

## **Reliability analysis with ill-known probabilities and dependencies**

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### **ABSTRACT**

In this work, we are interested in estimating the uncertainty that a system will work, given the probabilities that its components will be either failing or working. Usually, this estimation is done by propagating the component uncertainties through a structure function. This problem is well-studied when component probabilities are known and when the component are assumed to be independent. In this paper, we first recall some results obtained when considering that probabilities are unknown and when it is assumed that components are independent or that no knowledge about the components dependence is available. We also explore some structures not studied up to now, such as consecutive k-out-of-n structures.

### **INTRODUCTION**

In many cases, a system reliability, here assumed to be binary (working or failing) can be estimated from the reliabilities of its components and from a structure function linking the binary states (working or failing) of those components to the system state. Usually, this estimation is done by assuming:

- that probabilities of component failures are precisely known or given;
- that the components are (stochastically) independent.

The first assumption (preciseness) is quite strong, especially when few or no data concerning the components are available, or when one has to model some expert opinion about the component. In these cases, using a precise probability usually means adding some assumption not supported by available evidence (e.g., using maximum entropy principle (Jaynes 2003)). An alternative is to include the imprecision in the uncertainty model, by considering probability bounds.

The second assumption is in general more likely to hold, however the analyst may wish to relax this assumption, considering the case where the possible dependencies between components are unknown or only partially known.

Both issues have been investigated, in general settings, by imprecise probability theories (Walley 1991; Couso et al. 2000). However, the specific problem of assessing a system reliability under such conservative assumptions has only been explored in a very few works (Utkin 2004; Berleant and Zhang 2004; Pedroni and Zio 2013; Fetz

and Tonon 2008), and the case of partially specified independence in even less (Hill and Spall 1998; Troffaes and Coolen 2009). In this paper, we recall some of the main results of these previous papers, setting them in a general framework. We also provide some preliminary results about consecutive k-out-of-n systems (Lambiris and Papastavridis 1985), that have not been studied yet within an imprecise probabilistic framework.

## PRELIMINARIES AND NOTATIONS

We consider that we have a set of components  $X_1, \dots, X_N$ , whose values are binary and described by domain  $\mathcal{X} = \{1, 0\}$  where 1 usually stands for working and 0 for not working. We will denote by  $\mathcal{X}^N = \times_{i=1}^N \mathcal{X}$  the Cartesian product describing the set of all possible system states. We will denote by  $\mathbf{x} = (x_1, \dots, x_N) \in \mathcal{X}^N$  a state of the system.

The uncertainty about whether a given component  $X_i$  is working is given by an imprecisely specified probability, that is by two bounds

$$\underline{p}_i = \underline{p}(X_i = 1) \text{ and } \bar{p}_i = \bar{p}(X_i = 1)$$

Note that precise probabilities are retrieved when  $\underline{p}_i = \bar{p}_i$ .

*Example 1.* The assessment "component  $X_i$  has a probability of working that is between 0.8 and 0.9" corresponds to  $\underline{p}_i = 0.8, \bar{p}_i = 0.9$

For a given system, the structure function  $\phi : \mathcal{X}^N \rightarrow \{0, 1\}$  maps each system state  $\mathbf{x} \in \mathcal{X}^N$  to 1 if the system works in this state, and 0 if the system fails in this state.  $\phi^{-1}(0)$  and  $\phi^{-1}(1) \subseteq \mathcal{X}^N$  will respectively denote the set of states for which the system fails and the set of states for which it works (note that they form a partition of  $\mathcal{X}^N$ ). We also assume that the system is coherent, that is if  $\mathbf{x} \geq \mathbf{x}'$  (element-wise), then  $\phi(\mathbf{x}) \geq \phi(\mathbf{x}')$ , meaning that if more components work, then the system state can only improve.

