Introduction to linear programming

Linear programming

- Overview
  - Mathematical and linear programs
  - Linear programming and resolution methods;
  - Modeling in (integer) linear programming;
    - Scheduling problems;
    - Optimization problems in logistics and transportation
    - Examples, exercises.
  - Integer linear programming: arborescent methods, examples, exercises.

A linear program

- A mathematical program is an optimization problem with an *objective (or optimization) function* of n-variables satisfying m constraints. If at least one constraint or the objective function are not linear, it is called a non-linear *program,* otherwise it is *a linear program.*

Un program is convex if the optimization function is convex and the domain defined by the constraints is also convex : then, any local optimum is also a global optimum and it is achieved at some extreme point of the domain.

A real-valued function f(x) defined on an interval is called ***convex*** *if the graph of the function lies below the line segment joining any two points of the graph.*

A linear program is a convex program.

A linear program

- There can be distinguished:
  - Decision variables
  - Objective (or optimization) function
  - Constraints;
  - Domain of variables;

A linear program

General case

Suppose that a linear program is composed of :

- The objective function of ***n*** variables x_j (*maximization)* ;
- All variables take positive values.
- The constraints are linear functions bounded by some constant

That is :

Interpretation :

n activities, m resources…

$$Max \quad z = \sum_{j=1}^{n} c_j x_j$$

$$s.c. \quad A_{11}x_1 + A_{12}x_2 + \ldots + A_{1n}x_n \le b_1;$$
$$A_{21}x_1 + A_{22}x_2 + \ldots + A_{2n}x_n \le b_2;$$
$$\ldots$$
$$A_{m1}x_1 + A_{m2}x_2 + \ldots + A_{mn}x_n \le b_m;$$
$$x_1, x_2, \ldots, x_n \ge 0.$$

Geometrical interpretation

max: $Z = 2x_1 + x_2$

Subject to:

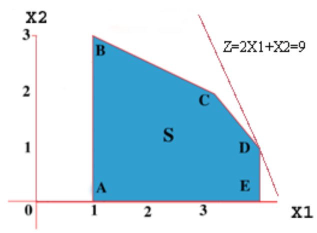
$x_1 + 2x_2 \le 7$

$x_1 + x_2 \le 5$

$x_1 \ge 1$

$x_1 \le 4$

$x_i \ge 0$



Introduction to linear programming

Linear programming (LP) is a technique for optimization of a linear objective function of variables x_1, x_2, …x_n, subject to linear equality and linear inequality constraints.

- **How to solve linear programming :**
  - The simplex algorithm (1951, 1963), developed by George Dantzig, solves LP problems by constructing an admissible solution at a vertex of the polyhedron and then walking along edges of the polyhedron to vertices with successively higher values of the objective function until the optimum is reached. (CPLEX, EXPRESS-MP, etc.).
  - **Alternative methods :**
    - the ellipsoid method by Leonid Khachiyan in 1979
    - In 1984, N. Karmarkar proposed a new interior point projective method for linear programming. (Karmarkar's algorithm)

Simplex algorithm : basics

The domain formed by linear constraints gives a polyhedron, and the search for optimal solutions can be restricted to extreme points.

The simplex method explores extreme points and improves the value of optimization function in each step.

Advantages :

- Efficiency (not polynomial but polynomial-time in average).
- Nice theoretical properties (duality).

Simplex algorithm : duality

Primal :

Max $c_1x_1 + c_2x_2$

s.t.:

$A_{11}x_1 + A_{12}x_2 \leq b_1$;

$A_{21}x_1 + A_{22}x_2 = b_2$;

$x_1 \geq 0$ *and* x_2 *unsigned;*

Dual :

Min $b_1y_1 + b_2y_2$

s.t.:

$A_{11}y_1 + A_{21}y_2 \geq c_1$;

$A_{12}y_1 + A_{22}x_2 = c_2$;

$y_1 \geq 0$; y_2 *unsigned;*

Transformation rules :

Take the transposed matrix A,

the right hand terms become coefficients of the objective function… etc.

