An extension of Universal Generating Function in Multi-State Systems considering epistemic uncertainties

Sebastien Destercke and Mohamed Sallak

Abstract

Many practical methods and different approaches have been proposed to assess Multi-State Systems (MSS) reliability measures. The universal generating function (UGF) method, introduced in 1986, is known to be a very efficient way of evaluating the availability of different types of MSSs. In this paper, we propose an extension of the UGF method considering epistemic uncertainties. This extended method allows one to model ill-known probabilities and transition rates, or to model both aleatory and epistemic uncertainty in a single model. It is based on the use of belief functions which are general models of uncertainty. We also compare this extension with UGF methods based on interval arithmetic operations performed on probabilistic bounds.

Index Terms

Belief functions theory, multi-state systems, epistemic uncertainties, random sets

ACRONYMS

MSS	Multi-State System
UGF	Universal Generating Function
BUGF	Belief Universal Generating Function
IUGF	Interval Universal Generating Function
BBA	Basic Belief Assignment
ODE	Ordinary Differential Equation
IDM	Imprecise Dirichlet Model

NOTATION

G^j	Component j
g_i^j	ith state of component j
$\mathscr{G}^j$	Space of states of component j
$\leq_j$	Order relation on $\mathscr{G}_j$
$\mathscr{R}$	Set of global performance rates of the system
$\leq_{\mathscr{R}}$	Order relation on $\mathscr{R}$
ϕ	Structure function
p^j	Probability distribution over $\mathscr{G}^j$
P^j	Probability measure over the power set of $\mathscr{G}^j$
m^j	BBA over the power set of $\mathscr{G}^j$
Bel^j	Belief function over the power set of $\mathscr{G}^j$
Pl^j	Plausibility function over the power set of $\mathscr{G}^j$
$\underline{x}_i$	Lower bound of variable x_i

S. Destercke and M. Sallak are with the Computer Engineering Department, Compiegne University of Technology, Heudiasyc Laboratory, UMR 7253, CNRS, Research Center of Royallieu, France (e-mail: sdestercke@gmail.com,sallakmo@utc.fr).

$\overline{x}_i$	Upper bound of variable x_i
p_i^j	Probability of being in state g_i^j
$\lambda_{i,k}^j$	Degrading rate from state g_i^j to g_{i-k}^j
$\mu_{i,k}^j$	Repair rate from g_i^j to g_{i+k}^j
$U_j(z)$	z function of the component G^j
$\mathbb{E}(f)$	Expectation of function f
A	Availability of the system

I. Introduction

Most of the literature in reliability theory deals with the binary theory where a system and its components can only be in one of two states at a time each: functioning, or failed. However, a system and its components can have different states characterized by different levels of performance. Such systems are referred to as Multi-State Systems (MSSs). The first main contributions to the theory of MSS are due to Barlow and Wu [1], and El-Neveihi *et al.* [2]. A comprehensive presentation of MSS reliability theory and its applications can be found in the first book devoted to the reliability analysis of MSSs [3], and a recent review of the literature can be found in [4], [5]. Practical methods of MSS reliability assessment are based on four different approaches [6], [7]: the structure function [8], [9], the Monte Carlo simulation technique [10], [11], the Markov approach [12], [13], and the Universal Generating Function (UGF) method [14], [15], [7], [16].

The structure function approach, based on the extension of Boolean models to multi-valued models, was the first method developed for MSS reliability assessment. This first approach and the Markov method both require to know all the possible states of the system, and provide exact results. Monte-Carlo simulations allow one to limit the number of simulations, but only provide an approximation whose quality depends on the number of simulations performed. Those three techniques can be extremely time consuming to large MSS due to the possibly high number of system states.

The Universal Generating Function (UGF) technique, first introduced by Ushakov [17], and greatly extended by Lisniansky [3], and Levitin [14], is an efficient technique to evaluate the availability of different types of MSSs. The UGF function extends the moment-generating function, and reduces the computational complexity of MSS reliability assessment. Thanks to its efficiency, the UGF technique is also suitable for solving different MSS reliability optimization problems, as it can quickly evaluate a systems reliability. Classical MSS reliability theory requires two major assumptions [7]:

- the state probabilities of a MSS components can be fully characterized by probability measures, and
- the performance rates of a MSS component can be precisely determined.

However, there are MSSs where different types of uncertainties about the state probabilities and performance rates of components [18] need to be modelled. There are different ways of classifying uncertainty, but one of the most widely used is to divide it into two types: aleatory uncertainty, and epistemic uncertainty. The former, also called irreducible uncertainty, arises from intrinsic variability of a phenomenon across space, through time, or among a population. The latter, also called reducible uncertainty, arises from incompleteness of knowledge or data [19], [20], [21]. Over the last few years, the reliability community has been increasingly aware that distinguishing between these types of uncertainty is important when evaluating the reliability of systems [22]. When data are sufficient, the classical probabilistic approach can be safely used in risk and reliability assessments [23], [24], [22]. However, there may be cases where the adequacy of the probabilistic model may be questioned [25]: components for which no or few data exist, for instance components that fail only rarely (e.g., nuclear systems, chemical processes, railway systems). In this case, either expert opinions must be collected (i.e., in case of no data), or parameters of probability distributions (i.e., transition rates) cannot be exactly estimated. To solve this issue, several methods have been proposed, some of them using uncertainty models where the epistemic uncertainty (the lack of knowledge) is explicitly modelled: interval approach, belief functions theory [26], possibility theory and fuzzy sets [27], etc. Again, most of these approaches concern binary systems, with the exception of Simon *et al.* [26], that considers multi-state systems modelled by Evidential networks (i.e., Bayesian networks where probabilities are replaced by belief functions).

In this work, we consider the problem of modelling epistemic uncertainty in multi-state systems and components (in contrast with other works that consider systems with multi-state performances and only binary components [28], [29]). There are only few other works that consider the modelling of epistemic uncertainty with multi-state systems and components.

- Huang *et al.* [30] propose the use of intervals and p-boxes to model ill-known probability distributions of component states, two models that are special cases of belief functions [31] (the model retained in this paper).
- Li *et al.* [32] propose an approach based on the use of interval arithmetic with interval-valued probability masses as a model of ill-known probability distributions [33], which are different from belief functions.
- Several methods [34], [35], [36] propose to describe each ill-known probability mass by a fuzzy set, propagating this epistemic uncertainty using Zadeh's extension principle and fuzzy arithmetic (see, e.g., [37]). In these latter proposals, the fuzzy sets are defined over probability masses, not over component states. Hence, they correspond to so-called hierarchical models [38], [39], i.e., uncertainty models defined over uncertainty models. These approaches are in contrast with the other approaches using p-boxes and interval-valued probability masses, and with the approach defined in the current paper, where the uncertainty representation is directly defined over the states. In fact, a possible extension of the work presented in this paper would be to combine the two approaches, using so-called fuzzy-valued belief structures [40].

To our knowledge, the extension of the UGF method when the uncertainty is represented by belief functions has never been investigated. In this paper, we study the extension of the UGF method in the belief function framework [41]. Apart from its novelty, there are a number of reasons why such an extension is appealing.

- Belief functions are uncertainty models that generalize many proposed models of epistemic uncertainty, including probability, sets, fuzzy sets (or, equivalently, possibility distributions [37]), p-boxes, clouds, and more. Hence, they encompass in one sweep all these models.
- Arguably, belief functions model imprecision and lack of knowledge more faithfully than classical probabilities do.
- As they encompass fuzzy sets and probabilities, belief functions can handle linguistic expert opinions as well as comprehensive statistics of components.
- Belief functions, as used in this paper, can be associated to probability bounds. Hence, they can be given a sensitivity analysis interpretation, in which case they correspond to a robust version of classical probabilistic analysis.

