New Operators for Computing with *Associative Nets*

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June 26, 1998

Abstract

We present new operators for computing with associative nets, a parallel computing model for irregular data using asynchronism. These results can be applied to distributed wave algorithms.

Plan

- Associative nets
- Computing entity, network
- Problem and method
- Distributed waves, s-operators
- New operators
  - r-operators
  - properties
  - theorem of termination
- Applications
  - distance, bread-first-search
  - single source shortest path
  - depth-first-search
- Complexity, safety hypotheses
- Conclusion

Associative nets

Parallel computing model

- especially for image analysis
- data-parallelism (data unit = image), pvar
- irregular (objects coded by a digraph)
- SPMD on a network of computing entities
- special electronic display / parallel programming library

Originality

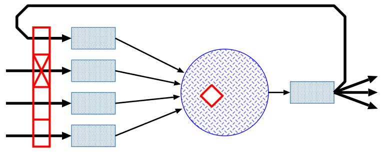
- iterative synchronous loops replaced by asynchronous global computations
- average delay instead of maximum delay
- not for all the operators (e.g. minimum / addition)

Our work

- extension of the operators allowing asynchronous computations instead of loops
- these results can be applied to distributed wave algorithms

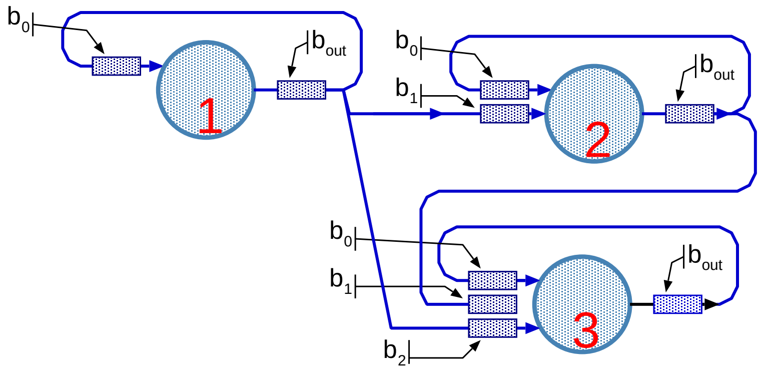
Computing entity, Network

Entity



$b_{\text{out}} \leftarrow b_0 \diamond \cdots \diamond b_{\delta^-(v)}$ $\qquad$ $b_{\text{out}} \leftarrow \diamond\left(b_0, \ldots, b_{\delta^-(v)}\right)$

Network



- asynchronous system / ABD (Tel 1994)
- one-to-many communications
- weak safety hypotheses, depending of the operator
- dynamic topology (irregular data)
- termination detection display / algorithm

Computing primitive: $\diamond$*–direct–association*(G, p) (wave)

Problem and method

Aim

- Properties of the operator $\diamond$ such that any $\diamond - \textit{direct} - \textit{association}(G, p)$ terminates with any G and any p

General properties for operators $\diamond$:

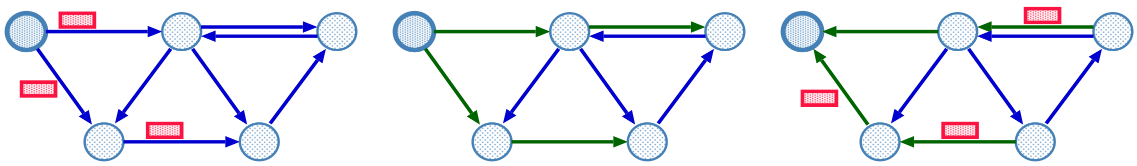
- A : set of possible values for registers/variables
- $\diamond$ internal law on A
- left-bracketing $(x \diamond y) \diamond z = x \diamond y \diamond z$
- right-identity element $x \diamond e_\diamond = x$
- faithful $(\forall x \in A, \quad x \diamond y = x \diamond z) \Leftrightarrow (y = z)$

Method

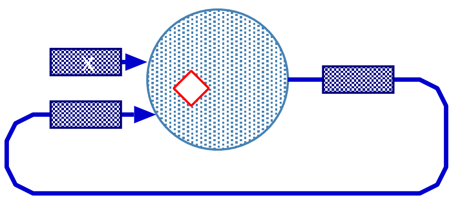
- entity: builds some expressions with those it received
- cycle: infinite length expressions with finite number of variables
- operator: reduction of these expressions to finite length deterministic expressions
  example: $x \diamond (y \diamond x) \diamond (x \diamond y) \rightsquigarrow x \diamond y$