Simplex algorithm: duality

Proposition. If x is feasible for the primal (maximum) problem and if y is feasible for its dual, then cTx ≤ yTb.

Proof. cTx ≤ yTAx ≤ yTb. The first inequality follows from x ≥ 0 and cT≤ yT. The second inequality follows from y ≥ 0 and Ax ≤ b.

Economic interpretation:

- Estimate the cost/price of resources…
- Estimate the profit increasing when changing the right hand terms;

Sensibility analysis

A simple example (I)

Example. Determine the quantities to be produced such that all the production constraints are satisfied and the benefit is maximized. We suppose that two products A and B can be produced, each of them passing through cutting and packing stages, respectively (C) and (P) :

	Cutting	**Packing**
Necessary time to produce 1 unity of A	2 hours	3 hours
Necessary time to produce 1 unity of B	2 hours	1 hours
Availability in working hours	200 hours	100 hours

The unitary benefits for A et B are respectively 20 € and 10 €.

A simple example (II)

Primal PL :

Max 20A + 10B

Subject to :

$2A + 2B \leq 200;$

$3A + B \leq 100;$

$A \geq 0$ *et* $B \geq 0;$

Dual :

Min 200a + 100b

Subject to :

$2a + 3b \geq 20;$

$2a + b \geq 10;$

$a \geq 0$ *et* $b \geq 0;$

Economic interpretation :
a will give the cost/price of one hour for the cutting process and *b* the price for the packing one.

Linear program modeling

General cases :

lower and upper bounds;

render inequalities to equalities;

express unsigned variables as nonnegative ones;

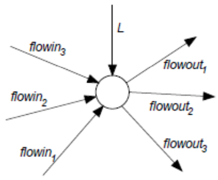
Linear program modeling

Expressing the objective function:

- maxmin or minmax functions;
  - Let be given variables x 1, x2, …, xn. The objective function consists in maximizing the minimum of them.
    - Intro duce a new variable y which is a lower bound of xj;
    - The objective function becomes: Max y;
- "multiplicative" functions
  - Min 1/x => Max x;
  - y=ax, where x binary variable and a $\in$ [0, .., M].
    - y <= a; y <= Mx; y >= a + (x-1)M.
    - how to do when both binary variables ?

Linear program modeling

- Expressing constraints :
  - Capacity constraints
    - Usually they give rise to upper bound constraints;
  - Demand constraints
    - Usually they give rise to lower bound constraints;
  - Mass balance constraints



  - Proportion constraints
    - a, b and c give all, and a is exactly 30%: a = 0,3 (a+b+c)

LP modeling techniques

$$Min \ \frac{\sum_j N_j x_j}{\sum_j D_j x_j}$$

$$s.t. \ \sum_j A_{ij} x_j \le R_i \qquad \forall i$$

$$Set: \quad d = \frac{1}{\sum_j D_j x_j}; \quad y_j = d\, x_j;$$

$$Obj.Function \qquad Min \ \sum_j N_j y_j$$

$$s.t. \qquad \sum_j D_j y_j = 1;$$

$$\sum_j A_j y_j \le dR_i;$$

Integer Linear Programming

Introduction to integer linear programming

- **Integer Linear Programming (ILP)**
  - An integer linear program is a linear programming problem with variables taking values in Z.
  - **Binary or 0-1 linear programming problems are a special case.**

- **How to solve integer linear programming :**
  - **Cutting planes, Branch and bound methods, branch and cut…**

