A new theoretical framework for Robust Optimization under multi-band uncertainty

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Abstract We provide an overview of our main results about studying Linear Programming Problems whose coefficient matrix is subject to uncertainty and the uncertainty is modeled through a multi-band set. Such an uncertainty set generalizes the classical one proposed by Bertsimas and Sim [3] and is particularly suitable in the common case of arbitrary non-symmetric distributions of the parameters. Our investigations were inspired by practical needs of our industrial partner in ongoing projects with focus on the design of robust telecommunications networks.

1 Introduction

A central assumption in classical optimization is that all parameters describing the problem are known exactly. However, many real-world problems consider data that are uncertain or not known with precision (for example, because of measurement methodologies which introduce an error or because of approximated numerical representations). Neglecting the uncertainty may have dramatic effects and turn optimal solutions into infeasible or very costly solutions. Since the groundbreaking investigations by Dantzig [11], many works have thus tried to find effective ways to deal with uncertainty (see [2] for an overview). During the last years, Robust Optimization (RO) has attracted a lot of attention as a valid methodology to deal with uncertainty affecting optimization problems. A key feature of RO is to take into account uncertainty as hard constraints, which are added to the original formulation of the problem in order to cut off solutions that are not *robust*, i.e. protected from deviations of the data. For an exhaustive introduction to the theory and applications

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of RO, we refer the reader to the book by Ben-Tal et al. [1], to the recent survey by Bertsimas et al. [2] and to the Ph.D. Thesis [6].

An approach to model uncertain data that has been highly successful and has been adapted to several applications is the so-called Γ-scenario set (BS) by Bertsimas and Sim [3]. The uncertainty model for a Linear Program (LP) considered in BS assumes that, for each coefficient a we are given a nominal value $\overline{a}$ and a maximum deviation d and that the actual value lies in the symmetric interval $[\overline{a}-d, \overline{a}+d]$. Moreover, a parameter Γ is introduced to represent the maximum number of coefficients that deviate from their nominal value and to control the conservativeness of the robust model. A central result of BS is that, under the previous characterization of the uncertainty set, the robust counterpart of an LP can be formulated as a compact linear problem. However, the use of a single and symmetric deviation band may greatly limit the power of modeling uncertainty, as it becomes evident when the deviation probability sensibly varies within the band: in such a case, if we neglect the inner-band behavior and we just consider the extreme values like in BS, we obtain a rough estimation of the deviations and thus an unrealistic uncertainty set. Reaching a higher modeling resolution would therefore be very desirable, as also highlighted and requested by our industrial partners. This can be accomplished by breaking the single band into multiple and narrower bands, each with its own Γ value. Such model is particularly attractive when historical data on the deviations are available, a very common case in real-world settings. Thus, a multi-band uncertainty set can effectively approximate the shape of the distribution of deviations built on past observations, guaranteeing a much higher modeling power than BS. The multi-band idea was first exploited by Bienstock for the special case of Robust Portfolio Optimization [4]. However, a general definition of the multi-band model applicable also in other contexts and a deep theoretical study of its properties have not yet been done. The main objective of our original study is to fill such a gap.

We remark that, while the present work was under revision, we have refined and extended our results, realizing a new paper [8] that include additional results about dominance among uncertainty scenarios, uncertain Binary Programs and probability bounds of constraint violation.

2 Multi-band uncertainty in Robust Optimization

We study the robust counterpart of a Linear Programming Problem (LPP) whose coefficient matrix is subject to multi-band uncertainty. The deterministic Linear Program that we consider is of the form:

$$\begin{aligned} \max \quad & \sum_{j \in J} c_j \, x_j & & \text{(LPP)} \\ & \sum_{j \in J} a_{ij} \, x_j \leq b_i & & i \in I \\ & x_j \geq 0 & & j \in J \end{aligned} \tag{1}$$

where $I = \{1,\ldots,m\}$ and $J = \{1,\ldots,n\}$ denote the set of constraints and variable indices, respectively. We assume that the value of each coefficient a_{ij} is uncertain and that such uncertainties are modeled through a set of scenarios $\mathscr{S}$. A scenario $S \in \mathscr{S}$ specifies the deviation d_{ij}^S experienced by each coefficient of the problem with respect to its nominal value $\bar{a}_{ij}$. The actual value a_{ij} is thus equal to $a_{ij} = \bar{a}_{ij} + d_{ij}^S$. The robust counterpart of (LPP) is the optimization problem providing *robust solutions* that are protected against deviations of a specified scenario set $\mathscr{S}$. A natural formulation of the robust counterpart of (LPP) can be obtained by replacing each constraint (1) with its counterpart which considers the deviations allowed by $\mathscr{S}$, namely $\sum_{j\in J} \bar{a}_{ij}\, x_j + d_{ij}^S\, x_j \leq b_i$, $i \in I, S \in \mathscr{S}$.