In this paper, we propose an extension of the UGF method integrating belief functions where exact computations can be performed (in contrast with methods using interval arithmetic or extensions of it [32], [30] that only provide approximations). A known drawback of belief functions is that they can be computationally hard to handle; and as the idea of UGF method is precisely to reduce computational costs of reliability analysis for MSSs, combining them with belief functions seems to be a good approach. Actually, we will see that in most practical cases, the complexity of using belief functions in UGF method is not much higher than the complexity of using probabilities.

We also propose some means to easily obtain belief functions from ill-known failure and repair rates.

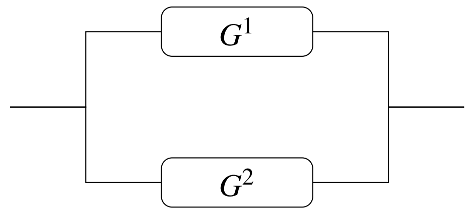

Fig. 1. An example of 2-element system.

Section II provides the necessary background to understand the rest of the paper. Section III explores a particularly interesting way of obtaining belief functions from state transition rates, and recalls some other propositions and means to obtain them. Section IV presents the extension of the UGF approach to belief functions, which is illustrated and compared to the Li *et al.* [32] approach in Section V.

II. Problem setting

In this section, we first explain the notations used for system modelling, before explaining those concerning belief functions.

A. System modelling

We assume that a system can be composed of n components G^j, $j = 1,\dots,n$, whose states are described by a finite ordered space $\mathscr{G}^j = \{g^j_1,\dots,g^j_{k_j}\}$ of k_j values. The value g^j_i is the ith state of component j. We let $\leq_j$ be the order relation on $\mathscr{G}_j$, and assume (without loss of generality) that states are indexed such that $g^j_i <_j g^j_k$ iff $i < k$. Typically, states are ordered according to their performance rates.

We denote by $\mathscr{G}^{1:n} = \times_{i=1}^{n} \mathscr{G}^j$ the Cartesian product of all possible combination of component states; that is, $\mathscr{G}^{1:n}$ corresponds to what is usually called the states of the (overall) system. We denote by $\mathscr{R} = \{r_1,\dots,r_R\}$ the set of global performance rates that can be reached by the system. We assume that $\mathscr{R}$ is totally ordered; and we denote by $\leq_{\mathscr{R}}$ the order on $\mathscr{R}$, and assume an indexing such that $r_i <_{\mathscr{R}} r_k$ iff $i < k$. This assumption covers the usual case where performances are real-valued and ordered according to the natural ordering of numbers. It would be interesting to relax this assumption, for example to consider multi-objective systems, in which case elements of $\mathscr{R}$ are usually only pre-ordered.

The structure function $\phi : \mathscr{G}^{1:n} \to \mathscr{R}$ links the system states to their global performance states. We assume that the system is *coherent*; that is, if one of the performance rates of a component (strictly) increases, all other things being equal, then the overall performance of the system increases. This assumption translates into the fact that $\phi(g^1_{i_1},\dots,g^\ell_{i_\ell},\dots,g^n_{i_n}) \leq_{\mathscr{R}} \phi(g^1_{i_1},\dots,g^\ell_{i'_\ell},\dots,g^n_{i_n})$ iff $i_\ell \leq i'_\ell$. That is, if the performance of one component increases, all other things being equal, then the global performance of the system can only increase or stay the same (note that this assumption is different from assuming that $\mathscr{R}$ is totally ordered).

Example 1. Fig. 1 provides an example of a 2-component system, where G^1 and G^2 are in parallel. G^1 has three possible states: $\mathscr{G}^1 = \{g^1_1 = 0, g^1_2 = 1, g^1_3 = 1.5\}$. The other component is such that $\mathscr{G}^2 = \{g^2_1 = 0, g^2_2 = 1.5, g^2_3 = 2\}$. In this example, we set $\phi(g^1,g^2)$ as the max operator. The set of possible output performances is $\mathscr{R} = \{0,1,1.5,2\}$ (note that knowing this space definition in advance is not necessary).

B. Probabilities and Belief functions

A probability distribution p^j over space $\mathscr{G}^j$ is a function $p : \mathscr{G}^j \to [0,1]$ such that $\sum_{g\in\mathscr{G}^j} p(g) = 1$, where $p(g)$ is the probability of the component G^j being in state g. For convenience, we will denote by $p^j_i := p(g^j_i)$ the probability of component G^j to be in state g^j_i.

The distribution p^j over $\mathscr{G}^j$ describes the uncertainty about the component state. A probability distribution p^j on $\mathscr{G}^j$ induces, on any set $A \subseteq \mathscr{G}^j$, a probability measure P^j such that

$$P^j(A) = \sum_{g \in A} p^j(g) \tag{1}$$

However, identifying the values of p^j requires a lot of information and data. Several authors [41], [25], [42] have argued that in case of severe uncertainty (imprecise data, lack of information, little sample, information given by expert opinions, ...) probabilities are not adequate to model the uncertainty, and that some imprecision should be introduced in the uncertainty model to take account of the lack of information.

Belief functions are one of these uncertainty models. They include as special cases classical sets, probability distributions, and possibility distributions (which are formally equivalent to fuzzy sets). The basic building block of belief functions is a Basic Belief Assignment (BBA), also called a mass distribution $m^j : 2^{|\mathscr{G}^j|} \to [0,1]$ ($|\mathscr{G}^j|$ denoting the cardinality of $\mathscr{G}^j$) that is a positive mapping from the power set $2^{|\mathscr{G}^j|}$ to the unit interval such that

$$\sum_{E \subseteq \mathscr{G}^j} m^j(E) = 1 \text{ , and } m(\emptyset) = 0. \tag{2}$$

A set E such that $m(E) > 0$ is called a *focal element*. Probability distributions correspond to masses where focal elements are singletons (i.e., only elements $\{g_i^j\}$ receive positive mass), and possibility distributions to masses where focal elements are nested (i.e., if A, B are focal elements, then either $A \subset B$ or $B \subset A$). Note that m^j can be seen as a probability distribution, not defined on singletons of $\mathscr{G}^j$ but on subsets of $\mathscr{G}^j$.

From a mass function m^j, it is possible to define two set functions on $2^{|\mathscr{G}^j|}$, namely the plausibility and belief functions that are such that, for any event $A \subseteq \mathscr{G}^j$, we have

$$Bel^j(A) = \sum_{E \subseteq A} m^j(E), \tag{3}$$

$$Pl^j(A) = \sum_{E \cap A \neq \emptyset} m^j(E). \tag{4}$$

$Bel^j(A)$ measures how much event A is implied by the information m^j, while $Pl^j(A)$ measures how much the event A is consistent with information m^j. They are monotone (i.e., if $A \subseteq B$, then $Bel^j(A) \leq Bel^j(B)$, and $Pl^j(A) \leq Pl^j(B)$), dual (for any A, we have $Bel^j(A) = 1 - Pl(\overline{A})$ with $\overline{A}$ the complement of A), and satisfy the inequality $Bel^j(A) \leq Pl^j(A)$ (what is credible is plausible). The duality relation $Bel^j(A) = 1 - Pl(\overline{A})$ tells us that, if A is certain ($Bel^j(A) = 1$), then its contrary ($\overline{A}$) is impossible ($Pl(\overline{A}) = 0$). The difference $Pl^j(A) - Bel^j(A)$ is a measure of our lack of information regarding event A uncertainty, and a complete lack of information (ignorance) corresponds to $[Bel^j(A), Pl^j(A)] = [0,1]$. Probability distributions correspond to the case where $Bel^j(A) = Pl^j(A)$ for all $A \subseteq \mathscr{G}^j$, and in this case the lack of information $Pl^j(A) - Bel^j(A)$ is zero for any event.

