Robust Optimization Under Multiband Uncertainty

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We provide an overview of the main results that we obtained studying uncertain mixed integer linear programs when the uncertainty is represented through the new multiband model [4]. Such model extends and refines the classical one proposed by Bertsimas and Sim [2] and is particularly suitable in the common case of arbitrary non-symmetric distributions of the uncertainty. Our investigations were inspired by the practical needs of our industrial partners in the German research project ROBUKOM [8].

1 Introduction

A central assumption in classical optimization is that all coefficients of the considered problem are known exactly. However, many real-world problems involve uncertain data and neglecting these uncertainties may have dramatic effects: optimal solutions may reveal to be of very bad quality, and solutions supposed to be feasible may turn out to be infeasible.

Over the past few years, Robust Optimization (RO) has increasingly gained attention as a valid methodology to tackle uncertainties affecting optimization problems. The key feature of RO is to take into account uncertainty as hard constraints which restrict the feasible set, thus maintaining only *robust* solutions, i.e. solutions protected from data deviations. For an exhaustive introduction to the theory and applications of RO, we refer the reader to the recent survey by Bertsimas et al. [1].

In this work, we focus on multiband uncertainty, a new approach to model uncertainty that refines and generalizes the widely known Γ-scenario set (BS) of Bertsimas and Sim [2]. The uncertainty model BS assumes that, for each coefficient a of the problem, we know the nominal value $\overline{a}$ as well as the maximum deviation d and that the actual value of a lies in the symmetric interval $[\overline{a} - d, \overline{a} + d]$. Moreover, a parameter Γ is introduced to represent the maximum number of coefficients that deviate from their nominal value. This parameter is also controls the conservativeness of the robust model. A central result of BS is that the robust counterpart of an LP can be formulated as a compact linear problem. However, the use of a single deviation band may greatly limit the power of modeling uncertainty, as noted even by

Sim and his colleagues in [7]. This is particularly evident in real-world problems, where it is common to have *asymmetric probability distributions* for the deviations, that are additionally defined over *asymmetric* intervals. In such cases, neglecting the inner-band behavior and just considering the extreme values like in BS leads to a rough estimate of the deviations and produce over-conservative robust solutions. Having a higher modeling resolution is therefore highly desirable. This can be accomplished by a simple operation: breaking the single band of BS into multiple narrower bands, each with its own Γ value, as we do in multiband uncertainty.

The idea of using multiple bands was originally proposed for portfolio optimization [3], but this applied-oriented study was (surprisingly) not followed by a theoretical study. Our main objective has thus been to close the knowledge gap about the use of multiband uncertainty in RO.

For a comprehensive presentation of the results we refer the reader to [4, 5, 6]. In [4] we have presented two of the fundamental results of the study, namely: 1) the robust counterpart of a MILP can be formulated as a compact linear program; 2) the separation of robustness cuts can be done by solving a min-cost flow problem. A refinement of the results is presented in [5] and finally in [6] the study was extended by investigating special properties of uncertain binary programs as well as the probability bounds of constraint violation.

We note that our results are not obtained by simply extending the proofs of [2] for single-band uncertainty, but required alternative proof strategies.

2 Multiband uncertainty in Robust Optimization

We study the robust counterpart of the following Mixed-Integer Linear Program (MILP):

$$\max \sum_{j \in J} c_j \, x_j \qquad \text{s.t.} \quad \sum_{j \in J} a_{ij} \, x_j \leq b_i \,, \;\; i \in I = \{1, \ldots, m\}, \tag{1}$$

$$x_j \geq 0 \,, \;\; j \in J = \{1, \ldots, n\}, \qquad x_j \in \mathbb{Z}^+, \;\; j \in J^{\mathbb{Z}} \subseteq J \,. \tag{2}$$

We assume that the value of each coefficient a_{ij} is uncertain and that the uncertainty is modeled through what we call a *multiband uncertainty set* $\mathcal{S}_M$. Specifically, we assume that, for each coefficient a_{ij}, we are given its nominal value $\overline{a}_{ij}$ and maximum negative and positive deviations $d_{ij}^{K^-}, d_{ij}^{K^+}$ from $\overline{a}_{ij}$, such that the actual value a_{ij} lies in the interval $[\bar{a}_{ij} + d_{ij}^{K^-}, \bar{a}_{ij} + d_{ij}^{K^+}]$. Moreover, we derive a generalization of the Bertsimas-Sim model by partitioning the single deviation band $[d_{ij}^{K-}, d_{ij}^{K+}]$ of each coefficient a_{ij} into K bands, defined on the basis of K deviation values: $-\infty < d_{ij}^{K^-} < \cdots < d_{ij}^{-2} < d_{ij}^{-1} \; < \; d_{ij}^{0} = 0 \; < \; d_{ij}^{1} < d_{ij}^{2} < \cdots < d_{ij}^{K^+} < +\infty$.

