Examples of LP problem solved by the Simplex Method

Linear Optimization 2016
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Exercise 2

Solve the following Linear Programming problem through the Simplex Method.

\[
\begin{align*}
\text{max} & \quad 3x_1 + 2x_2 - 5x_3 \\
\text{s.t} & \quad 4x_1 - 2x_2 + 2x_3 \leq 4 \\
& \quad -2x_1 + x_2 - x_3 \geq -1 \\
& \quad x_1, x_2, x_3 \geq 0
\end{align*}
\]

Solution

The first step is to rewrite the problem in standard form as follows:

\[
\begin{align*}
\text{min} & \quad -3x_1 - 2x_2 + 5x_3 \\
\text{s.t} & \quad 4x_1 - 2x_2 + 2x_3 + x_4 = 4 \\
& \quad 2x_1 - x_2 + x_3 + x_4 + x_5 = 1 \\
& \quad x_1, x_2, x_3, x_4, x_5 \geq 0
\end{align*}
\]

Having added the slack variables \(x_4, x_5\), it is easy to find the following initial basis:

\[
B = [A_4 \ A_5] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\]

and thus to split the decision variables in the following way:

\[
x_B = \begin{pmatrix} x_4 \\ x_5 \end{pmatrix}, \quad x_N = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}
\]

The solution associated with the basis \(B\) is \(x = (0, 0, 0, 4, 1)\) with value \(z = 0\).

We can then define the following simplex tableau:

\[
\begin{array}{ccccc}
0 & -3 & -2 & 5 & 0 \\
4 & 4 & -2 & 2 & 1 \\
1 & 2 & -1 & 1 & 0 & 1 \\
\end{array}
\]

The first thing to do is checking the value of the reduced costs in the 0th-row: if all reduced costs are non-negative, then we have a sufficient condition of optimality and the solution associated with the current basis is optimal.

In our case, three variables, namely \(x_1, x_2\) are associated with the negative reduced costs (-3, -2). The sufficient condition is thus not satisfied and we thus proceed to operate a change of basis.

**REMARK:** We could already conclude that the problem is unbounded, noticing that all the entries in the 2nd column are non-positive. However, we can reach the same conclusion following the algorithm and applying Bland’s rule.

Following Bland’s rule, we choose as variable entering the basis that with the smallest subscript: we then choose \(x_1\). Therefore, \(x_1\) enters the basis and column 1 of the tableau is the pivot column.

Looking at the entries of the pivot column, we can then derive the value \(\theta^*\) considering the values associated with the basic variables. So we have:

\[
\theta = \min_{k=1,2,\ u_k > 0} \left\{ \frac{x_k}{u_k} \right\} = \min \left\{ \frac{4}{1}, \frac{1}{2} \right\} = \frac{1}{2}
\]
So the minimum is attained for variable $x_5$ and $x_5$ exits the basis. The pivot row is thus the row 2 of the tableau and the pivot element is that at the intersection of row 2 and column 1.

In order to get the new tableau corresponding to the new basis:

$$B = [A_4 \ A_1] = \begin{pmatrix} 1 & 4 \\ 0 & 2 \end{pmatrix}$$

we operate the following row operations, aimed at transforming the first column $(2 \ 1 \ 2)^T$ of the tableau into the column $(1 \ 0 \ 0)^T$ using the entries of the pivot row (all entries but the pivot element must become null, while the pivot element must become equal to 1):

- $R_0 \leftarrow R_0 + \frac{3}{2} R_2$
- $R_1 \leftarrow R_1 - 2 R_2$
- $R_2 \leftarrow \frac{1}{2} R_2$

The new tableau that we obtain is:

\[
\begin{array}{cccccc}
3/2 & 0 & -7/2 & 13/2 & 0 & 3/2 \\
2 & 0 & 0 & 0 & 1 & -2 \\
1/2 & 1 & -1/2 & 1/2 & 0 & 1/2 \\
\end{array}
\]

associated with the solution $(1/2, 0, 0, 2, 0)$ of value $z = -3/2$.

Again, we look at the 0-th row to check the presence of negative reduced costs. We have a single variable associated with negative reduced cost, namely $x_2$. So the current basis is not optimal. Looking at the entries in the pivot column, we can notice that they are all non-positive, so we can conclude that the problem is unbounded.