The bounds $Bel^j(A), Pl^j(A)$ can be associated to a convex set of probabilities. This association allows us to give them a sensitivity analysis interpretation, where Bel^j, Pl^j bound some ideal, precise probability that is not completely known due to lack of knowledge. The probability set $\mathscr{P}_{m^j}$ associated to m^j is such that

$$\mathscr{P}_{m^j} = \{p^j | \forall A \subseteq \mathscr{G}^j, Bel^j(A) \leq P^j(A) \leq Pl^j(A)\}. \tag{5}$$

That is, $\mathscr{P}_{m^j}$ is the set of probability distributions whose induced measures are bounded by Bel^j, Pl^j.

Example 2. There are many types of mass functions. For instance, for component G^1 of Example 1, we could have

$$m^1(\{g_3^1\}) = 0.2 \quad m^1(\{g_2^1, g_3^1\}) = 0.5 \quad m^1(\{g_1^1, g_2^1, g_3^1\}) = 0.3$$

that could correspond to the assessment *at least half of the time the component is working, with perfect working condition being the most likely state value.* For component G^2 of Example 1, we could have

$$m^2(\{g_1^2\}) = 0.2, \quad m^2(\{g_2^2\}) = 0.3, \quad m^2(\{g_3^2\}) = 0.5$$

that is a classical probability that can result from a sufficiently large sample.

The next section details how mass functions on component states can be obtained in practice. We focus in particular on a method using (continuous) Markov chains expressed as Ordinary Differential Equations (ODEs).

III. Obtaining mass functions

In this section, we first recall some existing results concerning ODE systems with interval-valued parameters. We then apply those results to the estimation of component reliability uncertainty, and explain how mass functions can be obtained from these estimates. In the sequel of the paper, an interval-valued quantity x will be denoted $[x] = [\underline{x}, \overline{x}]$, with $\underline{x}$, and $\overline{x}$ its lower, and upper bounds, respectively.

A. Small reminder on ODE systems with interval-valued parameters

Let $\dot{x} = f(x,y,t)$ be an ODE system with ℓ variables, i.e. $x \in \mathbb{R}^\ell$, and k parameters, i.e., $y \in \mathbb{R}^k$. x_i is the i^{th} variable, while y_j is the j^{th} parameter, and we have

$$\dot{x}_i = f_i(x,y,t), \quad i = 1,\dots,\ell. \tag{6}$$

The (multi-dimensional) solution to such a system is a function $x(t)$ (that can be projected on each variable $x_i(t)$), and finding it requires solving the system given by (6). However, when parameters y are modelled by intervals $[y]_1,\dots,[y]_k$ with $[y]_j = [\underline{y}_j, \overline{y}_j]$, the solution of the system becomes set-valued, and can be approximated by a box specified by a time-dependent interval $[x]_i(t) = [\underline{x}_i, \overline{x}_i](t)$ on each variable x_i. Computing the bounds of $[x]_i(t)$ can be hard; however, Ramdani *et al.* [43] have shown that, if the following condition holds for every $i = 1,\dots,\ell$ such that

$$\forall j, \forall y_j \in [y]_j, \forall i^* \neq i, \forall t \geq t_0, \quad \frac{\partial f_i}{\partial x_{i^*}} \geq 0, \tag{7}$$

then the lower bound $\underline{x}_i(t)$ for $i = 1,\dots,\ell$ can be obtained by substituting the parameter y values in the system as follows. For every $i = 1,\dots,\ell$, replace y_j in $f_i(x,y,t)$ by

- $\underline{y_j}$ if $\partial f_i / \partial y_j \geq 0$ for all $t \geq 0, y_l \in [y]_l, l \neq j$; or
- $\overline{\overline{y_j}}$ if $\partial f_i / \partial y_j \leq 0$ for all $t \geq 0, y_l \in [y]_l, l \neq j$.

The upper bound $\overline{x}_i(t)$ for $i = 1,\dots,\ell$ can be obtained likewise by substituting the parameter y values:

- $y_j = \overline{y_j}$ if $\partial f_i / \partial y_j \geq 0$ for all $t \geq 0, y_l \in [y]_l, l \neq j$; or
- $y_j = \underline{y_j}$ if $\partial f_i / \partial y_j \leq 0$ for all $t \geq 0, y_l \in [y]_l, l \neq j$.

B. The case of reliability component

Consider a component G^j. We denote by

- p_i^j the probability of being in state g_i^j,
- $\lambda_{i,k}^j$ the transition or degrading rate from state g_i^j to g_{i-k}^j, and
- $\mu_{i,k}^j$ the repair rate from g_i^j to g_{i+k}^j.

In our framework, both transition and repair rates are given by intervals (the precise case is retrieved when lower and upper bounds coincide).

Let us now show that conditions (7) apply to reliability components, and that parameters $\lambda_{i,k}^j, \mu_{i,k}^j$ can be replaced by suitable bounds. The behavior of the multi-state component G^j can be described by an ordinary differential equation bearing on probabilities $p_1^j,\dots,p_n^j$ (the variables) whose evolution depends on $\lambda_{i,k}^j$, $i=1,\dots,n$, $k=1,ldots,i-1$, and $\mu_{i,k}^j$, $i=1,\dots,n$, $k=1,\dots,k_j-i$. In this system,

$$\dot{p}_i^j \;=\; \sum_{k=1}^{k_j-i} \lambda_{i+k,k}^j p_{i+k}^j + \Big(-\sum_{k=1}^{i-1} \lambda_{i,k}^j - \sum_{k=1}^{k_j-i} \mu_{i,k}^j\Big) p_i^j + \sum_{k=1}^{i-1} \mu_{i-k,k}^j p_{i-k}^j \tag{8}$$

with the convention $\sum_{k=1}^{0}=0$. Hence, the derivatives of the function $\dot{p}_i^j$ for any p_{i+k}^j, p_{i-k}^j are respectively ${\partial f_i}/{\partial p_{i+k}}^j = \lambda_{i,k}^j$, and ${\partial f_i}/{\partial p_{i-k}^j} = \mu_{i-k,i}^j$. As these values are always positive (being transition rates), our system satisfies (7) (note that the conditions comprised by (7) make no requirement about p_i^j itself).

As all numbers are positive, to get the lower bounds $\underline{p}^j(t)$ of the state probabilities, we just have to solve the system such that

$$\dot{\underline{p}}_i^j(t) = \sum_{k=1}^{k_j-i} \underline{\lambda}_{i+k,k}^j \underline{p}_{i+k}^j(t) + \Big(-\sum_{k=1}^{i-1} \overline{\lambda}_{i,k}^j - \sum_{k=1}^{k_j-i} \overline{\mu}_{i,k}^j\Big) \underline{p}_i^j(t) + \sum_{k=1}^{i-1} \underline{\mu}_{i-k,k}^j \underline{p}_{i-k}^j(t). \tag{9}$$

And to get the upper bounds $\overline{p}^j(t)$ of the state probabilities, we have to solve the system such that

$$\dot{\overline{p}}_i^j(t) = \sum_{k=1}^{k_j-i} \overline{\lambda}_{i+k,k}^j \overline{p}_{i+k}^j(t) + \Big(-\sum_{k=1}^{i-1} \underline{\lambda}_{i,k}^j - \sum_{k=1}^{k_j-i} \underline{\mu}_{i,k}^j\Big) \overline{p}_i^j(t) + \sum_{k=1}^{i-1} \overline{\mu}_{i-k,k}^j \overline{p}_{i-k}^j(t). \tag{10}$$

The interest of this approach is that it is time dependent, continuous, and relies on transition rates, while still providing quick answers (values can be computed by efficient ODE solvers). This approach can be compared to the Imprecise Dirichlet Model (IDM) approach proposed by Li *et al.* [32], which uses a fixed time t and hyper parameter s whose significance is not obvious in reliability problems. To simplify examples in the sequel, we will only consider cases where transitions can only be done between neighbouring states (i.e., $\lambda_{i,k}^j=0$, and $\mu_{i,k}^j=0$ if $k\neq 1$). Other similar approaches include the one followed by Mechri *et al.* [44] in a discrete setting, and the one proposed by Liu *et al.* [36]. This latter approach proposes a component-wise fuzzy Markov model for non-repairable components based on the extension principle, and on a Laplace-Stieltjes transform that is used to solve the differential equations.

