Construction of belief functions from statistical data about reliability under epistemic uncertainty

F. Aguirre, M. Sallak, and W. Schön

Abstract

It is recognized that probability theory is well adapted to handle aleatory uncertainties resulting from the variability of failure phenomena. Recently, several uncertainty theories, such as belief function theory, were introduced, in reliability assessments, to handle epistemic uncertainties resulting from lack of knowledge or insufficient data. In this paper, we propose some methods to construct belief functions of reliability parameters of components from statistical data about reliability. The proposed methods consider parametric estimation of reliability parameters.

Index Terms

Reliability data, belief function theory, epistemic uncertainty, parametric estimation.

ACRONYMS

BBA	Basic Belief Assignment
BBD	Basic Belief Density
MP	Minimal Path
D-S	Dempster-Shafer
PDF	Probability Density Function
ML	Maximum Likelihood

NOTATION

Bel	Belief function
Pl	Plausibility function
Q	Commonality function
Ω	Frame of discernment
m^{Ω}	BBA on the frame of discernment Ω
F_i	Failure state of component i
W_i	Working state of component i
R_S	System's reliability
$\mathbb{R}$	The set of real numbers
$\mathscr{R}$	The set of extended real numbers
$\mathscr{P}_{\mathrm{i}}$	Minimal path i
$\underline{\mathrm{E}}[\lambda]$	Lower expectation of failure rate λ
$\overline{\mathrm{E}}[\lambda]$	Upper expectation of failure rate λ
$\Delta\mathrm{E}[\lambda]$	Imprecision of the failure rate estimate λ

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I. INTRODUCTION

When testing some components, the number of tests must be limited for many reasons (e.g. the components are very expensive, the tests are time-consuming and expensive, some types of tests are simply impractical, etc.). As a consequence, the number of tests may be insufficient to generate enough statistical data about reliability. Several authors have proposed to make distinction between two types of uncertainty: epistemic uncertainty related to insufficient reliability data (or knowledge), which is also called reducible uncertainty, and aleatory uncertainty related to the stochastic behavior of lifetime of components [1], [2], [3], [4], [5]. The frequentist probabilistic approach introduced by Venn [6] was widely used in risk and reliability assessments to model aleatory uncertainties. It defined the probability of an event as the limit of its relative frequency in a large number of trials. However, in the case of components that fail only rarely (nuclear systems, chemical processes, railway systems, etc.) or components that have not been operated long enough to generate a sufficient quantity of reliability data, the frequentist approach becomes not suitable [7], [8], [9], [10], [11], [12]. For this reason, several methods were proposed to manage epistemic uncertainties such as:

- Bayesian approach (based on the use of subjective probabilities [13]);
- Interval approach [14];
- Belief function theory [15], [16];
- Possibility theory [17].

The belief function theory, also called Dempster-Shafer theory, is a generalization of the Bayesian theory of probabilities. It is used to model incomplete and imprecise information. Whereas the Bayesian theory requires to assign probabilities for each question of interest, according to Shafer [18], '*belief functions allow us to base degrees of belief for one question from probabilities for another*'. For example, consider a sensor S, which is used to indicate the state of a binary component c (working mode or failure mode). The degree of belief that the sensor is infallible is 0.8, and the degree of belief that the sensor is not infallible is 0.2. Consider the fact that the sensor S indicates that the component c is working perfectly. This information, which must be true if the sensor S is infallible, is not necessarily false if S is not infallible. We have a 0.8 degree of belief that the component c is working perfectly, but only a 0 degree of belief (not a 0.2 degree of belief) that the component c is down. The zero value indicates that there is no reason to believe that the component c is down. Thus, the belief interval that c is functioning is $[0.8, 1]$, and the belief interval that c is in a failure mode is $[0, 0.2]$. The length of the belief interval 0.2 represents the epistemic uncertainty (the imprecision) about the state of c. The values 0.8 and 1 represent the bounds of the correct value to be in the working state (aleatory uncertainty). In summary, we obtain degrees of belief for one question (the state of component c) from probabilities for another question (the reliability of sensor S). Furthermore, probabilities are additive and can be assigned only to elementary events (singletons). In contrast, belief functions are super-additive and can be assigned to sets of elementary events.

During the last years, several works have appeared in the reliability and risk assessment using belief function theory [19], [20], [21], [22]. However, there is no work concerning the construction of belief functions of reliability parameters from failure data. Our aim here is to apply statistical inference methods, introduced in the belief functions framework, to develop parametric and non-parametric methods to estimate reliability parameters of components based on a small number of failure data. In Section 2, we introduce some theoretical background of belief function theory. Sections 3 and 4 are devoted to parametric methods to estimate reliability parameters in the discrete and in the continuous cases, as well as some examples of application. In Section 5, we present our method to perform reliability evaluation of systems using belief function theory. Section 6 contains a numerical example of the reliability evaluation of a 12 components parallel-series system. For this example, components have exponential failure distributions with unknown failure rates. Concluding remarks and future works are given in Section 6.

II. Belief function theory

Belief function theory was described by Dempster [15] with the study of upper and lower probabilities and further developed by Shafer[16]. Belief function theory represents a framework for representing and manipulating aleatory and epistemic uncertainties. It can be interpreted as a generalization of probability theory where probabilities can be assigned to subsets instead of just singletons for the case of probability theory. The belief function theory was first developed for discrete frames of discernment. Then, Smets expanded belief functions to continuous frames of discernment where basic belief masses are generalized to basic belief densities [23]. In this section, basic notions, operations, and terminology of discrete and continuous belief function theory are explained. For a more detailed exposition see [15], [24], [23], [25].