One of the purposes of our work is to characterize the robust counterpart of (LPP) when the set of scenarios corresponds to what we call a *multi-band uncertainty set*. This set is denoted by $\mathscr{S}_M$ and generalizes the Bertsimas-Sim uncertainty model. Specifically, we assume that, for each coefficient a_{ij}, we are given its nominal value $\bar{a}_{ij}$ and maximum negative and positive deviations $d_{ij}^{K^-}, d_{ij}^{K^+}$ from $\bar{a}_{ij}$, such that the actual value a_{ij} lies in the interval $[\bar{a}_{ij} + d_{ij}^{K^-}, \bar{a}_{ij} + d_{ij}^{K^+}]$ for each scenario $S \in \mathscr{S}_M$. Moreover, we derive a generalization of the Bertsimas-Sim model by partitioning the single deviation band $[d_{ij}^{K-}, d_{ij}^{K+}]$ of each coefficient a_{ij} into K bands, defined on the basis of K deviation values:
$-\infty < d_{ij}^{K^-} < \cdots < d_{ij}^{-2} < d_{ij}^{-1} < d_{ij}^{0} = 0 < d_{ij}^{1} < d_{ij}^{2} < \cdots < d_{ij}^{K^+} < +\infty$.

Through these deviation values, we define: 1) a set of positive deviation bands, such that each band $k \in \{1,\ldots,K^+\}$ corresponds to the range $(d_{ij}^{k-1}, d_{ij}^{k}]$; 2) a set of negative deviation bands, such that each band $k \in \{K^- + 1,\ldots,-1,0\}$ corresponds to the range $(d_{ij}^{k-1}, d_{ij}^{k}]$ and band $k = K^-$ corresponds to the single value $d_{ij}^{K^-}$ (the interval of each band but $k = K^-$ is thus open on the left). With a slight abuse of notation, in what follows we indicate a generic deviation band through the index k, with $k \in K = \{K^-,\ldots,-1,0,1,\ldots,K^+\}$ and the corresponding range by $(d_{ij}^{k-1}, d_{ij}^{k}]$.

Additionally, for each band $k \in K$, we define a lower bound l_k and an upper bound u_k on the number of deviations that may fall in k, with $l_k, u_k \in \mathbb{Z}$ satisfying $0 \leq l_k \leq u_k \leq n$. In the case of band k = 0, we assume that $u_0 = n$, i.e. we do not limit the number of coefficients that take their nominal value. We also assume that $\sum_{k\in K} l_k \leq n$, so that there exists a feasible realization of the coefficient matrix.

The robust counterpart of (LPP) under a multi-band uncertainty set defined by $\mathscr{S}_M$ can be equivalently written as:

$$
\begin{aligned}
\max \quad & \sum_{j\in J} c_j\, x_j && && (2)\\
& \sum_{j\in J} \bar{a}_{ij}\, x_j + DEV_i(x, \mathscr{S}_M) \leq b_i && i \in I\\
& x_j \geq 0 && j \in J,
\end{aligned}
$$

where $DEV_i(x, \mathscr{S}_M)$ is the maximum overall deviation allowed by the multi-band scenario set $\mathscr{S}_M$ for a feasible solution x when constraint i is considered. Finding the value $DEV_i(x, \mathscr{S}_M)$ can be formulated as a 0-1 linear maximization problem, whose optimal solution defines a distribution of the coefficients among the bands that maximizes the deviation w.r.t. the nominal values, while respecting the bounds on the number of deviations of each band (see [7] for details about this problem). As a consequence, the robust counterpart (2) is actually a max-max problem. However, we prove that (2) can be reformulated as a compact and linear problem (we refer the reader to [7] for the complete proofs of the results presented in this section).

Theorem 1. *The robust counterpart of problem (LPP) under the multi-band scenario set $\mathscr{S}_M$ is equivalent to the following compact Linear Program:*

$$
\begin{aligned}
\max \quad & \sum_{j \in J} c_j \, x_j && \textit{(Rob-LP)} \\
& \sum_{j \in J} \bar{a}_{ij} \, x_j - \sum_{k \in K} l_k \, v_i^k + \sum_{k \in K} u_k \, w_i^k + \sum_{j \in J} z_i^j \leq b_i && i \in I \\
& -v_i^k + w_i^k + z_i^j \geq d_{ij}^k \, x_j && i \in I, j \in J, k \in K \\
& v_i^k, \; w_i^k \geq 0 && i \in I, k \in K \\
& z_i^j \geq 0 && i \in I, j \in J \\
& x_j \geq 0 && j \in J.
\end{aligned}
$$

In comparison to (LPP), this compact formulation uses $2 \cdot K \cdot m + n \cdot m$ additional variables and includes $K \cdot n \cdot m$ additional constraints.