C. Example

Consider a component G^j with three states:

1) State 1 g_3^j represents completely successful operation,
2) State 2 g_2^j represents degraded successful operation, and
3) State 3 g_1^j represents total failure.

Let the possible transition rates be:

- $\lambda_{3,1}^j = [10^{-5}h^{-1}, 3.10^{-4}h^{-1}]$,
- $\lambda_{2,1}^j = [4.10^{-5}h^{-1}, 5.10^{-4}h^{-1}]$,
- $\mu_{2,1}^j = [2.10^{-2}h^{-1}, 5.10^{-2}h^{-1}]$,
- $\mu_{1,1}^j = [4.10^{-2}h^{-1}, 8.10^{-2}h^{-1}]$.

The starting state is $p_1^j=1, p_2^j=p_3^j=0$. The component is in the perfect working state at the beginning (state 1). Then, we obtain the upper and lower bounds of component state probabilities by solving the following equations.

t (h)	State 1 (g_3^j)	State 2 (g_2^j)	State 3 (g_1^j)
0	1	0	0
1	[0.9997,1]	[0,0.0003]	0
2	[0.9994,1]	[0,0.0006]	0
3	[0.9991,1]	[0,0.0009]	0
4	[0.9988,1]	[0,0.0012]	0
5	[0.9985,1]	[0,0.0015]	0
10	[0.997,1]	[0.0001,0.003]	0
100	[0.987,0.9999]	[0.0001,0.013]	[0,0.0001]
1000	[0.7437,0.9999]	[0.0001,0.308]	[0,0.0004]

TABLE I
INTERVAL VALUES TO BE IN EACH STATE AT TIME t

$$\left\{\begin{array}{rcl}
\dot{\underline{p}}_3^j(t) & = & -\overline{\lambda}^j_{3,1}\underline{p}_3^j(t)+\underline{\mu}^j_{2,1}\underline{p}_2^j(t) = -3.10^{-4}\underline{p}_3^j(t)+2.10^{-2}\underline{p}_2^j(t)\\
\dot{\underline{p}}_2^j(t) & = & \underline{\lambda}^j_{3,1}\underline{p}_3^j(t)+(-\overline{\lambda}^j_{2,1}-\overline{\mu}^j_{2,1})\underline{p}_2^j(t)+\underline{\mu}^j_{1,1}\underline{p}_1^j(t)\\
 & = & 10^{-5}\underline{p}_3^j(t)-5.10^{-4}p_2^j(t)+4.10^{-2}\underline{p}_1^j(t)\\
\dot{\underline{p}}_1^j(t) & = & \underline{\lambda}^j_{2,1}\underline{p}_2^j(t)-\overline{\mu}^j_{1,1}\underline{p}_1^j(t) = 4.10^{-5}\underline{p}_2^j(t)-8.10^{-2}\underline{p}_1^j(t)
\end{array}\right.$$

$$\left\{\begin{array}{rcl}
\dot{\overline{p}}_3^j(t) & = & -\underline{\lambda}^j_{3,1}\overline{p}_3^j(t)+\overline{\mu}^j_{2,1}\overline{p}_2^j(t) = -10^{-5}\overline{p}_3^j(t)+5.10^{-2}\overline{p}_2^j(t)\\
\dot{\overline{p}}_2^j(t) & = & \overline{\lambda}^j_{3,1}\overline{p}_3^j(t)+(-\underline{\lambda}^j_{2,1}-\underline{\mu}^j_{2,1})\overline{p}_2^j(t)+\overline{\mu}^j_{1,1}\overline{p}_1^j(t)\\
 & = & 3.10^{-4}\overline{p}_3^j(t)-4.10^{-5}p_2^j(t)+0.08\overline{p}_1^j(t)\\
\dot{\overline{p}}_1^j(t) & = & \overline{\lambda}^j_{2,1}\overline{p}_2^j(t)-\underline{\mu}^j_{1,1}\overline{p}_1^j(t) = 5.10^{-4}\overline{p}_2^j(t)-4.10^{-2}\overline{p}_1^j(t).
\end{array}\right.$$

These equations can then be solved for any time value using classical ODE solvers. Table I presents the obtained interval bounds of state probabilities as a function of the time.

D. Getting masses from probability intervals

Assume that we have a component G^j with different states, $\mathscr{G}^j = \{g_1^j, g_2^j, \ldots g_{k_j}^j\}$. Consider the probability intervals of being in each state $[\underline{p}_1^j, \overline{p}_1^j], \ldots, [\underline{p}_{k_j}^j, \overline{p}_{k_j}^j]$, obtained by solving the ODE system for a fixed time. In the case of ordered spaces (recall that $\mathscr{G}^j$ is ordered), Denoeux [45] has proposed an efficient way to obtain a mass function from such probability intervals.

Let $E_{r,s} = \{g_r^j, g_{r+1}^j, \ldots, g_s^j\}$ with $r \leq s$ be the set containing all states from index r to index s. Then, a mass function can be defined from intervals $[\underline{p}_i^j, \overline{p}_i^j]$, $i = 1, \ldots, k_j$ such that, for any set $E_{r,s}$, $1 \leq r \leq s \leq k_j$, we have

$$m(E_{r,s}) = \left\{\begin{array}{cl}
\underline{p}_r^j & \text{if } r = s\\
\underline{P}(E_{r,s}) - \underline{P}(E_{r+1,s}) - \underline{P}(E_{r,s-1}) & \text{if } s = r+1\\
\underline{P}(E_{r,s}) - \underline{P}(E_{r+1,s}) - \underline{P}(E_{r,s-1}) + \underline{P}(E_{r+1,s-1}) & \text{if } s > r+1,
\end{array}\right. \tag{11}$$

where $\underline{P}(E_{r,s}) = \max(\sum_{i \in \langle r,s \rangle} \underline{p}_i^j, 1 - \sum_{i \notin \langle r,s \rangle} \overline{p}_i^j)$ with $\langle r,s \rangle := \{r, r+1, r+2, \ldots, s\}$ denoting a set of indices. Denoeux [45] has shown that, if focal elements are restricted to sets of the type $E_{r,s}$, then the mass given by (11) is the most precise that can be built from bounds $[\underline{p}_i^j, \overline{p}_i^j]$.

However, in situations where computational efficiency is a priority, one may find that mass functions given by (11) still have a too many focal elements. In this case, there is a simpler, faster approximation that will give fewer focal elements, and will correspond to the so-called linear-vacuous mixture. It consists of computing the mass

$$m(E) = \begin{cases} \underline{p}_k^j & \text{if } E = g_k^j \\ 1 - \sum_{k=1}^{k_j} \underline{p}_k^j & \text{if } E = \mathscr{G}^j \\ 0 & \text{else} \end{cases} \tag{12}$$

This model, which limits the focal elements to the singletons g_k^j and the complete space $\mathscr{G}^j$, correspond to a mixture between a precise probability (with weight $1 - m(\mathscr{G}^j)$) and the set of all possible probabilities on $\mathscr{G}^j$ (with weight $m(\mathscr{G}^j)$). It has been widely used in robust Bayesian analysis as a practical tool, and we refer to Walley [25, Sec. 2.9.2] and references therein for further details. Let us consider the example at time $t = 1000h$. We have

$$\begin{aligned} [\underline{p}_3^j, \overline{p}_3^j] &= [0.7437, 0.9999] \\ [\underline{p}_2^j, \overline{p}_2^j] &= [0.0001, 0.308] \\ [\underline{p}_1^j, \overline{p}_1^j] &= [0, 0.0004] \end{aligned}$$