A. Discrete belief functions

A Basic Belief Assignment (BBA) on the frame of discernment Ω is a function, $m^{\Omega} : 2^{\Omega} \rightarrow [0,1]$, which maps belief masses not only on events but also on subsets of events, such that:

$$\sum_{A \in 2^{\Omega}} m^{\Omega}(A) = 1 \tag{1}$$

The mass $m^{\Omega}(A)$ is interpreted as the degree of belief of knowing only A as containing the actual answer to the problem under concern. If A is a singleton, then $m^{\Omega}(A) = p(A)$. Every subset A to which a belief different from zero is allocated ($m^{\Omega}(A) > 0$) is called a focal element. A clear distinction has to be made between probabilities and BBA: probability distribution functions are defined on Ω and BBA on the power set 2^{Ω}. The number of possible hypotheses is then $2^{card(\Omega)}$ in belief function theory, while in probability theory it is $card(\Omega)$. Moreover, belief function theory ignores the sub-additivity hypothesis required for probability functions in probability theory. As an example, let us consider $\Omega = \{x_1, x_2, x_3\}$ as our frame of discernment. Then, x_1, x_2 and x_3 are elementary propositions and mutually exclusive to each other. Through the use of belief function theory, a BBA allocates masses to the subsets belonging to the power set $2^{\Omega} = \{\{\emptyset\}, \{x_1\}, \{x_2\}, \{x_3\}, \{x_1,x_2\}, \{x_1,x_3\},$
$\{x_2,x_3\}, \Omega\}$ in such a way that (1) is respected.

A BBA having a singleton $\{x\}$ ($x \in \Omega$) as a unique focal set represents full knowledge. A BBA having only singletons as focal sets is equivalent to a probability distribution. A BBA having Ω as a unique focal set represents complete ignorance and is called vacuous. In addition, BBAs have further properties, which distinguish them from probability functions:

- It is not required that $m^{\Omega}(\Omega) = 1$.
- It is not required that $m^{\Omega}(A) \leq m^{\Omega}(B)$ when $A \subseteq B$.
- $m^{\Omega}(A) + m^{\Omega}(\bar{A}) \leq 1$.

The belief Bel, commonality Q, and plausibility Pl functions for a subset A are, respectively, defined by

$$Bel(A) = \sum_{B \subseteq A} m^{\Omega}(B) \tag{2}$$

$$Q(A) = \sum_{A \subseteq B} m^{\Omega}(B) \tag{3}$$

$$Pl(A) = \sum_{B \cap A \neq \emptyset} m^{\Omega}(B) \quad \forall A \subseteq \Omega, \forall B \subseteq \Omega \tag{4}$$

$Bel(A)$ is interpreted as the total mass of information implying the occurrence of A, whereas $Pl(A)$ is interpreted as the total mass of information consistent with A. The value $Pl(A) - Bel(A)$ represents the epistemic uncertainty about A. Note that $Pl(A)$ may also be defined as the extent to which we fail to disbelieve the hypothesis of A. Thus

$$Pl(\overline{A}) = 1 - Bel(A) \tag{5}$$

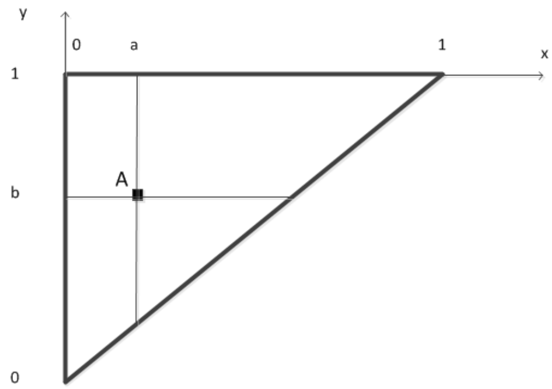

Fig. 1: Graphical representation of the interval $[a,b] \subseteq [0,1]$

The Commonality function Q is a measure that is generally used to simplify the operations used in belief function theory (combination, marginalization, etc.).

B. Continuous belief functions

1) Interval and graphical representation: First, suppose $r < s$ and let us define the sets

$$\mathscr{I}_{[r,s]} = \{[a,b],(a,b],[a,b),(a,b) : a,b \in [r,s]\}$$
$$\mathscr{I} = \{[a,b],(a,b],[a,b),(a,b) : a,b \in \mathscr{R}\}$$

as the set of closed, half open and open intervals in $[r,s]$ or $\mathscr{R}$, respectively, where $\mathscr{R} = \mathbb{R} \cup \{-\infty, \infty\}$ is the set of extended real numbers obtained from the set of real numbers by adding two elements:$-\infty$ and ∞.

In the graphical representation, the intervals in $[r,s]$ or $\mathscr{R}$ are represented by points in a space (o,ox,oy) and we define the sets $\mathscr{T}_{[r,s]}$ and $\mathscr{T}$ as follow

$$\mathscr{T}_{[r,s]} = \{(a,b) : a,b \in [r,s], a \leq b\}$$
$$\mathscr{T} = \{(a,b) : a,b \in \mathscr{R}, a \leq b\}$$

Fig. 1 illustrates graphically this representation. The diagonal represents the domain [0,1]. Each interval in [0,1] can be represented by a point in the triangle. For example, the point A represents the interval $[a,b] \subseteq [0,1]$ where a is in the *ox* axis and b is in the *oy* axis. This representation can be extended to the intervals of $\mathscr{I}_{[r,s]}$ and $\mathscr{I}$. For more details about belief functions' operations (combination, marginalization, etc.), see the Appendix.

2) Belief functions on $\mathscr{R}$: Let $\mathscr{A}$ be a finite collection of intervals in $[r,s] : \mathscr{A} = \{\mathscr{A}_i : \mathscr{A}_i \in \mathscr{I}_{[r,s]}, i = 1,\ldots,n\}$. Consider a BBA $m^{\mathscr{A}} : \mathscr{A} \to [0,1]$ which satisfies $\sum_{i=1,\ldots,n} m^{\mathscr{A}}(\mathscr{A}_i) = 1$. These $\mathscr{A}_i$s which verify $m^{\mathscr{A}}(\mathscr{A}_i) > 0$ are the focal elements of the BBA $m^{\mathscr{A}}$. To each point (a,b) in $\mathscr{T}_{[r,s]}$ that corresponds to a focal element $[a,b] \in \mathscr{I}_{[r,s]}$ of $m^{\mathscr{A}}$, a mass $m^{\mathscr{A}}([a,b])$ is allocated. Then, we define a probability density function (pdf) $f^{\mathscr{T}_{[r,s]}}$ on $\mathscr{T}_{[r,s]}$ by

$$f^{\mathscr{T}_{[r,s]}}(x,y) = \sum_{i=1,\ldots,n} m^{\mathscr{A}}(A_i = [a_i,b_i])\delta(x-a_i)\delta(y-b_i),$$

where the function δ is the Dirac function.