Distributed waves algorithms

- Waves



  - diffusing computation (Dijkstra, Scholten, 1980)
  - echo algorithms (Chang, 1982)
  - PI / PIF (Segall, 1983)
  - extension to oriented networks (Tel, 1987)
  - waves on special topologies (Raynal and Hélary, 1988)
  - total, wave algorithms (Tel, 1991, 1994)
- $\rightsquigarrow$ unification : s-operator $\star$
  - $\star$ associative, commutative, idempotent
  - partial order $x \preceq_\star y \Leftrightarrow x \star y = x$ *sort*
  - $\wedge$, $\vee$, $\mathbf{min}$, $\mathbf{max}$, $\mathbf{gcd}$, $\mathbf{lcm}$, $\cap$, $\cup$, $\ldots$
  - Theorem $b_{\text{out}}(v) \leftarrow \star \left\{ b_0(u), u \in \widehat{\Gamma_G^{-1}}(v) \right\}$
- Necessary condition: idempotency of order k



Classical properties revisited: r-operators

- More general properties

r-associativity	$r : A \to A$
	$(x \diamond y) \diamond r(z) = x \diamond (y \diamond z)$
example:	in $\mathbb{Z}$, $x \oplus y \equiv x + 2y$ with $r(z) = 2z$
r-commutativity	$r : A \to A$
	$r(x) \diamond y = r(y) \diamond x$
example:	$\mathbf{op}(x) - y = -x - y = \mathbf{op}(y) - x$
r-idempotency	$r : A \to A$
	$r(x) \diamond x = r(x)$
example:	$x\mathbf{p}y = 0$ if x and y have same parity, $r(2k) = 0$, $r(2k+1) = 1$, $k \in \mathbb{Z}$

- r-operators

  $\triangleleft$ is an r-operator on A if there exists $r : A \to A$ bijective mapping such that $\triangleleft$ is:

r-associative	$(x \triangleleft y) \triangleleft r(z)$	$=$	$x \triangleleft (y \triangleleft z)$
r-commutative	$r(x) \triangleleft y$	$=$	$r(y) \triangleleft x$
r-idempotent	$r(x) \triangleleft x$	$=$	$r(x)$

Properties of the r-operators (1)

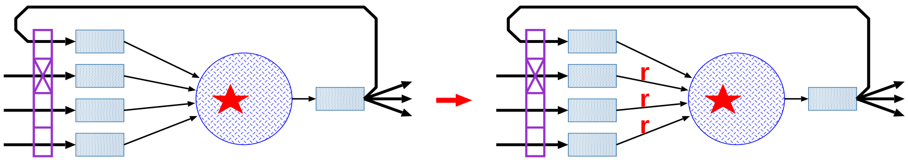
Properties:

- $r(x) = e_{\triangleleft} \triangleleft x$
- $\mathrm{Ker}(r) = \{t \in A, r(t) = e_{\triangleleft}\} = \{e_{\triangleleft}\}$
- r isomorphism of $(A, \triangleleft)$: $r(x \triangleleft y) = r(x) \triangleleft r(y)$
- $\diamond$ r-associative and r-idempotent with r bijective $\Rightarrow$ $x \diamond y \diamond y = x \diamond y$
- $\triangleleft$ r-operator $\Rightarrow$ 2-commutative: $x \triangleleft y \triangleleft z = x \triangleleft z \triangleleft y$

Associated s-operator:

- Theorem $\triangleleft$ r-operator on A with $r : A \to A$ a bijective mapping $\Rightarrow$ $\star$ defined by $x \star y = x \triangleleft r^{-1}(y)$ is an s-operator.
- $\triangleleft$ r-operator $\rightsquigarrow$ $\preceq_{\triangleleft}$ defined by $x \triangleleft r^{-1}(y) = x$
- r is an isomorphism of $(A, \star)$: $r(x \star y) = r(x) \star r(y)$
- $e_{\triangleleft} = e_{\star}$

Interest:



Properties of the r-operators (2)