The mass obtained with Denoeuxs approach (11) is

$$\begin{aligned} m^j(\{g_3^j\}) &= 0.7437, \\ m^j(\{g_2^j\}) &= 0.0001, \\ m^j(\{g_1^j\}) &= 0, \\ m^j(\{g_2^j, g_3^j\}) &= 0.9996 - 0.7437 - 0.0001 = 0.2558, \\ m^j(\{g_1^j, g_2^j\}) &= 0.0001 - 0 - 0.0001 = 0, \\ m^j(\{g_1^j, g_2^j, g_3^j\}) &= 0.0004, \end{aligned}$$

and the mass obtained by using (12) is

$$\begin{aligned} m^j(\{g_3^j\}) &= 0.7437, \\ m^j(\{g_2^j\}) &= 0.0001, \\ m^j(\{g_1^j, g_2^j, g_3^j\}) &= 0.2562, \end{aligned}$$

which is indeed more imprecise than the one obtained with Denoeux's approach, but contains fewer focal elements. Actually, the number of focal elements grows linearly with the size $|\mathscr{G}^j|$ with transformation (12), while it grows quadratically with transformation (11). This relationship means that Denoeux's transformation should be preferred to (12) when computational tractability is only a secondary issue, while the linear-vacuous transformation should be used when facing important computational issues (e.g., dealing with limited power or with huge systems). Note that, in the case of the IDM model (Li *et al.* [32]), both transformations yield the same results, and no loss of information is endured.

There are yet other methods to obtain mass functions that we will not detail in this paper. For example, the probability intervals given by the Imprecise Dirichlet model [32] are linear-vacuous mixtures, and they correspond to models given by (12). Also, instead of expressing the Markov model as an ODE system, one can use (discrete) Markov chains with imprecise parameters, a topic studied by many authors [9], [46], [47]. Finally, because belief functions encompass possibility distributions and probabilities, any fuzzy and probabilistic representation on the component states can be, in principle, directly embedded in the framework.

Let us now study how the UGF method can be extended to deal with mass functions and belief functions instead of probability distributions.

IV. Extended Universal generating functions

Generally speaking, the complexity of reliability analysis increases exponentially with the number n of components. Universal generating functions allows one to somehow reduce this exponential growth by taking advantage of the system modularity. It is based on polynomial-like formulas. In this section, we first recall the classical probabilistic approach before extending it to belief functions.

A. Probabilistic case

Given p^j, the information about the state of each component G^j is transformed into the z function $U_j(z)$ [17] such that

$$U_j(z) = \sum_{i=1}^{k_j} p_i^j z^{g_i^j} \tag{13}$$

The z variable is used to provide polynomial-like formulas, the performance information being stored in the exponent. This notation also allows one to apply different functions to the z variable to estimate various features of the system (availability, expected performances). The entire system is then given by the z function

$$\begin{aligned} U(z) &= \Omega_\phi\left(U_1(z),\ldots,U_n(z)\right) \\ &= \sum_{i_1=1}^{k_1}\sum_{i_2=1}^{k_2}\cdots\sum_{i_n=1}^{k_n}\prod_{j=1}^{n} p_{i_j}^j z^{\phi(g_{i_1}^1,\ldots,g_{i_n}^n)}. \end{aligned} \tag{14}$$

Ω_ϕ is a combination operator developed in the second line. Note that the global performances provided by the structure function ϕ are stored in the z exponent. Various questions can then be answered through the function $U(z)$. Two typical ones concern the availability and the average performances of the system.

- The availability consists in computing the likelihood that the system will deliver some minimal performance r_i, that the system performance will lie in the event $A = \{r_i, r_{i+1}, \ldots, r_R\} \subseteq \mathscr{R}$. This calculation is performed by applying the operator δ_A to $U(z)$ such that

$$\begin{aligned} \delta_A(U(z)) &= \delta_A\Big(\sum_{i_1=1}^{k_1}\sum_{i_2=1}^{k_2}\cdots\sum_{i_n=1}^{k_n}\prod_{j=1}^{n} p_{i_j}^j z^{\phi(g_{i_1}^1,\ldots,g_{i_n}^n)}\Big) \\ &= \sum_{i_1=1}^{k_1}\sum_{i_2=1}^{k_2}\cdots\sum_{i_n=1}^{k_n}\prod_{j=1}^{n} p_{i_j}^j \delta_A\big(z^{\phi(g_{i_1}^1,\ldots,g_{i_n}^n)}\big) \end{aligned} \tag{15}$$

  with

$$\delta_A\big(z^{\phi(g_{i_1}^1,\ldots,g_{i_n}^n)}\big) = \begin{cases} 1 & \textbf{if } r_i \leq_{\mathscr{R}} \phi(g_{i_1}^1,\ldots,g_{i_n}^n) \\ 0 & \textbf{else} \end{cases} . \tag{16}$$

- The average performance consists of computing the expected performance of the system. This calculation is performed by applying the operator $\delta_{\mathbb{E}}$ to $U(z)$ such that

$$\begin{aligned} \delta_{\mathbb{E}}(U(z)) &= \delta_{\mathbb{E}}\Big(\sum_{i_1=1}^{k_1}\sum_{i_2=1}^{k_2}\cdots\sum_{i_n=1}^{k_n}\prod_{j=1}^{n} p_{i_j}^j z^{\phi(g_{i_1}^1,\ldots,g_{i_n}^n)}\Big) \\ &= \sum_{i_1=1}^{k_1}\sum_{i_2=1}^{k_2}\cdots\sum_{i_n=1}^{k_n}\prod_{j=1}^{n} p_{i_j}^j \delta_{\mathbb{E}}\big(z^{\phi(g_{i_1}^1,\ldots,g_{i_n}^n)}\big) \end{aligned} \tag{17}$$

  with

$$\delta_{\mathbb{E}}\big(z^{\phi(g_{i_1}^1,\ldots,g_{i_n}^n)}\big) = \phi(g_{i_1}^1,\ldots,g_{i_n}^n). \tag{18}$$

Note that $\delta_{\mathbb{E}}(U(z))$ can only be computed when applying the addition operation over $\phi(g^1_{i_1},\dots,g^n_{i_n})$ makes sense (most of the time $\phi(g^1_{i_1},\dots,g^n_{i_n})$ is real-valued, but it could be complex or vector-valued).

One of the main interest of this formulation is that, when ϕ can be decomposed into sub-systems (e.g., into parallel and series subsystems), so can be the computation of $U(z)$ that can be performed step-by-step (Section V provides an example). However, $U(z)$ can also be evaluated globally, meaning that if ϕ is the (known) structure function of a complex system (i.e. not decomposable into parallel and series subsystems), it can be treated by this method as well.

B. *Belief function case: preliminaries*

In this section, we explore how the UGF can be extended to belief functions. To make this extension, we first need to introduce some concepts related to belief functions and their interpretation as imprecise probabilities, as well as some notions of set analysis.

To make a parallel with interval analysis, we will denote by $[\cdot]$ the subsets of a (discrete) ordered space: for example, $[g]^i \subseteq \mathscr{G}^i$ will denote a subset of $\mathscr{G}^i$.

We will introduce three concepts in this section: (i) the notion of evidential independence (also known as random set independence) between two masses m^i, m^j, (ii) the notion of lower and upper expectations, and (iii) the notion of set-analysis over coherent systems.

First, assume we have two belief functions m^i, m^j defined over $\mathscr{G}^i, \mathscr{G}^j$, and let $[g]^i_1,\dots,[g]^i_{F_i}$, $[g]^j_1,\dots,[g]^j_{F_j}$ be their respective set of focal elements (i.e. subsets having positive mass). Then the joint model m^{ij} over $\mathscr{G}^i \times \mathscr{G}^j$ defined, for all $k \in \{1,\dots,F_i\}$, and $\ell \in \{1,\dots,F_j\}$ as

$$m^{ij}([g]^i_k \times [g]^j_\ell) = m^i([g]^i_k) m^j([g]^j_\ell) \tag{19}$$

corresponds to the model obtained under the evidential independence assumption [48] (also called random set independence [49]). Within the belief function framework, this random set independence assumption is the most natural, straightforward extension of classical stochastic independence between probability distributions. Note also that the obtained joint mass function can again be given a sensitivity analysis interpretation, as the probability set induced by m^{ij} is bigger than the set that would be obtained by taking the stochastic product between all probabilities of $\mathscr{P}_{m^i}$ and $\mathscr{P}_{m^j}$ [50] (a notion usually called strong independence).