Example 1. Let us consider the three focal elements $[a_i,b_i]$ $(i \in \{1,2,3\})$ of a component failure rate λ given by

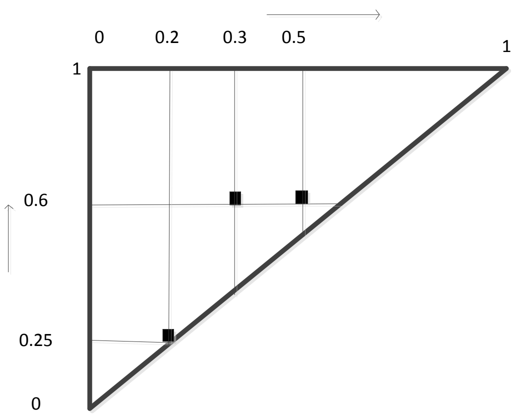

Fig. 2: Graphical representation of the focal elements corresponding to example 1

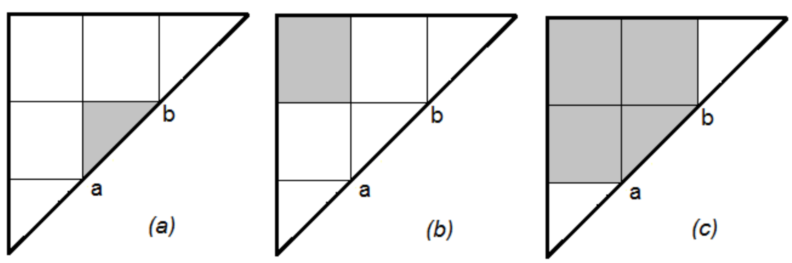

Fig. 3: Graphical representation of (a) belief; (b) commonality; (c) plausibility.

- $m^{\mathscr{A}}([0.3,0.6]) = 0.7$
- $m^{\mathscr{A}}([0.5,0.6]) = 0.1$
- $m^{\mathscr{A}}([0.2,0.25]) = 0.2$

where $\mathscr{A} = \{[0.3,0.6],[0.5,0.6],[0.2,0.25]\}$.

For example, the value $m([0.3,0.6]) = 0.7$, means that the probability of knowing only that λ belongs to the interval $[0.3,0.6]$ is 0.7. The focal elements are represented by a small black squares in Fig. 2. Note that, in the following sections, we will explain how to obtain the intervals of λ from statistical data about reliability. In this section, we aim only to obtain the functions $Bel^{\mathscr{A}}, Pl^{\mathscr{A}}, Q^{\mathscr{A}}$ such that $\lambda \in [0.22,0.62]$.

The function $Bel^{\mathscr{A}}(X)$ represents the sum of the masses allocated to the subsets C such that $C \subseteq X = [a,b]$. The triangle shown in Fig. 3(a) is created by drawing an horizontal and a vertical line from point (a,b) toward the diagonal line. To obtain $Bel^{\mathscr{A}}(X)$, we have to sum the masses of the focal elements located on the triangle

$$\begin{aligned} Bel^{\mathscr{A}}([0.22,0.62]) &= \sum_{C \subseteq [0.22,0.62]} m(C) \\ &= m^{\mathscr{A}}([0.3,0.6]) + m^{\mathscr{A}}([0.5,0.6]) \\ &= 0.8 \end{aligned}$$

The function $Q^{\mathscr{A}}(X)$ for $X = [a,b]$ is represented by the sum of the masses allocated to the intervals $\mathscr{A}_i = [a_i,b_i]$ where $[a,b] \subseteq [a_i,b_i]$. The triangle shown in figure 3(b) is created by drawing an horizontal from point (a,b) toward the left border of $\mathscr{T}_{[0,1]}$, and a vertical line from point (a,b) up to the upper border of $\mathscr{T}_{[0,1]}$, defining thus a rectangle shown is figure 3(b). To obtain $Q^{\mathscr{A}}(X)$, one adds the masses of the focal elements located on the rectangle

$$\begin{aligned} Q^{\mathscr{A}}([.22,.62]) &= \sum_{[.22,.62]\subseteq C} m(C) \\ &= 0 \end{aligned}$$

The function $Pl^{\mathscr{A}}(X)$ for $X=[a,b]$, is defined as the sum of the masses allocated to the intervals $\mathscr{A}_i=[a_i,b_i]$ where $[a,b]\cap[a_i,b_i]\neq\emptyset$. We use the triangle drawn for $Bel^{\mathscr{A}}(X)$, we then draw an horizontal line from its lower corner up to the left border of $\mathscr{T}_{[0,1]}$, and a vertical line from its upper corner up to the upper border of $\mathscr{T}_{[0,1]}$, delimiting so an area shown in figure 3(c). To obtain $Pl^{\mathscr{A}}(X)$, one adds the masses of the focal elements located on this area

$$\begin{aligned} Pl^{\mathscr{A}}([.22,.62]) &= \sum_{C\cap[0.22,0.62]\neq\emptyset} m(C) \\ &= m^{\mathscr{A}}([0.2,0.25])+m^{\mathscr{A}}([0.3,0.6])+m^{\mathscr{A}}([0.5,0.6]) \\ &= 1 \end{aligned}$$

The values $Bel^{\mathscr{A}}([0.22,0.62])=0.8$ and $Pl^{\mathscr{A}}([0.22,0.62])=1$ represent the bounds of the correct value that $\lambda\in[0.22,0.62]$ (aleatory uncertainty). The length of the interval [0.8,1] represents the epistemic uncertainty (imprecision) about the fact that $\lambda\in[0.22,0.62]$.

