)} \left(b_0(u) \right), u \in \widehat{\Gamma_G^{-1}}(v) \right\}$$

- n-ary r-operators

$$b_{\text{out}}(v) = \star \left\{ r_P \left(b_0(u) \right), u \in \widehat{\Gamma_G^{-1}}(v), P \text{ dipath from } u \text{ to } v \right\}$$

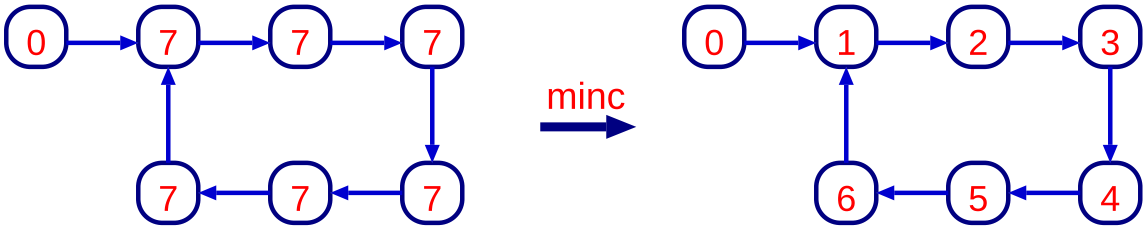
- generalized r-operators

- New possibilities for waves

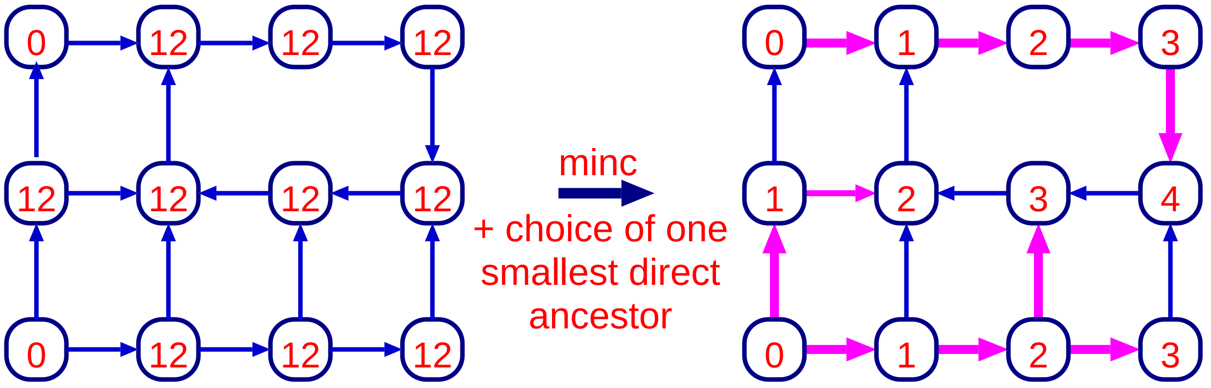


Applications (2). Distance, bfs

- $\mathbf{minc}(x, y) = \mathbf{min}(x, y+1)$ in $\overline{\mathbb{Z}}$



- bread-first-search, several sources



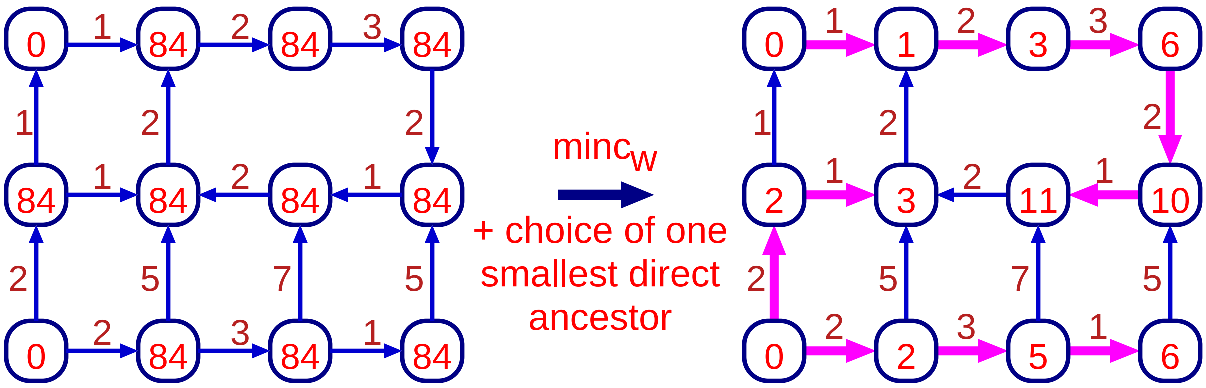
Applications (3). Weighted digraph

Single source shortest path

- digraph with weighted edges $w\left((u,v)\right)$
- $\mathbf{minc}_w[x_0,\ldots,x_n] = \mathbf{min}[x_0, r_v^{u_1}(x_1),\ldots,r_v^{u_n}(x_n)]$

$r_v^{u_i}(x) = x + w\left((u_i,v)\right)$

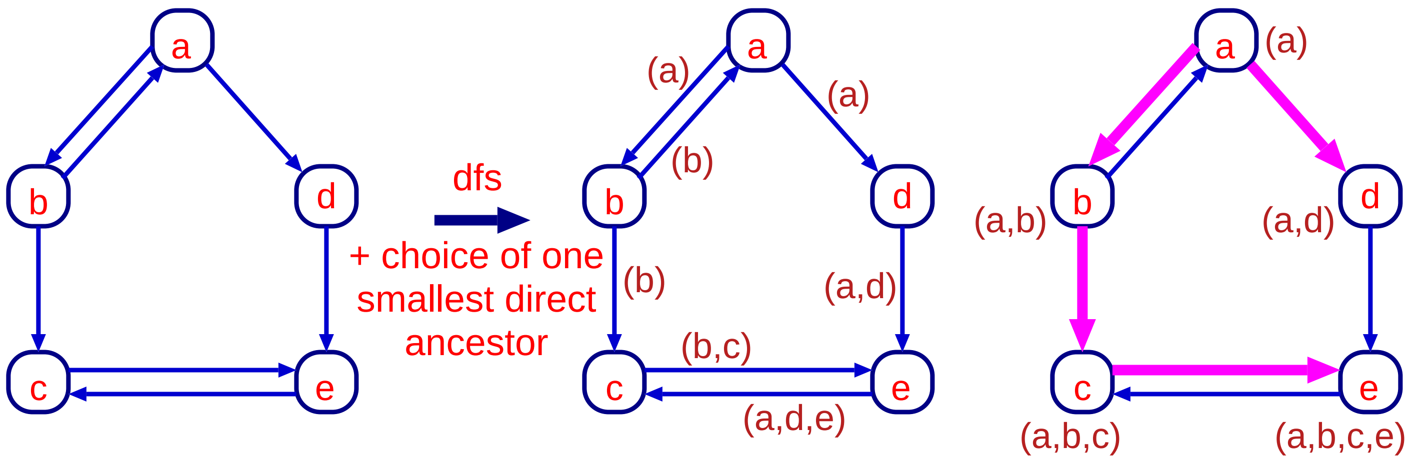
Example



Applications (4). dfs

- operator lexicat $\triangleleft$:
  - vertices completely ordered (non anonymous computation)
  - A: set of lists of vertices
  - $\star$: lexicographic order on the lists
    examples: $(a, b, d) \star (a, b, c, d) = (a, b, c, d)$
    $(a, a) \star (a) = (a)$
  - $r : A \rightarrow A$, $r(l) = l \cup (v)$ on the vertex v
  - $\triangleleft$ defined by $l \triangleleft l' = l \star r(l')$ is idempotent

- Application:



Conclusion

- New operators: r-operators
  - $\{$ s-operators $\} \subsetneq \{$ r-operators $\}$
  - necessary condition: idempotency (of order k)
  - the idempotent r-operators still insure the termination of the waves.
  - extension to the n-ary and generalized r-operators
- Applications: distance, bfs, sssp, dfs, etc.
- Interest: extended theory for the waves
  - distributed algorithms $\rightsquigarrow$ waves
  - simplification of the protocol, weak hypotheses for links
- Perspective
  - self-stabilization