As an alternative to the direct solution of (Rob-LP), we also investigated the possibility of adopting a cutting-plane approach [16]: in this case, given a solution $\bar{x} \in \mathbb{R}^n$ we want to test if $\bar{x}$ is robust feasible, i.e. $a_i^S \bar{x} \leq b_i$ for every scenario $S \in \mathscr{S}_M$ and $i \in I$. Specifically, our solution strategy is the following: we start by solving the nominal problem (LPP) and then we check if the optimal solution is robust. If not, we add a cut that imposes robustness (*robustness cut*) to the problem. This initial step is then iterated as in a typical cutting plane method [16].

In the case of the Bertsimas-Sim model, the problem of separating a robustness cut for a given constraint is very simple and essentially consists in sorting the deviations in increasing-order and choose the worst $\Gamma > 0$ (see [12] for details). In the case of multi-band uncertainty, this simple approach does not guarantee the robustness of a computed solution. However, we prove that the separation can be done in polynomial time by solving a *min-cost flow problem* (see [7] for the the detailed description of how we build the instance of this problem and structure the corresponding proof), as formalized in the following theorem.

Theorem 2. *Let $x \in \mathbb{R}_+^n$ and let $\mathscr{S}_M$ be a multi-band scenario set. Moreover, let $(G,c)_x^i$ be the min-cost flow instance corresponding to a solution x and a constraint $i \in I$ of (LPP). The solution x is robust for constraint i w.r.t. $\mathscr{S}_M$ if and only if $\bar{a}_i' x - c_i^*(x) \leq b_i$, where $c_i^*(x)$ is the minimum cost of a flow of the instance $(G,c)_x^i$.*

The proof is based on showing the existence of a one-to-one correspondence between the integral flows and the non-dominated feasible solutions of the binary program expressing the maximum deviation allowed by the multi-band uncertainty set.

3 Application to Wireless Network Design

We applied our new theoretical results about Robust Optimization to the design of wireless networks, considering the *Power Assignment Problem* (PAP): this is the problem of dimensioning the power emission of each transmitter in a wireless network, to provide service coverage to a number of users, while minimizing the overall power emission. The PAP is particularly important in the (re)optimization of networks that are updated to new generation digital transmission technologies. For a detailed introduction to the PAP and the general problem of designing wireless networks, we refer the reader to [9, 10, 15].

A classical LP formulation for the PAP can be defined by introducing the following elements: 1) a vector of non-negative bounded continuous variables p that represent the power emissions of the transmitters; 2) a matrix A of the coefficients that represent signal attenuation (*fading coefficients*) for each transmitter-user couple; 3) a vector of r.h.s. δ (signal-to-interference thresholds) that represents the minimum power values that guarantee service coverage. If the objective is to minimize the overall power emission, the PAP can be written as: $\min \mathbf{1}'p$ s.t. $Ap \geq \delta,\ 0 \leq p \leq P^M$, where exactly one constraint $a_i' p \geq \delta_i$ is introduced for each user i to represent the corresponding service coverage condition.

Each fading coefficient of the matrix A summarizes the different factors which influence propagation (e.g., distance between transmitter and receiver, terrain features) and is classically computed by a propagation model. However, the exact propagation behavior of a signal cannot be evaluated and thus each fading coefficient is naturally subject to uncertainty. Neglecting such uncertainty may lead to unexpected coverage holes in the resulting plans, as devices may be actually uncovered for bad deviations affecting the fading coefficients. For a detailed presentation of the technical aspects of propagation, we refer the reader to [17]. Following the ITU recommendations (e.g., [13]), we assume that the fading coefficients are mutually independent random variables and that each variable is log-normally distributed. The adoption of the Bertsimas-Sim model would provide only a very rough modeling of the deviations associated with such distribution. Contrarily, the multi-band uncertainty model provides a much more refined representation of the fading coefficient deviations. In our computational study, we considered realistic instances corresponding to region-wide networks based on the Terrestrial Digital Video Broadcasting technology (DVB-T) [13] and were taken as reference for the design of the new Italian DVB-T national network. We built the uncertainty set taking into account the ITU recommendations [13] and discussions with our industrial partners in past projects about wireless network design. For a description of our benchmark instances and the corresponding computational results, we refer the reader to [7]. The main purpose of our tests was to compare the efficiency of solving directly the compact formulation (Rob-LP) with that of a cutting-plane method based on the robustness cuts presented in the previous section. For the Bertsimas-Sim model, such comparison led to contrasting conclusions in past works (e.g., [12, 14]). In our case, we found that the cutting-plane approach produced optimal solutions in less time for the bigger instances, while in all the other cases the compact formulation performed better. Concerning the price of robustness, we noted that imposing robustness with the multi-band model led to a sensible increase in the overall power emission, that was anyway lower than that of the Bertsimas-Sim model in all but two cases. On the other hand, the power increase under multi-band uncertainty was compensated by a very high average protection factor against deviations. The good overall performance encourages further investigations. In particular, future research will be focused on refining the cutting plane method and enlarging the computational experience to other relevant real-world problems.

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