3) Basic Belief Densities: In this subsection, we generalize the classical BBA into a 'Basic Belief Density' (BBD) on $\mathscr{I}$. This function $m^{\mathscr{I}}$ plays the role of the BBA except now it is a density.

The elements A of $\mathscr{I}$ such that $m^{\mathscr{I}}(A)>0$ are called the focal elements of $\mathscr{I}$. We consider that $m^{\mathscr{T}}$ is a normalized BBD, i.e. $m^{\mathscr{I}}(\emptyset)=0$. Then, we define the Probability density function as follows

Definition 1. The function $f^{\mathscr{T}}$ defined on $\mathscr{R}^2$ such that for all $a,b\in\mathscr{R}$:

$$\begin{aligned} f^{\mathscr{T}}(a,b) &= m^{\mathscr{I}}([a,b]), \quad \text{if} \quad a\leq b \\ &= 0 \qquad\qquad\quad \text{if} \quad a>b, \end{aligned}$$

is called a pdf.

The case where the domain is a finite interval $[\alpha,\beta]$ is considered as a particular case of pdf. It is obtained by considering $f^{\mathscr{T}}(x,y)=0$ when $(x,y)\notin\mathscr{T}_{[\alpha,\beta]}$.

The Belief, Plausibility and commonality functions are, respectively, defined by

$$Bel^{\mathscr{T}}(A)=\int\int_{[x,y]\subseteq A} f^{\mathscr{T}}(x,y)dxdy$$

$$Pl^{\mathscr{T}}(A)=\int\int_{[x,y]\cap A\neq\emptyset} f^{\mathscr{T}}(x,y)dxdy$$

$$Q^{\mathscr{T}}(A)=\int\int_{A\subseteq[x,y]} f^{\mathscr{T}}(x,y)dxdy$$

Particularly

$$Bel^{\mathscr{T}}([a,b])=\int_{x=a}^{x=b}\int_{y=x}^{y=b} f^{\mathscr{T}}(x,y)dxdy \tag{6}$$

$$Pl^{\mathscr{T}}([a,b])=\int_{x=-\infty}^{x=a}\int_{y=a}^{y=+\infty} f^{\mathscr{T}}(x,y)dxdy+\int_{x=a}^{x=b}\int_{y=x}^{y=+\infty} f^{\mathscr{T}}(x,y)dxdy \tag{7}$$

$$Q^{\mathscr{T}}([a,b]) = \int_{x=-\infty}^{x=a} \int_{y=b}^{y=+\infty} f^{\mathscr{T}}(x,y)dxdy \tag{8}$$

The upper and lower expectation over a parameter λ with a pdf $f^{\mathscr{T}}$ and a cost functions C from $\mathscr{T}$ to the real numbers, are, respectively, defined by

$$\underline{E}^{\mathscr{T}}[C(\lambda)] = \int_{x=-\infty}^{x=+\infty} \int_{y=x}^{y=+\infty} f^{\mathscr{T}}(x,y) inf_{x \leq \lambda \leq y} C(\lambda) dxdy \tag{9}$$

$$\overline{E}^{\mathscr{T}}[C(\lambda)] = \int_{x=-\infty}^{x=+\infty} \int_{y=x}^{y=+\infty} f^{\mathscr{T}}(x,y) sup_{x \leq \lambda \leq y} C(\lambda) dxdy \tag{10}$$

The upper and lower expectations are two measures which express the imprecision (epistemic uncertainty) of the parameter estimate λ (in the case $C(\lambda) = \lambda$).

III. Parametric construction of belief functions of reliability data in the discrete case

In this section, we present a method to build BBAs, belief and plausibility functions about failure rates of components from reliability data. This method is based on the inference methods presented by Dempster when he introduced upper and lower probabilities [26]. We propose to use two theorems, to obtain general expressions of belief and plausibility functions, instead of the proposed general measures used by Dempster [26]. Moreover, we propose some examples of construction of failure rates in reliability studies.

A. Dempster's model

The construction of the sampling model of Dempster is based on the definition of two spaces. The population being sampled which is explicitly represented by a space Ω, and the space of possible observations $\mathscr{X}$. The individuals are randomly sampled from Ω according to a probability measure μ. Thus, the sample model of Dempster can be regarded as a measure space $(\Omega, \mathscr{A}, \mu)$ where $\mathscr{A}$ is the σ-field corresponding to the measure μ over Ω. For example, if each individual ω from a finite population is equally likely to be observed, the sampling probability is then represented by a uniform discrete distribution with the values $\mu = \frac{1}{N}$ assigned to each individual $\omega \in \Omega$. Then, a multivalued mapping M from Ω to $\mathscr{X}$ is introduced. A probability distribution over $2^{\mathscr{X}}$ is then induced because we assigned to each individual $w \in \Omega$ a subset of the observation space $\mathscr{X}$. If M is a single-valued mapping, we obtain a unique probability measure over $\mathscr{X}$, whereas if M is multi-valued mapping, the measure μ is induced over $2^{\mathscr{X}}$. Then, belief and plausibility functions are defined on $\mathscr{X}$ such that, for all $A \subseteq \mathscr{X}$:

$$Bel(A) = \mu(M_*(A)) \tag{11}$$

$$Pl(A) = \mu(M^*(A)) \tag{12}$$

Where

$M_*(A) = \{x : x \in \Omega, M(x) \subseteq A, M(x) \neq \emptyset\}$

and

$M^*(A) = \{x : x \in \Omega, M(x) \cup A \neq \emptyset\}$

We can also define a BBA m because there is a one to one correspondence between Bel, Pl and m.