Idempotent r-operators

- $\triangleleft$ is idempotent $\Rightarrow \forall x \in A, \forall k \in \mathbb{N},\ x \preceq_\star r^k(x)$
- $\forall x \in A, x \preceq_\star r(x) \Leftrightarrow \star\ \left(r^{-1}\right)$-idempotent
  $\Leftrightarrow \triangleleft$ idempotent

Building r-operators

Theorem $\triangleleft$ and $\star$ binary operators on A,

$$r : A \to A,\ x \triangleleft y = x \star r(y)$$

then equivalence between:

1. r is surjective, $\triangleleft$ has a right identity element $e_\triangleleft \in A$, admits the faithful property and is r-associative, r-commutative and r-idempotent on A
2. $\triangleleft$ is an r-operator on A admitting the faithful property and has a right identity element $e_\triangleleft \in A$
3. $\star$ is an s-operator on A admitting a right-identity element $e_\star$, and r is an isomorphism of $(A, \star)$

Example

in $\overline{\mathbb{Z}} = \mathbb{Z} \cup \{-\infty, +\infty\}$, $\star = \mathbf{min}$ and $r(x) = x + 1$

Fundamental property

- Theorem Any $\triangleleft - \mathit{direct} - \mathit{association}(G, p)$ based on an idempotent r-operator $\triangleleft$ terminates on any network G with any initial data p and

$$\forall v \in V, \quad b_{\text{out}}(v) = \star \left\{ r^{\mathrm{d}_G(u,v)}\left(b_0(u)\right), u \in \widehat{\Gamma_G^{-1}}(v) \right\}$$

- Lemma *suppression of the repeated terms*
  $x \triangleleft x = x$, $x \triangleleft r^k(x) = x$, ...
  $a \triangleleft r^p(x) \triangleleft r^q(x) = a \triangleleft r^{\min(p,q)}(x)$, ...

- Lemma *distance*
  $u \in \widehat{\Gamma_G^{-1}}(v), \mathrm{d}_G(u, v) = k$
  $\Leftrightarrow$ the expression in $b_{\text{out}}(v)$ contains $r^k\left(b_0(u)\right)$

- Corollary *wave*
  Any wave based on r-operators terminates.

Applications (1). New possibilities

- s-operators

$$b_{\text{out}}(v) \leftarrow \star \left\{ b_0(u), u \in \widehat{\Gamma_G^{-1}}(v) \right\}$$

- binary r-operators

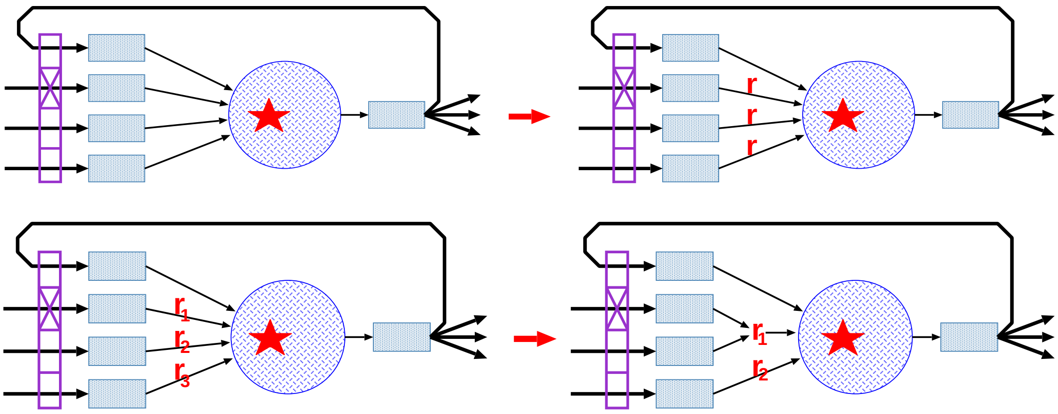
$$b_{\text{out}}(v) = \star \left\{ r^{\mathrm{d}_G(u,v)}\left(b_0(u)\right), u \in \widehat{\Gamma_G^{-1}}(v) \right\}$$

- n-ary r-operators

$$b_{\text{out}}(v) = \star \left\{ r_P\left(b_0(u)\right), u \in \widehat{\Gamma_G^{-1}}(v), P \text{ dipath from } u \text{ to } v \right\}$$