Modeling by binary variables

At most one of A, B,...,H	$a+b+c+d+e+f+g+h \le 1$
Exactly two of A, B,...,H	$a+b+c+d+e+f+g+h = 2$
If A then B	$b \ge a$
Not B	$\overline{b} = 1 - b$
If A then not B	$a + b \le 1$
If not A then B	$a + b \ge 1$
If A then B, and if B then A	$a = b$
If A then B and C	$b \ge a$ and $c \ge a$
If A then B or C	$b + c \ge a$
If B or C then A	$a \ge b$ and $a \ge c$
	or alternatively: $a \ge \frac{1}{2} \cdot (b + c)$
If B and C then A	$a \ge b + c - 1$
If two or more of B, C, D or E then A	$a \ge \frac{1}{3} \cdot (b + c + d + e - 1)$
If M or more of N projects (B, C, D, ...) then A	$a \ge \frac{b+c+d+\ldots-M+1}{N-M+1}$

Linear programming modeling

Modeling constraints in scheduling problems:

disjunctive constraints :

task A before task B or task B before task A;

conjunctive constraints;

task A before task B; or task A before B and C...

Expressing disjunctive constraints:

Either $2 \cdot x_1 + x_2 \ge 6$ (Constraint 1) or $x_1 + 2 \cdot x_2 \ge 7$ (Constraint 2)

$$2 \cdot x_1 + x_2 \ge 6 \cdot b$$
$$x_1 + 2 \cdot x_2 \ge 7 \cdot (1 - b)$$

$$b = 0: \quad 2 \cdot x_1 + x_2 + x_3 \ge 0 \quad x_1 + 2 \cdot x_2 + 3 \cdot x_3 \ge 7$$
$$b = 1: \quad 2 \cdot x_1 + x_2 + x_3 \ge 6 \quad x_1 + 2 \cdot x_2 + 3 \cdot x_3 \ge 0$$

Scheduling problems

Problem formalization:

Data: tasks (ready time, processing time, due time), number of machines, precedence constraints, preemption, etc …

Problem modeling

- data: ri (ready time), pi (processing time), di (due time);
- Decision variables : ti (start date of execution), xij (execution order between i and j), ci (completion execution time).
- Precedence/conjunctive constraints: task i before task j : ti + pi <= tj:
- Disjunctive constraints: ti + pi <= tj or tj + pj <= ti :
  ti + pi <= tj + M(1-b) ; tj + pj <= ti + Mb ; b $\in$ {0,1}
- Objective function: minimize the completion time (makespan), total lateness, number of delayed tasks, etc.

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The one machine scheduling problem

We consider the *n*-job one-machine scheduling problem with ready time *ri*, processing time *pi*, and due time *di* for each job *i*. Preemption is not allowed, and precedence constraints among jobs are not considered.

- Decision variables:
  - xij (boolean variables): task i executed at the j-th order ;
  - tj (>= 0) start time for task placed at order j ;
- LP formulation

f.obj. : Min tn + Σ_i *xin*pi ;*

s.t. for all i : Σ_j *xij=1 ;*

for all j :Σ_i *xij=1 ;*

for all j : tj >= Σ_i *xij*ri ;*

for all j : tj+1 >= tj + Σ_i *xij*pi ;*

for all j : tj + Σ_i *xij*pi <=* Σ_i *xij*di ;*

Examples in LP modeling

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Problem. *Computer production planning*

The company ORDI produces laptops. Selling previsions (in thousands of unities) for the next six months are reported in the following table. The production capacity of the company is 30000 per month. It is possible to produce more at higher cost, 300 Euros per piece instead of 250.

january	*february*	*march*	*april*	*may*	*june*	*july*
30	*15*	*15*	*25*	*33*	*40*	*45*

Selling previsions for the next six months

There are 2000 laptops in stock. The storing cost is of 25 Euros per unity for what is in stock at the end of the month. We suppose that the stocking capacities are unlimited. We begin at the 1^st^ January. What is the production plan for the next six months such that all demands are satisfied and the cost is minimized.