Second, we know that m^i induces a corresponding set $\mathscr{P}_{m^i}$ of probabilities. Hence, if we want to compute the expectation of some (real-valued) function $f : G^i \to \mathbb{R}$ (e.g., the performances rate of component i), we can search the lower and upper bounds this expectation would reach by considering all probabilities inside $\mathscr{P}_{m^i}$. In the belief function framework, these bounds $[\underline{\mathbb{E}}(f), \overline{\mathbb{E}}(f)]$ can be easily computed using

$$\underline{\mathbb{E}}(f) = \sum_{j=1}^{F_i} m^i([g]^i_j) \inf_{g^i \in [g]^i_j} f(g^i), \tag{20}$$

$$\overline{\mathbb{E}}(f) = \sum_{j=1}^{F_i} m^i([g]^i_j) \sup_{g^i \in [g]^i_j} f(g^i). \tag{21}$$

Third, as mass functions bear on sets, we need to deal with set-valued parameters in the structure function ϕ. Given sets $[g]^i \subseteq \mathscr{G}^i$, $i = 1,\dots,n$, let us denote by $[g]^{i,+} = \max_{\leq_i}\{g^i | g^i \in [g]^i\}$, and $[g]^{i,-} = \min_{\leq_i}\{g^i | g^i \in G^i\}$ the maximal, and minimal elements of $[g]^i$, respectively. If performance levels become set-valued, so does ϕ, and we have

$$\phi([g]^1,\dots,[g]^n) = \{\phi(g^1,\dots,g^n) | g^i \in [g]^i\}. \tag{22}$$

When the system is coherent, computing the lower and upper bounds of $\phi([g]^1,\dots,[g]^n)$ is very easy, as one can focus on extreme values of sets; that is,

$$\phi([g]^1,\dots,[g]^n)^+ = \phi([g]^{1,+},\dots,[g]^{n,+}), \tag{23}$$

$$\phi([g]^1,\dots,[g]^n)^- = \phi([g]^{1,-},\dots,[g]^{n,-}). \tag{24}$$

Note that the same applies to any sub-function.

We now have all the elements that will allow us to extend UGF to belief functions.

C. Belief Universal Generating Function (BUGF)

For a given component G^i, let $[g]^i_1,\dots,[g]^i_{F_i}$ be the focal elements of the mass m^i describing our uncertainty about G^i state, and denote m^i_j as the mass of $[g]^i_j$. The z transform for this component becomes

$$u_i(z) = \sum_{j=1}^{F_i} m_i^j z^{[g]_i^j} \tag{25}$$

This equation is very similar to the one of the probabilistic case, except that **only** the focal elements are in the summation, and that the exponents of z can be sets (and not individual states). In the probabilistic case (13), exponents of z are individual states, but usually all of them receive non-null probabilities. All the information concerning the uncertainty of component G^i are in this transform: mass functions are in the summation, and focal sets are in the exponent of z.

The whole transform of the n component system is then

$$\begin{aligned} U(z) &= \Omega_\phi\left(u_1(z),\dots,u_n(z)\right) \\ &= \sum_{i_1=1}^{F_1}\sum_{i_2=1}^{F_2}\cdots\sum_{i_n=1}^{F_n}\prod_{j=1}^{n} m^j_{i_j} z^{\phi([g]^1_{i_1},\dots,[g]^n_{i_n})}. \end{aligned} \tag{26}$$

Again, this formulation is pretty close to the one we have in the probabilistic case. We can now define operators allowing us to compute availability and performance expectations. Note that as belief functions are interval-valued uncertainty measures, there will be two evaluations for each task: a lower one, and an upper one.

- Concerning the availability of the event $A = \{r_i,\dots,r_R\}$, we define two operators δ^+_A and δ^-_A that compute plausibility and belief values of A. δ^+_A is defined as

$$\begin{aligned} \delta^+_A(U(z)) &= \delta^+_A\Big(\sum_{i_1=1}^{F_1}\sum_{i_2=1}^{F_2}\cdots\sum_{i_n=1}^{F_n}\prod_{j=1}^{n} m^j_{i_j} z^{\phi([g]^1_{i_1},\dots,[g]^n_{i_n})}\Big) \\ &= \sum_{i_1=1}^{F_1}\sum_{i_2=1}^{F_2}\cdots\sum_{i_n=1}^{F_n}\prod_{j=1}^{n} m^j_{i_j} \delta^+_A\big(z^{\phi([g]^1_{i_1},\dots,[g]^n_{i_n})}\big) \end{aligned} \tag{27}$$

  with

$$\delta^+_A\big(z^{\phi([g]^1_{i_1},\dots,[g]^n_{i_n})}\big) = \begin{cases} 1 & \textbf{if } r_i \leq_{\mathscr{R}} \phi([g]^1,\dots,[g]^n)^+ \\ 0 & \qquad\textbf{else} \end{cases} \tag{28}$$

  and $\delta^-_A(U(z))$ is obtained by replacing the plus signs by minus signs. We have that $\delta^+_A(U(z)) = Pl(A)$, and $\delta^-_A(U(z)) = Bel(A)$. This can be seen by noticing that $r_i \leq_{\mathscr{R}} \phi([g]^1,\dots,[g]^n)^+$ means that at least one element of the interval-valued performance $\phi([g]^1,\dots,[g]^n)$ is above (or equal to) performance r_i, hence $\phi([g]^1,\dots,[g]^n)\cap A \neq \emptyset$, while if $r_i \geq_{\mathscr{R}} \phi([g]^1,\dots,[g]^n)^+$, no elements of $\phi([g]^1,\dots,[g]^n)$ is in A. Similarly, $r_i \leq_{\mathscr{R}} \phi([g]^1,\dots,[g]^n)^-$ is equivalent to $\phi([g]^1,\dots,[g]^n) \subseteq A$.

This result means that the product of masses in $\delta_A^+(U(z))$ ($\delta_A^-(U(z))$) are counted only in the case of non-empty intersection (inclusion).

- Similarly, we define two expectation operators $\delta_{\underline{\mathbb{E}}}$ and $\delta_{\overline{\mathbb{E}}}$, with $\delta_{\underline{\mathbb{E}}}$ as

$$\begin{aligned}\delta_{\underline{\mathbb{E}}}(U(z)) &= \delta_{\underline{\mathbb{E}}}(\sum_{i_1=1}^{F_1}\sum_{i_2=1}^{F_2}\cdots\sum_{i_n=1}^{F_n}\prod_{j=1}^{n} m^j_{i_j} z^{\phi([g]^1_{i_1},\ldots,[g]^n_{i_n})}) \\ &= \sum_{i_1=1}^{F_1}\sum_{i_2=1}^{F_2}\cdots\sum_{i_n=1}^{F_n}\prod_{j=1}^{n} m^j_{i_j}\delta_{\underline{\mathbb{E}}}(z^{\phi([g]^1_{i_1},\ldots,[g]^n_{i_n})}) \end{aligned} \tag{29}$$

with $\delta_{\underline{\mathbb{E}}}(z^{\phi([g]^1_{i_1},\ldots,[g]^n_{i_n})}) = \phi([g]^1,\ldots,[g]^n)^-$. $\delta_{\overline{\mathbb{E}}}$ is defined similarly with $\delta_{\overline{\mathbb{E}}}(z^{\phi([g]^1_{i_1},\ldots,[g]^n_{i_n})}) = \phi([g]^1,\ldots,[g]^n)^+$. As $\phi([g]^1,\ldots,[g]^n)^-$, and $\phi([g]^1,\ldots,[g]^n)^+$ are the minimal, and maximal values of the focal elements $\phi([g]^1,\ldots,[g]^n)$ of the joint mass $\prod_{j=1}^{n} m^j_{i_j}$, these operators do correspond respectively to lower, and upper expectations (under a random set independence assumption).