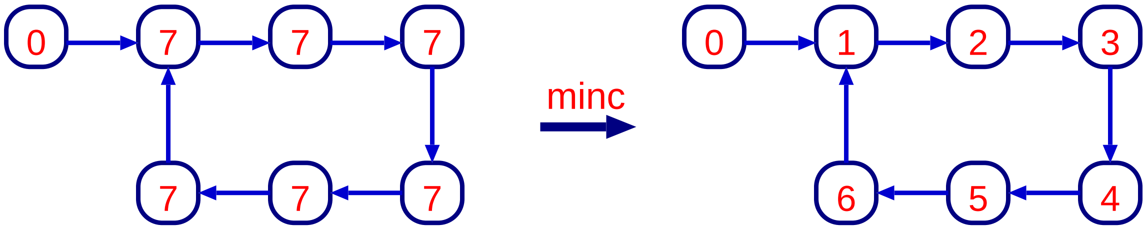
- generalized r-operators

- New possibilities for *direct−associations* and waves

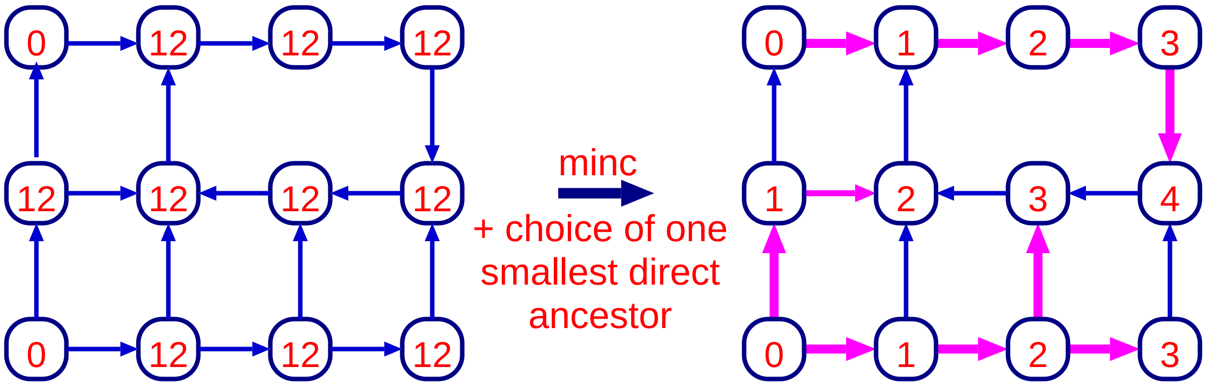


Applications (2). Distance, bfs

- $\mathbf{minc}(x, y) = \mathbf{min}(x, y+1)$ in $\overline{\mathbb{Z}}$



- bread-first-search, several sources

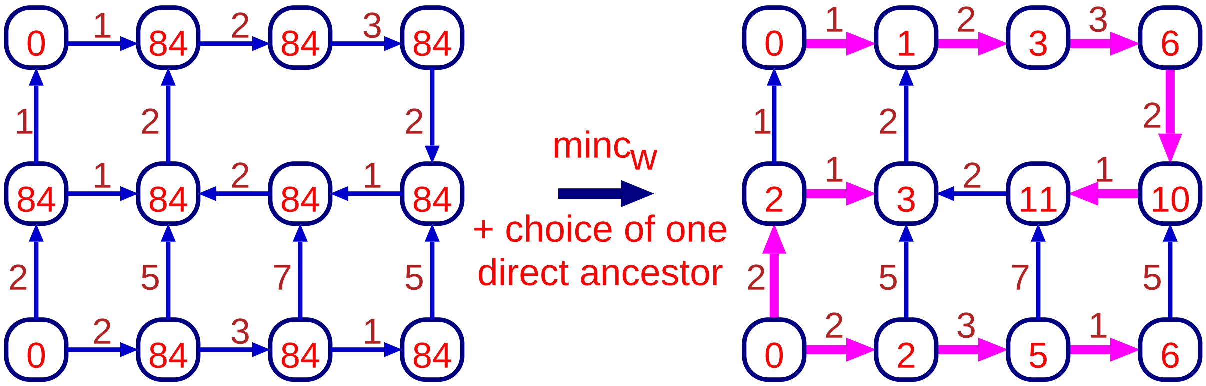


Applications (3). Weighted digraph

- Single source shortest path
  - digraph with weighted edges $w((u,v))$
  - $\mathbf{minc}_w[x_0, \ldots, x_n] = \mathbf{min}[x_0, r_v^{u_1}(x_1), \ldots, r_v^{u_n}(x_n)]$