B. Belief and plausibility functions about failure rates

Let $X_1, X_2, ..., X_n$ be an i.i.d sample with parent variable $X \in \mathscr{X} = \{0,1\}$ ($\mathscr{X}$ is the space of possible observations) following a Bernoulli process with unknown failure rate $\lambda \in \Lambda = [0,1]$. The variable X_i is equal to one if the component is down at the ith observation. The problem is to infer λ from $X_1, X_2, ..., X_n$. Particularly, We aim to build BBA, belief and plausibility functions of λ.

Thus, we introduce a series of pivotal variables $a_1, a_2, ..., a_n$. The pivotal variable a_i for λ is a random variable $a_i = g(X_i, \lambda)$ that is a function of the observation variable X_i and the parameter λ, but whose distribution does not depend on λ. We also assume that each a_i follows a uniform distribution over [0,1] and is associated with an observation X_i such that

$$X_i = \begin{cases} 1 & \text{if } a_i \leq \lambda \\ 0 & \text{if } a_i > \lambda \end{cases} \tag{13}$$

The uniform distribution of a_i can be regarded as modeling random sampling from an infinite population assimilated to the interval $\Omega = [0,1]$. The equation (13) represents a relationship among observables X_i, pivotal variables a_i, and the unknown failure rate λ. Marginalizing out the pivotal variables reduces the problem to the standard likelihood for a Bernoulli process.

We then invert the equation (13) into the multivalued mapping from Ω to $\mathscr{X} \times \Lambda$. This mapping can be then represented by a BBA m on the joint space $\Omega \times \mathscr{X} \times \Lambda$. Thus, having observed a realization x_i of each X_i, a BBA over λ can be obtained after a marginalization on Λ.

Now, we will demonstrate how to obtain the formulas of the BBA m over λ.

Let X be the total number of failures observed in the n observations, i.e. $\sum_{i=1}^{n} X_i = X$. We consider the case when we have observed k failures. Let $\underline{a} = a_{(k)}$ and $\overline{a} = a_{(k+1)}$ be respectively the k and $k+1$ order statistics of the pivotal variables a_i. In this case, we have exactly k pivotal variables less than λ, i.e.

$$\underline{a} \leq \lambda < \overline{a} \tag{14}$$

Theorem 1. *Order statistics of the uniform distribution [27]*
Let $X_1, ..., X_n$ be an i.i.d. sample from $U(0,1)$. Let $X_{(k)}$ be the k^{th} order statistic from this sample. Then, the probability distribution of $X_{(k)}$ is a Beta distribution with parameters k and $n-k+1$. The expected value of $X_{(k)}$ is

$$E(X_{(k)}) = \frac{k}{n+1} \tag{15}$$

Theorem 2. *Joint distribution of the order statistics of the uniform distribution [27]*
The joint probability density function of the two order statistics X_i and Y_j that follow a Beta distribution is given by

$$f_{X_i,Y_j}(x,y) = n! \frac{x^{i-1}(y-x)^{j-i-1}(1-y)^{n-j}}{(i-1)!(j-i-1)!(n-j)!} \tag{16}$$

Using Theorem 1, the variable $\underline{a}$ follows a Beta distribution with parameters k and $n-k+1$, and the variable $\overline{a}$ follows a Beta distribution with parameters $k+1$ and $n-k$.

Then, using Theorem 2, the joint probability density function of the two order statistics $\underline{a} = a_{(k)}$ and

$\overline{a} = a_{(k+1)}$ is given by

$$\begin{aligned} f^{\mathscr{T}}(\underline{a},\overline{a}) &= f^{\mathscr{T}}(a_{(k)},a_{(k+1)}) \\ &= n!\frac{\underline{a}^{k-1}(1-\overline{a})^{n-k-1}}{(k-1)!(n-k-1)!} \end{aligned}$$

Where $0 < k < N$ and $\mathscr{T}$ denotes the set of closed intervals in $\mathscr{R}$.

We consider the case when we have observed k failures, i.e, the total number of failures observed in the n demands is $X = k$. Thus, for $0 < X < n$, we have

$$f^{\mathscr{T}}(\underline{a},\overline{a}) = \frac{n!}{(X-1)!(n-X-1)!}\underline{a}^{X-1}(1-\overline{a})^{n-X-1}$$

The case when $X = 0$ (No failures are observed) and $X = n$ (All the X_i represent failure events) lead easily to the following formulas

$$f^{\mathscr{T}}(0,\overline{a}) = n(1-\overline{a})^{n-1}$$

$$f^{\mathscr{T}}(\underline{a},1) = n\underline{a}^{n-1}$$

Then using Definition 1, we can define a BBA over λ such that

$$\begin{aligned} &m^{\mathscr{T}}([\underline{a},\overline{a}]) = f^{\mathscr{T}}(\underline{a},\overline{a}) = \frac{n!}{(X-1)!(n-X-1)!}\underline{a}^{X-1}(1-\overline{a})^{n-X-1} \quad 0 < X < n; \\ &m^{\mathscr{T}}([0,\overline{a}]) = n(1-\overline{a})^{n-1} \quad X = 0; \\ &m^{\mathscr{T}}([\underline{a},1]) = n\underline{a}^{n-1} \quad X = n \end{aligned} \tag{17}$$

Using (6) and (7), the belief and plausibility function over λ are, respectively, obtained as follows

$$Bel^{\mathscr{T}}([\alpha,\beta]) = \int_{\underline{a}=\alpha}^{\underline{a}=\beta}\int_{\overline{a}=\underline{a}}^{\overline{a}=\beta} f^{\mathscr{T}}(\underline{a},\overline{a})d\underline{a}d\overline{a}$$

$$Pl^{\mathscr{T}}([\alpha,\beta]) = \int_{\underline{a}=-\infty}^{\underline{a}=\alpha}\int_{\overline{a}=\alpha}^{\overline{a}=+\infty} f^{\mathscr{T}}(\underline{a},\overline{a})d\underline{a}d\overline{a} + \int_{\underline{a}=\alpha}^{\underline{a}=\beta}\int_{\overline{a}=\underline{a}}^{\overline{a}=+\infty} f^{\mathscr{T}}(\underline{a},\overline{a})d\underline{a}d\overline{a}$$