Example 3. Let us proceed with our small illustrative example. The mass functions for G^1 are

$$m^1(\{g^1_3\}) = 0.2, \quad m^1(\{g^1_2, g^1_3\}) = 0.5, \quad m^1(\{g^1_1, g^1_2, g^1_3\}) = 0.3$$

. The mass functions for G^2 are

$$m^2(\{g^2_1\}) = 0.2, \quad m^2(\{g^2_2\}) = 0.3, \quad m^2(\{g^2_3\}) = 0.5$$

. The component equations are

$$\begin{aligned} U_1(z) &= 0.2z^{\{g^1_3\}} + 0.5z^{\{g^1_3, g^1_2\}} + 0.3z^{\{g^1_3, g^1_2, g^1_1\}} \\ &= 0.2z^{\{1.5\}} + 0.5z^{\{1.5,1\}} + 0.3z^{\{1.5,1,0\}}, \\ U_2(z) &= 0.2z^{\{g^2_3\}} + 0.3z^{\{g^2_2\}} + 0.5z^{\{g^2_3\}} \\ &= 0.2z^{\{0\}} + 0.3z^{\{1.5\}} + 0.5z^{\{2\}}. \end{aligned}$$

Applying first $\phi = \max$ to combine $U_1(z)$ and $U_2(z)$, we get

$$\begin{aligned} U_{12}(z) = {} & 0.04z^{\{1.5\}} + 0.06z^{\{1.5\}} + 0.1z^{\{2\}} + 0.1z^{\{1,1.5\}} + 0.15z^{\{1.5\}} + \\ & 0.25z^{\{2\}} + 0.06z^{\{0,1,1.5\}} + 0.09z^{\{1.5\}} + 0.15z^{\{2\}} \\ = {} & 0.34z^{\{1.5\}} + 0.5z^{\{2\}} + 0.1z^{\{1,1.5\}} + 0.06z^{\{0,1,1.5\}} \end{aligned}$$

We can then answer various questions about the system. For instance, applying δ_A^+ and δ_A^- for $A = \{1.5, 2\}$, that tells us the system availability for a performance of more than 1.5. We get

$$\delta_A^+(U(z)) = Pl(A) = 1, \quad \delta_A^-(U(z)) = Bel(A) = 0.84.$$

We can see that there is a lot of uncertainty in this system. Similarly, applications of $\delta_{\underline{\mathbb{E}}}$ and $\delta_{\overline{\mathbb{E}}}$ give

$$\delta_{\underline{\mathbb{E}}}(U(z)) = 1.51 \quad \delta_{\overline{\mathbb{E}}} = 1.75.$$

Again, the gap between the lower and upper bounds is rather large.

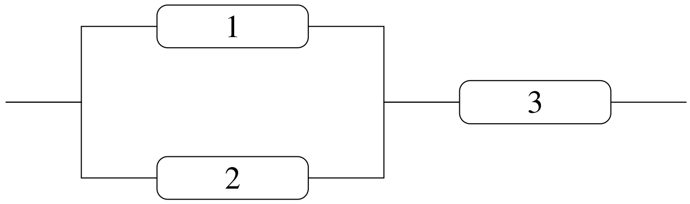

Fig. 2. Flow transmission system

D. About extension complexity

The extended UGF equations are, in the worst case, harder to solve than probabilistic ones, as they may contain a maximal amount of $2^{|\mathscr{G}^{1:n}|}$ elements (a potentially huge number), and require one to perform a set-analysis rather than classical function evaluations. Thus, in the worst case, and if one does make a naive set-analysis (enumerating all possible combinations), this extension is intractable.

However, this intractability is only in worst-case naive computations. Indeed, in the equations, only the focal elements have to be considered, and in practice their numbers are most of the time comparable to the number of states (hence to the number of probabilistic values). For instance, if one restricts himself or herself to possibility distributions and probability distributions (making so-called "hybrid" calculi [51]), then the number of focal elements for any element G^i is equal to the number of state k_i. In some cases, the number of focal elements may even be lower. For example, simple support functions (where the mass is such that $m^i(A) = \alpha$ for some A, and $m^i(\mathscr{G}_i) = 1 - \alpha$) always contains two focal elements (whatever the value k_i).

Concerning the set-analysis, we have seen that, if the system is coherent, interesting questions can be answered by focusing on bounds only. This means that, in practice, making these set computations and computing the bounds only doubles the number of required evaluations (when compared to a probabilistic analysis).

It results from these remarks that the complexity of applying generalized UGF mainly depends on the number of focal elements, which is often not much higher than the number of states of elements $\mathscr{G}^i$, and sometimes even lower (ignorance model, simple support functions, ...). This means that, in most practical applications, using belief functions with UGF will have a computational cost comparable to the probabilistic case.

V. Illustrative example

We illustrate the BUGF extension on a complete example, inspired from Ding and Lisnianski [34]. The results of the belief function approach will be compared to the ones obtained using the probability interval approach to UGF proposed by Li *et al.* [32][1].

In this example, we evaluate the availability of a flow transmission system design presented in Fig V and made of three pipes. The flow is transmitted from left to right, and the performances of the pipe are measured by their transmission capacity (tons of per minute). It is supposed that components 1 and 2 have three states: a state of total failure corresponding to a capacity of 0, a state of full capacity, and a state of partial failure. The component 3 only has two states: a state of total failure, and a state of full capacity. All state performances of the components are precise.

[1]A comparison of Li *et al.* [32] and Ding and Lisnianski [34] can be found in Li *et al.* [32])

Component G_j	1	2	3
p_1^j	[0.096,0.106]	[0.095,0.105]	-
p_2^j	[0.095,0.105]	[0.195,0.205]	[0.032,0.042]
p_3^j	[0.799,0.809]	[0.7,0.71]	[0.958,0.968]
g_1^j	0	0	-
g_2^j	1	1.5	0
g_3^j	1.5	2	4

TABLE II
PARAMETERS OF THE FLOW TRANSMISSION SYSTEM

A. Interval UGF approach (IUGF)

According to Li *et al.* [32], the IUGF of a component G^j with M_j states is defined by

$$U_j(z) = \sum_{i=1}^{M_j} [p_i^j].z^{g_i^j} \tag{30}$$

The IUGF of a system composed of n components is obtained as follows:

$$\begin{aligned} U(z) &= \Omega(U_1(z),\ldots,U_n(z)) \\ &= \Omega(\sum_{\ell_1=1}^{k_1}[p^1_{\ell_1}].z^{g^1_{\ell_1}},\ldots,\sum_{\ell_n=1}^{k_n}[p^n_{\ell_n}].z^{g^n_{\ell_n}}) \\ &= \sum_{\ell_1=1}^{k_1}\sum_{\ell_2=1}^{k_2}\cdots\sum_{\ell_n=1}^{k_n}[p^1_{\ell_1}].[p^2_{\ell_2}]\ldots[p^n_{\ell_n}]z^{\phi(g^1_{\ell_1},g^2_{\ell_2},\ldots,g^n_{\ell_n})} \\ &= \sum_{\ell_1=1}^{k_1}\sum_{\ell_2=1}^{k_2}\cdots\sum_{\ell_n=1}^{k_n}[\prod_{i=1}^{n}\underline{p}^i_{\ell_i},\prod_{i=1}^{n}\overline{p}^i_{\ell_i}]z^{\phi(g^1_{\ell_1},g^2_{\ell_2},\ldots,g^n_{\ell_n})} \\ &= \sum_{i=1}^{R}[p_i].z^{r_i} \end{aligned} \tag{31}$$