LP model

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Decision variables :

xi : quantity produced with nominal tariff during month i ;
yi : quantity produced with secondary tariff during month i ;
si : stock of month i, and s0 initial stock (=2000) ;

Mass conservation constraints :

For i from 1 to 6 : si-1 + xi + yi = Di + si ;

Capacity constraint:

xi <= 30000 ;

Positivity and integrality constraints:

xi, yi, si >= 0 ; xi, yi, si are integers

Objective function:

Min (*Σi xi*250 + Σi yi*300 + Σi si*25).*

Modeling optimization problems in logistics and transportation

Modeling optimization problems in logistics and transportation

Related problems:

- Partitioning, generalized matching, covering, facility location problems;
- Lp models for transportation problems:
  - Transportation programs,
  - Minimal cost flows and multiflows, vehicle routing problems, etc.

Partitioning, covering

- Partitioning problems ask whether is possible to split a given set in subsets satisfying some criteria.
  - (bi)partition problem : the problem is to decide whether a given a set of integers can be partitioned into two "halves" that have the same sum.
- Covering problems ask whether a certain combinatorial structure 'covers' another, or how large the structure has to be to do that.
  - A cover of a set *X* is a collection of sets such that *X* is a subset of the union of sets in the collection. An exact cover is a subcollection such that each element in *X* is contained in *exactly one* subset in it.
  - The vertex cover problem : a vertex cover of a graph is a set of vertices such that each edge of the graph is incident to at least one vertex of the set.
    - Propose an ILP model for the vertex cover problem.

Covering problems total cover

- Data:
  - *n* candidate sites (indexed by *i*)
  - *m* demands (indexed by *j*)
  - D_c maximum cover (reaching) distance.
  - constants (0,1) noted as a_{ij} indicating if *j* can be covered by *i*.
  - c_i installation cost of *i*.
  - d_{ij} distances between any *i* and *j*.

Goal: compute which sites will be open such that all clients are covered at minimal installation cost of sites.

PL01 Formulation:

$$Min \sum_i c_i x_i$$

$$\sum_i a_{i,j} x_i \geq 1 \qquad \forall j$$

$$x_i \in \{0,1\} \qquad \forall i.$$

maximal cover

- It occurs when one wants to maximize the area/demand covered when the number of allowed sites is bounded.

  additional coefficients and variables:

  - *p* maximum number of sites allowed to be installed;
  - d_j demand *j* (known data)
  - z_j binary variable indicating if some *j* is covered.

PL01 formulation:

$$
\begin{aligned}
& Max \ \sum_j d_j z_j \\
& z_j \le \sum_i a_{i,j} x_i \qquad \forall j \\
& \sum_i x_i \le p \\
& x_i \in \{0,1\} \quad \forall i \\
& z_j \in \{0,1\} \quad \forall j.
\end{aligned}
$$

p-center and p-median problems

Data:

- A set of demands/clients to satisfy
- A set of *n* sites, such that only *p* are allowed to open
- Distance d_{ij} between demands/clients and sites;

“p-center” problems

Goal:

- Where placing sites in order to minimize the maximal distance;
- Example : placing firestations, etc.

“p-median” problems

Goal:

- Minimize the average distance
- Example : distribution centers...

p-center problem

LP model:

$$
\begin{aligned}
& Min \quad z \\
& \sum_i y_{i,j} = 1; \qquad \forall j \\
& \sum_i x_i \le p; \\
& y_{i,j} \le x_i, \qquad \forall i, \forall j \\
& \sum_i d_{i,j} y_{i,j} \le z, \qquad \forall j, \\
& x_i \in \{0,1\} \qquad \forall i \\
& y_{i,j} \in \{0,1\} \qquad \forall i, \forall j, \\
& z \ge 0;
\end{aligned}
$$

x_i -> site i;
$y_{i,j}$ -> demand j covered by i

p-median problem

LP model:

$$
\begin{aligned}
& Min \quad \sum_j q_j \sum_i d_{i,j} y_{i,j} \\
& \sum_i y_{i,j} = 1; \qquad \forall j \\
& \sum_i x_i \le p; \\
& y_{i,j} \le x_i, \qquad \forall i, \forall j \\
& x_i \in \{0,1\} \qquad \forall i \\
& y_{i,j} \in \{0,1\} \qquad \forall i, \forall j,
\end{aligned}
$$

q_j -> quantity of demand j;
x_i -> site i;
$y_{i,j}$ -> demand j covered by i

A transportation program

The transportation problem deals with sources where a supply of some commodity is available, and destinations where the commodity is demanded.