Through these deviation values, we define: 1) a set of positive deviation bands, such that each band $k \in \{1, \ldots, K^+\}$ corresponds to the range $(d_{ij}^{k-1}, d_{ij}^{k}]$; 2) a set of negative deviation bands, such that each band $k \in \{K^- + 1, \ldots, -1, 0\}$ corresponds to the range $(d_{ij}^{k-1}, d_{ij}^{k}]$ and band $k = K^-$ corresponds to the single value $d_{ij}^{K^-}$ (the interval of each band but $k = K^-$ is thus open on the left). With a slight abuse of notation, we denote a generic deviation band by the index k, with $k \in K = \{K^-, \ldots, -1, 0, 1, \ldots, K^+\}$ and the corresponding range by $(d_{ij}^{k-1}, d_{ij}^{k}]$.

Additionally, for each band $k \in K$, we define a lower bound l_k and an upper bound u_k on the number of deviations that may fall in k, with $l_k, u_k \in \mathbb{Z}$ satisfying $0 \leq l_k \leq u_k \leq n$. In the case of band k = 0, we assume that $u_0 = n$, i.e. we do not limit the number of coefficients that take their nominal value. We also assume that $\sum_{k \in K} l_k \leq n$, so that there exists a feasible realization of the coefficient matrix.

Consider a feasible solution x and a constraint i and denote by $DEV_i(x, \mathcal{S}_M)$ the maximum overall deviation allowed by the multiband uncertainty set $\mathcal{S}_M$, then the robust counterpart of MILP can be defined by adding $DEV_i(x, \mathcal{S}_M)$ to each constraint $i \in I$, namely $\sum_{j \in J} a_{ij}\, x_j + DEV_i(x, \mathcal{S}_M) \leq b_i$. Since $DEV_i(x, \mathcal{S}_M)$ corresponds to a binary maximization program (see [4] for details), the resulting robust counterpart is actually a (non-linear) max-max problem. However, we prove that this problem can be reformulated as a compact and linear problem. For lack of space in the present extended abstract, we state only informally the main results of our investigations. We refer the reader to [4, 5, 6] for the formal complete statements and proofs of the presented theorems.

Theorem 2.1. *The robust counterpart of a MILP under the multiband uncertainty set is equivalent to a compact MILP, which includes $K \cdot m + n \cdot m$ additional variables and $K \cdot n \cdot m$ additional constraints.*

As an alternative to the direct solution of the compact and linear robust counterpart, we have also investigated the possibility of adopting a cutting-plane approach. Given a solution to MILP, we want to test if the solution is robust feasible. If not, we separate a cut that imposes robustness (*robustness cut*), we add it to the problem and we solve again the problem including the new cut. This basic step can be iterated as in a typical cutting-plane method until a robust feasible solution is found. In the case of the Bertsimas-Sim model, the problem of separating a robustness cut for a given constraint is very simple and essentially consists in sorting the deviations in increasing order and choose the worst $\Gamma > 0$. In the case of multiband uncertainty, this simple approach does not guarantee the robustness of a computed solution. However, we prove the following result:

Theorem 2.2. *The separation of a robustness cut for a constraint of a MILP can be done in polynomial time by solving a* min-cost flow problem*.*

We refer again the reader to [4, 5, 6] for the formal statement and the detailed description of how we build the min-cost flow instance and structure the corresponding proof.

2.1 Binary Programs with cost uncertainty

In the case of pure binary programs where the uncertainty only affects the objective function, the results presented above can be refined. To this end, consider the following Binary Program (BP):

$$\begin{aligned} \min \quad & \sum_{j \in J} c_j\, x_j \qquad\qquad (BP) \\ & x \in X \subseteq \{0,1\}^n \end{aligned}$$

with non-negative cost vector, i.e. $c_j \geq 0$, for all $j \in J = \{1, \ldots, n\}$. Relevant problems such as the minimum spanning tree problem, the maximum weighted matching problem and the shortest path problem belong to this class of problems.

We have studied the robust version of the previous problem when only the cost coefficients are uncertain and uncertainty is modeled through a multiband set. More formally, for each element $j \in J$, we are given the nominal cost $\bar{c}_j$ and a sequence of $K^+ + 1$ deviation values

d_j^k, with $k \in K = \{0, \ldots, K^+\}$, such that $0 = d_j^0 < d_j^1 < \ldots < d_j^{K+} < \infty$ (note that in contrast to the previous section we consider only positive deviations). Through these values, we define: 1) the zero-deviation band corresponding to the single value $d_j^0 = 0$; 2) a set K^+ of positive deviation bands, such that each band $k \in K \backslash \{0\}$ corresponds to the range $(d_j^{k-1}, d_j^k]$. Furthermore, integer values $l_k, u_k \in \mathbb{Z}$, with $0 \leq l_k \leq u_k \leq n$, represent the lower and upper bounds on the number of deviations falling in each band $k \in K$.

Since BP is a special case of MILP, we can solve it by referring to its compact robust counterpart or by adopting a cutting-plane algorithm based on the separation of robustness cuts, as shown above. However, as an alternative to these two approaches, we have proved the following result:

Theorem 2.3. *The robust optimal solution of a Binary Program with cost uncertainty modeled through a multiband set can be computed by solving a polynomial number of nominal problems BP with modified objective function, if the number of bands is constant. Tractability and approximability of BP are maintained.*

We refer the reader to [6] for the formal statement of the result. Our study has been completed by computational experiments on realistic network instances, defined in collaboration with our industrial partners in past and ongoing research projects. In particular, the experiments have highlighted a reduction in the price of robustness, thanks to the refined representation of the uncertainty obtained through the multiband model.

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