Then, the belief function is given by

$$Bel^{\mathscr{T}}([\alpha,\beta]) = \begin{cases} X\binom{n}{X}\int_{\underline{a}=\alpha}^{\underline{a}=\beta}\underline{a}^{X-1}(1-\underline{a})^{n-X}d\underline{a} - \binom{n}{X}(\beta^X-\alpha^X)(1-\beta)^{n-X} & 0 < X < n \\ 0 & X = 0, \quad \alpha > 0 \\ 1-(1-\beta)^n & X = 0, \quad \alpha = 0 \\ 0 & X = n, \quad \beta < 1 \\ 1-\alpha^n & X = n, \quad \beta = 1 \end{cases} \tag{18}$$

The plausibility function is given by

$$Pl^{\mathscr{T}}([\alpha,\beta]) = \begin{cases} X\binom{n}{X}\int_{\underline{a}=\alpha}^{\underline{a}=\beta}\underline{a}^{X-1}(1-\underline{a})^{n-X}d\underline{a} + \binom{n}{X}\alpha^X(1-\alpha)^{n-X} & 0 < X < n \\ (1-\alpha)^n & X = 0 \\ \beta^n & X = n \end{cases} \tag{19}$$

Remark 1. Let us consider the special case $\alpha = \beta$. Applying the formulas (18) and (19), we obtain

$$Bel^{\mathscr{T}}(\lambda = \alpha) = 0$$

failure rate λ	Belief interval
0.1	$[0, 0.0196]$
$[0, 0.2]$	$[0.5173, 0.6623]$
$[0.1, 0.5]$	$[0.8002, 0.9854]$
$[0.1, 0.2] \cup [0.4, 0.5]$	$[0.5173, 0.6623]$

TABLE I: Interval belief of the example 2

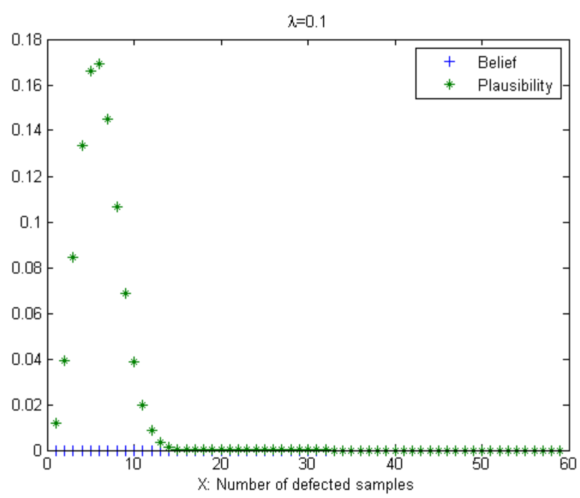

Fig. 4: Belief and Plausibility functions ($\lambda = 0.1$)

$$Pl^{\mathscr{T}}(\lambda = \alpha) = \binom{n}{X} \alpha^X (1-\alpha)^{n-X}$$

That means that, the lower bound of the interval obtained by belief function theory is zero, and the upper bound corresponds to the probability mass function obtained in the probabilistic approach for the binomial distribution.

Example 2. Assume that a producer has tested 60 samples of a product c and 11 were found to be defective. What are the plausibility and belief measures over the failure rate λ such that

- $\lambda = 0.1$.
- $\lambda \leq 0.2$.
- $\lambda \in [0.1, 0.5]$.
- $\lambda \in [0.1, 0.2] \cup [0.4, 0.5]$.

Using (18) and (19), we plot the plausibility and belief functions as a function of the number of defected samples X in the $n = 60$ samples for $0 < X < 60$ and $\lambda \in [\alpha, \beta]$ where $[\alpha, \beta] \in \{[0.1, 0.1], [0, 0.2], [0.1, 0.5], [0.1, 0.2] \cup [0.4, 0.5]\}$ (cf. Fig.7). The results are presented in Table I.

Using Theorem 1, we obtain the following formulas for upper and lower expectation of λ

$$\underline{E}[\lambda] = \frac{X}{n+1}, \qquad \overline{E}[\lambda] = \frac{X+1}{n+1} \tag{20}$$

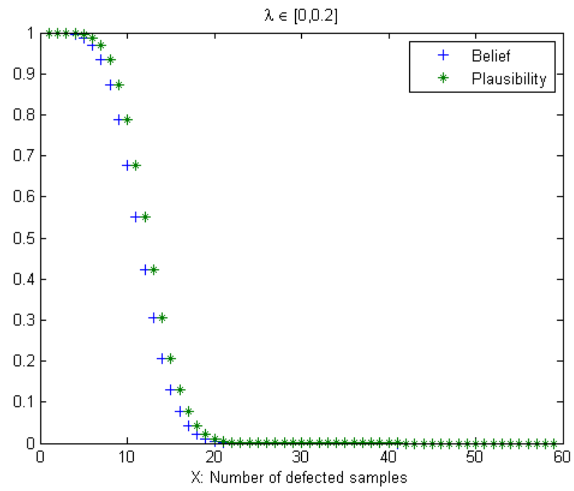

Fig. 5: Belief and Plausibility functions ($\lambda \in [0, 0.2]$)

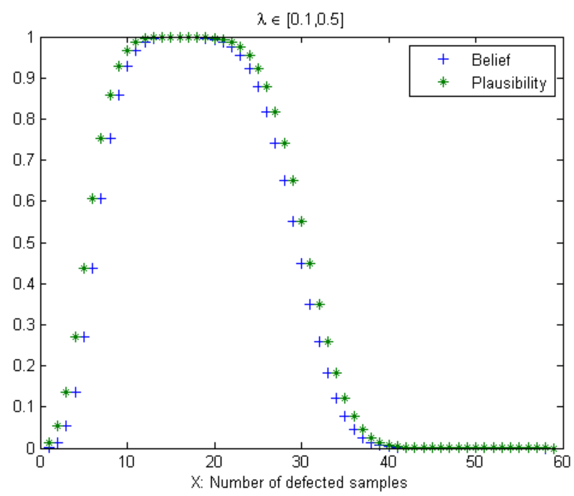

Fig. 6: Belief and Plausibility functions ($\lambda \in [0.1, 0.5]$)