where $\phi(g^1_{\ell_1},g^2_{\ell_2},...,g^n_{\ell_n})$ depends on the component states, and where the last line simply summarizes the last big sum into a sum of probabilities over system performance levels. For a demand level w, the system availability $[A]$ is finally computed using

$$[A] = \sum_{i=1}^{R}[p_i|r_i \geq w] \tag{32}$$

The state performance levels and the state probabilities of the flow transmitter system are given in Table II. These probabilities were chosen so that they could have been obtained using the imprecise Dirichlet model considered in Li *et al.* [32] (as it is necessary to do a fair comparison of the two approaches). Using (30), the IUGF equations of components 1, 2, and 3 are

$$U_1(z) = [0.799,0.809]z^{\{1.5\}} + [0.095,0.105]z^{\{1\}} + [0.096,0.106]z^{\{0\}}, \tag{33}$$

$$U_2(z) = [0.7,0.71]z^{\{2\}} + [0.195,0.205]z^{\{1.5\}} + [0.095,0.105]z^{\{0\}}, \tag{34}$$

$$U_3(z) = [0.958,0.968]z^{\{4\}} + [0.032,0.042]z^{\{0\}}. \tag{35}$$

The performance level of the subsystem 12 composed of elements 1 and 2 is equal to the sum of the performance of components 1 and 2. Applying (31) to the system (33), (34) with $\phi_{12}(g^1_{\ell_1}, g^2_{\ell_2}) = g^1_{\ell_1} + g^2_{\ell_2}$, the IUGF of this subsystem is given by

$$\begin{aligned}
U_{12}(z) &= \Omega(U_1(z), U_2(z)) \\
&= \sum_{\ell_1=1}^{3}\sum_{\ell_2=1}^{3}[\underline{p}^1_{\ell_1}\underline{p}^1_{\ell_1}, \overline{p}^1_{\ell_1}\overline{p}^2_{\ell_2}]z^{\phi_{12}(g^1_{\ell_1}, g^2_{\ell_2})} \\
&= [0.55593, 0.5744]z^{\{3.5\}} + [0.2223, 0.24035]z^{\{3\}} + [0.0185, 0.00215]z^{\{2.5\}} \\
&\quad +[0.0672, 0.075]z^{\{2\}} + [0.0946, 0.1066]z^{\{1.5\}} + [0.009, 0.011]z^{\{1\}} \\
&\quad +[0.0091, 0.0111]z^{\{0\}} \qquad (36)
\end{aligned}$$

The performance level of the overall system combining subsystem $1,2$ and component 3 can be computed using the fact that $\phi(g^1_{\ell_1}, g^2_{\ell_2}, g^3_{\ell_3}) = \phi_{123}(g^3_{\ell_3}, \phi_{12}(g^1_{\ell_1}, g^2_{\ell_2}))$ with $\phi_{123}() = \min()$. The IUGF of the overall system U_{123} can then be obtained by applying (31) to (36) and (35). To estimate the availability of the system when $w = 1.5$, we use (32) on the results, obtaining

$$[A] = [0.9183, 0.9665]$$

B. BUGF approach

Similarly to what is done in Example 3, we apply our UGF extension to the system illustrated by Fig. V, and detailed in Table II. Mass functions of probability intervals described in Table II can be obtained using (12). The UGF equations of components 1, 2, and 3 obtained by using (25) are

$$\begin{aligned}
&U_1(z) = 0.799z^{\{1.5\}} + 0.095z^{\{1\}} + 0.096z^{\{0\}} + 0.01z^{\{0,1,1.5\}} \\
&U_2(z) = 0.7z^{\{2\}} + 0.195z^{\{1.5\}} + 0.095z^{\{0\}} + 0.01z^{\{0,1.5,2\}} \\
&U_3(z) = 0.958z^{\{4\}} + 0.032z^{\{0\}} + 0.01z^{\{0,4\}}
\end{aligned}$$

Applying the ϕ_{12} to combine U_1 and U_2, we get

$$\begin{aligned}
U_{12}(z) = {}& 0.5593z^{\{3.5\}} + 0.2223z^{\{3\}} + 0.0185z^{\{2.5\}} + 0.0672z^{\{2\}} + 0.0946z^{\{1.5\}} + 0.009z^{\{1\}} + \\
& 0.0091z^{\{0\}} + 0.00799z^{\{1.5,3,3.5\}} + 0.00095z^{\{1,2.5,3\}} + 0.00096z^{\{0,1.5,2\}} + 0.007z^{\{2,3,3.5\}} + \\
& 0.00195z^{\{1.5,2.5,3\}} + 0.00095z^{\{0,1,1.5\}} + 0.0001z^{\{0,1,1.5,2,2.5,3,3.5\}}
\end{aligned}$$

The overall UGF of the system is similarly computed using the min operator to the resulting equation with the UGF of component 3. Then, applying δ^+_A and δ^-_A for $w = 1.5$, we get $[Bel(A), Pl(A)] = [0.9377, 0.9505]$.

As we can see, the range of interval-valued availability is overestimated compared to the range obtained using belief functions (as we have $[Bel(A), Pl(A)] \subseteq [A]$). Note that this result will always be the case, as the joint uncertainty model induced by considering random set s-independence will be included (and therefore will give more precise inferences) in the joint model induced by considering products of lower and upper bounds over singletons. Indeed, the joint model obtained with random set independence (see (19)) has lower and upper probability bounds that correspond to products of marginal probability bounds [49]. Hence, just considering the product over singletons (what is done in (31)) corresponds to considering a conservative approximation.

VI. CONCLUSION AND PERSPECTIVES

We have proposed an extension of the UGF method handling generic belief functions in MSSs. The proposed solution is exact (as opposed to approximations done with interval arithmetic and extensions of such arithmetic). The method comes down to solving a pair of UGF methods, one to get the lower bound, and one to get the upper bound (compared to just one in the probabilistic case). Provided the number of focal elements is not high, the method complexity is therefore not much higher than its probabilistic counterpart. To our knowledge, this is the first study to do so.

We have also proposed an easy way to build belief functions from failure and repair rates of MSS components. This method is done by computing bounds of an interval-valued ODE system. Few works dealing with uncertainty including imprecision (fuzzy sets, probability intervals, ...) consider such time-dependent models of component states.

We can mention many possible perspectives or extensions to this study, either from the point of view of uncertainty modelling, or of reliability analysis. Concerning uncertainty modelling, extensions are several.

- Replacing precise performance rates of components by set-valued or fuzzy-valued ones [52] would provide more robust solutions.
- Studying the implication of recent works [53] indicating that considering the product of masses (the central notion used to extend UGF to BUGF) also makes sense with more general uncertainty models than belief functions.
- Considering fuzziness or imprecision in the definition of mass functions can be dealt with by using proper extensions [40], [54], [55], [56].

From a reliability analysis point of view, perspectives include

- using this efficient method to include epistemic uncertainty in reliability allocation problems for MSS [57], and
- deriving (possibly imprecise) importance measures whose goal would be to identify which component(s) whose uncertainty impacts the most the epistemic uncertainty of the entire system.

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Sebastien Destercke is a CNRS researcher in the Heuristique et Diagnostic des Systèmes Complexes research unit in France since 2011. He graduated from the Faculté Polytechnique de Mons as an Engineer with a specialization in computer science and applied mathematics. His thesis was done under the supervision of Didier Dubois and Eric Chojnacki in France. His current research interests concern uncertainty modeling, and treatment with the help of imprecise probabilistic approaches (Lower previsions theory, Dempster-Shafer theory, Possibility theory).

Mohamed Sallak is an Associate Professor with the Computer Engineering Department in Compiègne University of Technology in France since 2009. He received a Ph.D. degree from Polytechnic Institute of Lorraine in 2007. His current research interests concern reliability assessment of complex systems under uncertainties using several uncertainty theories (belief function theory, imprecise probability theory, and fuzzy set theory), reliability allocation methods, and components importance measures.