## GENERIC PROBLEM

The generic problem we aim to tackle is to estimate the uncertainty bounds of  $\phi^{-1}(1)$ , that is  $\underline{p}(\phi^{-1}(1)) = 1 - \bar{p}(\phi^{-1}(0))$  and  $\bar{p}(\phi^{-1}(1)) = 1 - \underline{p}(\phi^{-1}(0))$ , given our knowledge about the component uncertainties. Provided the bounds  $\underline{p}_i, \bar{p}_i$  are all we know, the problem of estimating  $\underline{p}(\phi^{-1}(1))$  can be expressed as:

$$\min_p \sum_{\mathbf{x} \in \mathcal{X}^N, \phi(\mathbf{x})=1} p(\mathbf{x}) \quad (1)$$

under the constraints

$$\underline{p}_i \leq \sum_{\mathbf{x} \in \mathcal{X}^N, x_i=1} p(\mathbf{x}) \leq \bar{p}_i, \forall i \in [1, N] \quad (2)$$

$$\sum_{\mathbf{x} \in \mathcal{X}_{1:N}} p(\mathbf{x}) = 1, p(\mathbf{x}) \geq 0 \quad \forall \mathbf{x} \in \mathcal{X}^N.$$

The upper bound is obtained by taking the  $\max$  instead of the  $\min$ . The above linear program expresses that we search for the joint probability that minimizes  $p(\phi^{-1}(1))$  and respects the marginal information  $\underline{p}_i, \bar{p}_i$ , without making any assumption about the dependence. It is interesting to note that under such assumptions, the obtained inference on  $p(\phi^{-1}(1))$  will be imprecise, even if one starts from precise marginal probabilities  $\underline{p}_i = \bar{p}_i$ .

Although the above problem is a linear program, the number of variables increases exponentially with the number of components, making it NP-hard to solve (as is the fundamental reliability problem (Myers 2010, Ch 3.)) for generic systems. It is therefore interesting to seek specific cases displaying a lower computational complexity.

In the sequel, we will detail some results for such specific system structures, both for the case of independent components and unknown independence. We will stick to the easier but often considered case of components with identical uncertainty (i.e.,  $\underline{p}_i = \underline{p}_w$  and  $\bar{p}_i = \bar{p}_w$  for all  $i$ ). Also note, that, as the system is coherent, the minimum in (1) is obtained by considering  $p_i = \underline{p}_i$  for every  $i$ , that is we can replace (2) by

$$\underline{p}_i = \sum_{\mathbf{x} \in \mathcal{X}_{1:N}, x_i=1} p(\mathbf{x})$$

for every  $i$ . This makes the problem slightly simpler to solve, and basically says that we can consider it as a precise case problem where component precise probabilities of working are equal to their lower bounds. The same is true for the upper bounds, this time taking the upper probabilities  $\bar{p}_i$  as precise values.

## THE CASE OF INDEPENDENT COMPONENTS

In the case of independence, we have that

$$p(\mathbf{x}) = \prod_{i, x_i=1} p_i \prod_{i, x_i=0} (1 - p_i), \quad (3)$$

therefore in this case obtaining upper and lower bounds  $\underline{p}(\phi^{-1}(1)), \bar{p}(\phi^{-1}(1))$  simply consists in replacing the probability that a component will be working by the appropriate bound in Eq. (3), for instance take  $p_i = \underline{p}_i$  to compute  $\underline{p}(\phi^{-1}(1))$ .

Even in this case, computing  $\underline{p}(\phi^{-1}(1)), \bar{p}(\phi^{-1}(1))$  in general remains hard. However, there are special configurations of systems for which easy-to-use formulas exist. In this paper, we will be interested in the following types of structures:

- **k-out-of-n:F** ( $k/n : F$ ) systems where a system of  $n$  components fails if  $k$  of them fails.;
- **Consecutive linear k-out-of-n:F** ( $L : k/n : F$ ) systems, where components

$k/n : F$	$\bar{p}(\phi^{-1}(1)) = \sum_{i=0}^{k-1} \binom{n}{i} (\bar{p}_w)^{n-i} (1 - \bar{p}_w)^i$ $\underline{p}(\phi^{-1}(1)) = \sum_{i=0}^{k-1} \binom{n}{i} (\underline{p}_w)^{n-i} (1 - \underline{p}_w)^i$
$L : k/n : F$	$\bar{p}(\phi^{-1}(1)) = \sum_{i=0}^n \binom{n-i \cdot k}{i} (-1)^i (\bar{p}_w (1 - \bar{p}_w)^k)^i$ $- (1 - \bar{p}_w)^k \sum_{i=0}^{k-1} \binom{n-k(i+1)}{i} (-1)^i (\bar{p}_w (1 - \bar{p}_w)^k)^i$ $\underline{p}(\phi^{-1}(1)) = \sum_{i=0}^n \binom{n-i \cdot k}{i} (-1)^i (\underline{p}_w (1 - \underline{p}_w)^k)^i$ $- (1 - \underline{p}_w)^k \sum_{i=0}^{k-1} \binom{n-k(i+1)}{i} (-1)^i (\underline{p}_w (1 - \underline{p}_w)^k)^i$
$C : k/n : F$	$\bar{p}(\phi^{-1}(1)) = \sum_{i=0}^n \binom{n-i \cdot k}{i} (-1)^i (\bar{p}_w (1 - \bar{p}_w)^k)^i$ $+ k \sum_{i=0}^{k-1} \binom{n-k(i+1)-1}{i} (-1)^i (\bar{p}_w (1 - \underline{p}_w)^k)^{i+1} - (1 - \bar{p}_w)^n$ $\underline{p}(\phi^{-1}(1)) = \sum_{i=0}^n \binom{n-i \cdot k}{i} (-1)^i (\underline{p}_w (1 - \underline{p}_w)^k)^i$ $+ k \sum_{i=0}^{k-1} \binom{n-k(i+1)-1}{i} (-1)^i (\underline{p}_w (1 - \underline{p}_w)^k)^{i+1} - (1 - \underline{p}_w)^n$