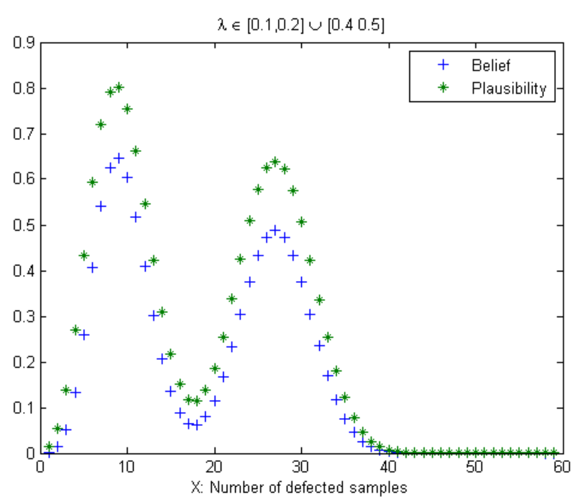

Fig. 7: Belief and Plausibility functions ($\lambda \in [0.1, 0.2] \cup [0.4, 0.5]$)

Remark 2. The imprecision (epistemic uncertainty) of the parameter estimate λ is given by

$$\Delta E[\lambda] = \overline{E}[\lambda] - \underline{E}[\lambda] \quad (21)$$
$$= \frac{1}{n+1} \quad (22)$$

As we can see, $\Delta E[\lambda]$ tends to 0 as $n \to \infty$.

Remark 3. Let us compare, the upper and lower belief expected values of λ, with estimations based on some non informative prior defined in the Bayesian approach.

The Maximum Likelihood (ML) estimate is given by $\hat{\lambda}_{ML} = \frac{X}{n}$. Thus

$$\hat{\lambda}_{ML} - \underline{E}[\lambda] = \frac{X}{n(n+1)} > 0$$
$$\overline{E}[\lambda] - \hat{\lambda}_{ML} = \frac{n-X}{n(n+1)} > 0$$

The uniform prior ($\beta(1,1)$) produces the value $\hat{\lambda}_{Uniform\ prior} = \frac{X+1}{n+2}$. It follows that

$$\hat{\lambda}_{Uniform\ prior} - \underline{E}[\lambda] = \frac{n-X+1}{(n+1)(n+2)} > 0$$
$$\overline{E}[\lambda] - \hat{\lambda}_{Uniform\ prior} = \frac{X+1}{(n+1)(n+2)} > 0$$

The Jeffrey's prior ($\beta(1/2,1/2)$) produces the value $\hat{\lambda}_{Jeffrey\ prior} = \frac{X+1/2}{n+1}$. Thus

$$\hat{\lambda}_{Jeffrey\ prior} - \underline{E}[\lambda] = \frac{1}{2(n+1)} > 0$$
$$\overline{E}[\lambda] - \hat{\lambda}_{Jeffrey\ prior} = \frac{1}{2(n+1)} > 0$$

As a conclusion, all these three Bayesian estimations lie between upper and lower expected belief values.

Let Y be an indicator variable of component failure. Then

$$Bel(Y=1) = \underline{E}[\lambda], \qquad Pl(Y=1) = \overline{E}[\lambda] \quad (23)$$

Then, the belief and plausibility functions of component reliability are given by

$$Bel(Y=0) = 1 - Pl(Y=1) = \frac{n-X}{n+1} \quad (24)$$

$$Pl(Y=0) = 1 - Bel(Y=1) = \frac{n+1-X}{n+1} \quad (25)$$

Example 3. In a large lot of component parts, the acceptance sampling plan for lots of these parts is to randomly select an important number of component parts for inspection, and accept the lot, if the proportion of a defective part does not exceed 10%. However, because the inspection is time-consuming and expensive, we inspect only 30 parts of a lot. What are the belief and plausibility of accepting the lot in the case that the number of defective parts in these 30 parts is 2?