Example. Cars to rent

A company specialized in car's rent has two garages with respectively 12 et 8 cars in stock, and three rent shops demanding for respectively 8, 7 and 5 cars. The unitary costs of transport are given in the following table. What would be the transport plan of minimal cost ?

Garages ↓ / Shops →	*1*	*2*	*3*	Supply
1	3	5	3	12
2	2	7	1	8
Demand→	*8*	*7*	*5*	

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LP model

Minimize $3x_{1,1} + 5x_{1,2} + 3x_{1,3} + 2x_{2,1} + 7x_{2,2} + x_{2,3}$

Subject to:

$$x_{1,1} + x_{2,1} \geq 8;$$
$$x_{1,2} + x_{2,2} \geq 7;$$
$$x_{1,3} + x_{2,3} \geq 5;$$
$$x_{1,1} + x_{1,2} + x_{1,3} \leq 12;$$
$$x_{2,1} + x_{2,2} + x_{2,3} \leq 8;$$
$$x_{i,j} \in Z+;$$

x_{ij} : decision variables : number of cars to be sent from i to j.

Exercise : Give the general LP model for the transportation program : There are a1, a2, ... , am units in the warehouses 1,2 ,..., m. We want to transport these units to the destinations 1,2 ,..., n where the requests b1, b2 ,..., bn occur, with Σai = Σbj.

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Transportation program

transport unitary cost matrix | supply

					supply
	c_{11}	c_{12}	......	c_{1n}	a_1
	c_{21}	c_{22}	......	c_{2n}	a_2
	$\vdots$			$\vdots$	$\vdots$
	c_{m1}	c_{m2}	......	c_{mn}	a_m
demand	b_1	b_2	......	b_n	

$$Min \sum_{i,j} c_{i,j} x_{i,j}$$

$$\sum_j x_{i,j} \le a_i \qquad i = 1, \dots, m$$

$$\sum_i x_{i,j} \ge b_j \qquad j = 1, \dots, n$$

$$x_{i,j} \ge 0 \qquad \forall i, \forall j.$$

Path problems LP formulations

LP model for the shortest path problem

$$Min \sum_i w_{i,j} x_{i,j}$$

$$(!) \quad \sum_{j\ succ\ i} x_{i,j} = \sum_{j\ prec\ i} x_{j,i}; \quad \forall i, j \in X, i, j \ne s, t.$$

$$(!!) \quad \sum_{j\ succ\ s} x_{s,j} = \sum_{j\ prec\ t} x_{j,t} = 1;$$

$$x_{i,j} \in \{0,1\} \qquad \forall i, \forall j.$$

The directed cycle with minimum weight/length is called the minimum cycle mean.
The LP formulation :

$$Min \sum_i w_{i,j} x_{i,j}$$

$$(!) \quad \sum_{j\ succ\ i} x_{i,j} = \sum_{j\ prec\ i} x_{j,i}; \quad \forall i, j \in X.$$

$$(!!) \quad \sum_{all\ (i,j)} x_{i,j} = 1;$$

$$x_{i,j} \in \{0,1\} \qquad \forall i, \forall j.$$

The art gallery problem

- The art gallery or museum problem originates from a real-world problem of guarding an art gallery with the minimum number of guards which together can observe the whole gallery.
- Propose an ILP model.

Exercise

Employment planning for a restaurant

- The manager of a restaurant needs to ensure permanence and service in his restaurant on the basis of some statistics (say for day i ($1 \leq i \leq 7$), are needed ai employees):
- He needs to find the minimal number of employees when any of them should work 5 consecutive days and next takes two days of break. Give an LP model for this problem.

END