**Table 1. Reliability bound formulas in the independent case**

are linear and the system fails if  $k$  consecutive elements in the line fail. Such systems can model, for instance, communication relays (Chiang and Chiang 1986) or pumping stations;

- **Consecutive circular k-out-of-n:F** ( $C : k/n : F$ ) systems, where components are placed around a circle and the system fails if  $k$  consecutive elements in the circle fail. Such systems can model, for instance, particle accelerators (Kao 1982).

All these systems generalize well-known series and parallel systems, as these latter systems correspond respectively to 1-out-of-n:F and n-out-of-n:F systems (whether they are consecutive or not). It should also be noted that, once  $n$ ,  $k$  and the component uncertainty are fixed, a  $L : k/n : F$  system is more reliable than a  $C : k/n : F$  system, itself more reliable than a  $k/n : F$  system (in the sense that  $p(\phi^{-1}(1))$  increases and is the highest for  $L : k/n : F$  system). It means that formulas for  $k/n : F$  systems can be used as pessimistic bounds for the other systems.

$k/n : F$ ,  $L : k/n : F$  and  $C : k/n : F$  systems under an assumption of component independence have been well studied in the literature (Lambiris and Papastavridis 1985; Utkin 2004; Chao et al. 1995; Eryilmaz 2010), hence obtaining lower and upper bounds case simply consists in replacing precise probabilities by appropriate bounds. Table 1 summarizes the formulas for the different systems. Note that, although trivial, the formulas for  $L : k/n : F$  and  $C : k/n : F$  systems considering imprecise probabilities

have not been investigated before.

*Example 2.* Consider components such that  $[\underline{p}_w, \bar{p}_w] = [0.95, 0.99]$  and  $2/4$  systems. Then, using the formulas of Table 1, we obtain the following bounds:

$$2/4 : F \quad [\underline{p}(\phi^{-1}(1)), \bar{p}(\phi^{-1}(1))] = [0.9859, 0.9993];$$

$$C : 2/4 : F \quad [\underline{p}(\phi^{-1}(1)), \bar{p}(\phi^{-1}(1))] = [0.9905, 0.9996].$$

$$L : 2/4 : F \quad [\underline{p}(\phi^{-1}(1)), \bar{p}(\phi^{-1}(1))] = [0.9927, 0.9997];$$

## THE CASE OF UNKNOWN INDEPENDENCE

In this section, we interest ourselves to the case of unknown independence, that is the case where we have to solve the optimization program given by (1) and (2). For the  $k/n : F$  systems, we can rely on previous results (Utkin 2004) indicating that

$$\underline{p}(\phi^{-1}(1)) = \max(0, (n - k + 1)\underline{p}_w + k - n);$$

$$\bar{p}(\phi^{-1}(1)) = \min(1, k\bar{p}_w).$$

In the case of series and parallel systems, we retrieve the well-known Frechet bounds (Pedroni and Zio 2013; Fréchet 1935). The cases of  $L : k/n : F$  and  $C : k/n : F$  systems have not been investigated up to now, but obtaining bounds under an assumption of unknown independence for such systems is harder than for the assumption of independence. However, the two following propositions show that, for  $L : 2/3 : F$  systems and  $C : 2/4 : F$  systems ( $C : 2/3 : F$  systems are equivalent to  $2/3 : F$  systems), closed formulas can be obtained.