  $r_v^{u_i}(x) = x + w((u_i, v))$

- Example

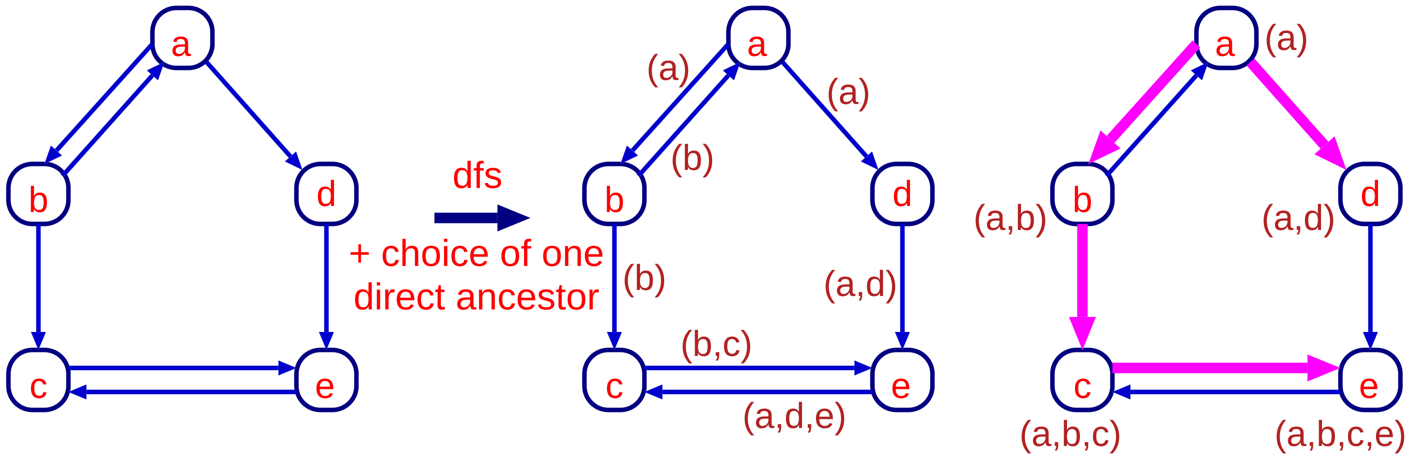


Applications (4). dfs

operator lexicat $\lhd$:

- vertices completely ordered (non-anonymous computation)
- A: set of lists of vertices
- $\star$: lexicographic order on the lists
  examples: $(a, b, d) \star (a, b, c, d) = (a, b, c, d)$
  $(a, a) \star (a) = (a)$
- $r : A \to A$, $r(l) = l \cup (v)$ on the vertex v
- $l \star r(l) = l$ thus $\lhd$ defined by $l \lhd l' = l \star r(l')$ is idempotent

Application:



Complexity, Safety

- Time for convergence (ABD):
  - building of the result v on u
  - delivery of this result on v
- General case: depends on the operator
  example: $\mathbf{min}(x+1, y+1, 100)$, contention
  - s-operator: total order / partial order
  - r-operator: idem (same computation)
- Message complexity:
  time complexity $\times |E|$ / minimal delay to send a message
- Links properties
  - Duplication (idempotency)
  - Loops on the network
  - Disordered messages (asynchronism)
  - $\rightsquigarrow$ loss of messages easily admitted (ABD)

Conclusion

- New operators: r-operators
  - $\{$ s-operators $\} \subsetneq \{$ r-operators $\}$
  - necessary condition: idempotency (of order k)
  - the idempotent r-operators still insure the termination of the *direct − associations* and waves.
  - extension to the n-ary and generalized r-operators

- Applications: distance, bfs, sssp, dfs, image analysis, etc.

- Interest
  - extension for the associative nets
    more synchronous iterations replaced by asynchronous computations
  - extension for the waves
    distributed algorithms $\leadsto$ waves
    simplification of the protocol, weak hypotheses for links
    self-stabilization