**Proposition 1.** *Given component uncertainty  $\underline{p}_w, \bar{p}_w$  and unknown independence, the lower and upper bounds  $\underline{p}(\phi^{-1}(1)), \bar{p}(\phi^{-1}(1))$  for a  $L : 2/3 : F$  system are*

$$\underline{p}(\phi^{-1}(1)) = \underline{p}_w$$

$$\bar{p}(\phi^{-1}(1)) = \min(1, 2\bar{p}_w)$$

*Example 3.* Consider components such that  $[\underline{p}_w, \bar{p}_w] = [0.95, 0.99]$  and  $2/3 : F$ ,  $L : 2/3 : F$  systems with an assumption of unknown independence. Then, using the corresponding formulas for these systems we obtain :

$$2/3 : F \quad [\underline{p}(\phi^{-1}(1)), \bar{p}(\phi^{-1}(1))] = [0.9, 1];$$

$$L : 2/3 : F \quad [\underline{p}(\phi^{-1}(1)), \bar{p}(\phi^{-1}(1))] = [0.95, 1].$$

The lower bound of the  $L : 2/3 : F$  being indeed slightly higher than the bound of the  $2/3 : F$  system.

**Proposition 2.** *Given component uncertainty  $\underline{p}_w, \bar{p}_w$  and unknown independence, the lower and upper bounds  $\underline{p}(\phi^{-1}(1)), \bar{p}(\phi^{-1}(1))$  for a  $C : 2/4 : F$  system are*

$$\underline{p}(\phi^{-1}(1)) = \max(0, 2\underline{p}_w - 1)$$

$$\bar{p}(\phi^{-1}(1)) = \min(1, 2\bar{p}_w)$$

*Example 4.* Consider components such that  $[\underline{p}_w, \bar{p}_w] = [0.95, 0.99]$  and  $2/4 : F$ ,  $C : 2/4 : F$  systems with an assumption of unknown independence. Then, using the corresponding formulas for these systems we obtain :

$$2/4 : F \quad [\underline{p}(\phi^{-1}(1)), \bar{p}(\phi^{-1}(1))] = [0.85, 1];$$

$$C : 2/4 : F \quad [\underline{p}(\phi^{-1}(1)), \bar{p}(\phi^{-1}(1))] = [0.9, 1].$$

The lower bound of the  $C : 2/4 : F$  being indeed slightly higher than the bound of the  $2/4 : F$  system.

In comparison with the formulas given in Table 1 for the independent case, the above formulas look extremely simple and are straightforward to use. This suggests that the case of unknown independence for consecutive systems may actually be simpler to handle. We also think that Proposition 2 will be more easy to extend to generic  $k/n$  systems than Proposition 1, due to the symmetry of circular systems that is not present in linear systems, where components at the beginning and end of the line play particular roles. It would also be interesting to relax the assumption that our knowledge about each component is identical (i.e.,  $p_i = p_w$  for all  $i \in [1, N]$ ).

Related to this last remark, it should also be noted that bounds for  $k/n : F$  and for  $C : k/n : F$  could be used to provide approximations of the  $L : k/n : F$  systems.

## CONCLUSION

We have recalled some results regarding the evaluation of lower and upper probabilities that a given system will work. In particular, we have settled them as a generic optimization problem, and have mentioned some structures for which this evaluation can be done through a closed-form formula.

Additionally, we have started to investigate the case of consecutive k-out-of-n:F systems, which to our knowledge have never been studied in the framework of imprecise probabilities. Our results are still preliminary for the case of unknown independence, but suggest that relatively easy formulas can be obtained for such systems as well. As such systems are used in a lot of modern applications, finding such formulas would be interesting, at least to have an idea of the extreme bounds one can obtain for such systems.

In future works, we intend to study how to integrate some known dependency information (Hunter 1976) in the constrained problem (2) so that it can still be easily

solved, and in particular for which case computations can still be achieved through closed form formulas.

Another interesting future line of work is to study other aspects of consecutive k-out-of-n systems when probabilities or dependencies are ill-known, such as for instance importance measures (Zhu et al. 2012), multi-state versions of the systems (Radwan et al. 2011) or design optimization (Shingyoch et al. 2010).

## ACKNOWLEDGMENT

This work was carried out and funded in the framework of the Labex MS2T. It was supported by the French Government, through the program "Investments for the future" managed by the National Agency for Research (Reference ANR-11-IDEX-0004-02).

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