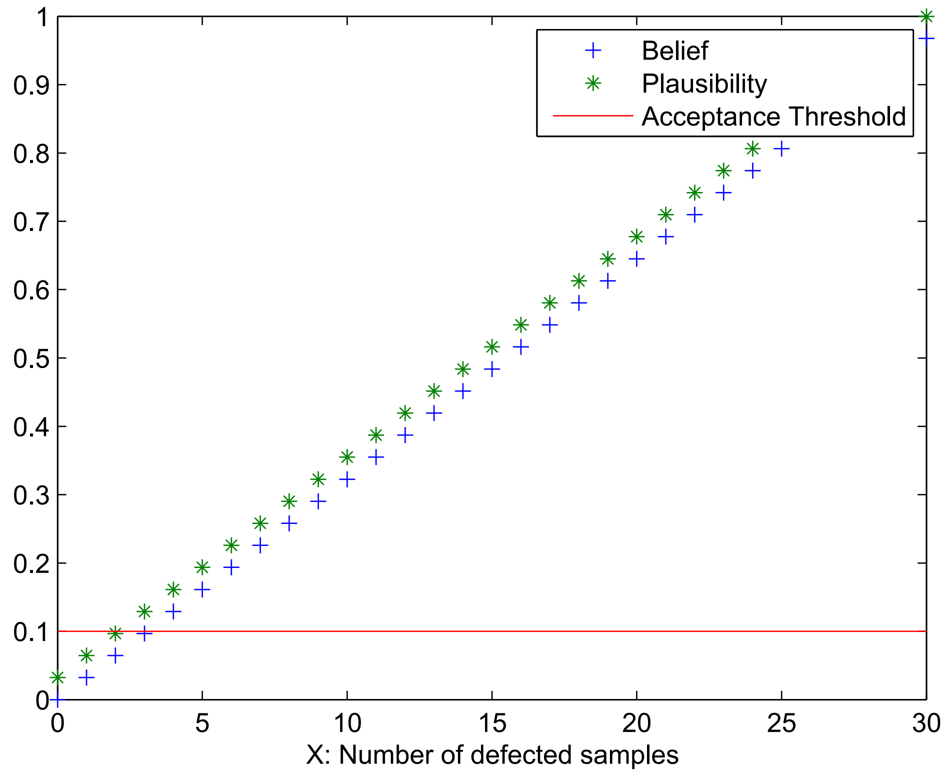

Fig. 8: Acceptance threshold of the lot

In this example, we have $X = 2$ and $n = 30$. Using (24) and (25), the obtained belief interval is

$$\begin{aligned} [Bel(Y=1), Pl(Y=1)] &= [\frac{2}{30+1}, \frac{2+1}{30+1}] \\ &= [0.0645, 0.0968] \end{aligned}$$

Thus, we have to accept the lot. In Fig.8, we plot the belief and plausibility of acceptence of the lot as a function of the number of defected samples.

IV. Parametric construction of belief functions of reliability data in the continuous case

In this section, we present a method to build BBAs, belief and plausibility functions of failure rates of components from reliability data when using exponential distributions with unknown failure rate λ. The proposed method is based on the use of a special form of a multivariate gamma distribution introduced by Mathal *et al.* [28].

Let us consider the variables W_i be the waiting times between the ith and $(i-1)th$ failures of the Poisson process of a component. The variables $W_1, W_2, ..., W_n$ are i.i.d. sample with parent variable $W \in \mathscr{X} = \mathscr{R}$ following an exponential distribution with unknown rate parameter $\lambda \in \Lambda = [0,1]$ ($\mathscr{X}$ is the sample of observations). The problem is to infer λ from $W_1, W_2, ..., W_n$. Thus we introduce pivotal variables v_i such that

$$v_i = \lambda W_i \tag{26}$$

Then, the variables v_i follow a unit exponential distribution. The unit exponential distribution of v_i can be thought of as modeling an exponential sampling from an infinite population assimilated to the real line $\Omega = \mathscr{R}$.

The equation (26) represents a relationship among observables W_i, pivotal variables v_i, and the unknown rate λ. Marginalizing out the pivotal variables reduces the problem to the standard likelihood for an exponential process. We then inverts the equation (26) into the multivalued mapping from Ω to $\mathscr{X} \times \Lambda$. This mapping can be represented by a BBA m on the joint space $\Omega \times \mathscr{X} \times \Lambda$. Thus, having observed a realization w_i of each W_i, a belief function over λ can be obtained after a marginalization on Λ.

Now, we will demonstrate how to obtain the formulas of the BBA over λ.

Let us consider the variable t which represents the total time periods of observations and X the number of events generated by the Poisson process ($X = \sum X_i$ where X_i represents the number of events observed between W_{i-1} and W_i), we have

$$\sum_{i=1}^{X} W_i \leq t < \sum_{i=1}^{X+1} W_i$$

Then

$$\lambda \sum_{i=1}^{X} W_i \leq \lambda t < \lambda \sum_{i=1}^{X+1} W_i$$

Finally

$$\frac{1}{t}\sum_{i=1}^{X} v_i \leq \lambda < \frac{1}{t}\sum_{i=1}^{X+1} v_i \tag{27}$$

As v_i follow a unit exponential distribution, $\sum_{i=1}^{X} v_i$ follow a Gamma(X,1) distribution and then $\frac{1}{t}\sum_{i=1}^{X} v_i$ follow Gamma(X,t) distribution. As a conclusion

- $\underline{a} = \frac{1}{t}\sum_{i=1}^{X} v_i$ follows a Gamma(X,t) distribution
- $\overline{a} = \frac{1}{t}\sum_{i=1}^{X+1} v_i$ follows a Gamma(X+1,t) distribution

Theorem 3. *[28]*
Let V_i $(i = 1,\ldots,k)$, be independent Gamma random variables with shape α_i, scale β_i and location parameter γ_i. The density function of V_i is given by

$$f(x,\alpha_i,\beta_i,\gamma_i) = \frac{(x-\gamma_i)^{\alpha_i - 1} exp(-\frac{x-\gamma_i}{\beta_i})}{\beta_i^{\alpha_i}\Gamma(\alpha_i)}, \qquad x > \gamma_i, \qquad \alpha_i > 0, \qquad \beta_i > 0 \tag{28}$$

Let us consider the partial sums $Z_1 = V_1$, $Z_2 = V_1 + V_2$, ..., $Z_k = V_1 + \ldots + V_k$. When the scale parameters β are all equals, each partial sum is again distributed as Gamma, and the joint distribution of the partial sums may be called a multivariate gamma and is given by

$$f(z_1,\ldots,z_k) = \frac{(z_1-\gamma_1)^{\alpha_1 - 1}}{\beta^{\alpha_k^*}\prod_{i=1}^{k}\Gamma(\alpha_i)}.(z_2 - z_1 - \gamma_2)\ldots(z_k - z_{k-1} - \gamma_k)^{\alpha_{k-1}}.exp(-\frac{(z_k - (\gamma_1 + \ldots + \gamma_k)}{\beta}) \tag{29}$$

where $\alpha^* = \sum_{i=1}^{k}\alpha_i$

By using the notation used in Theorem 3, we have $Z_1 = \underline{a} = V_1$ where $\underline{a}$ follows a Gamma(X,t) distribution ($\alpha_1 = X$, $\beta_1 = 1/t$, and $\gamma_1 = 0$). We have also $Z_2 = \overline{a} = V_1 + V_2$ where V_2 follows a Gamma(1,t) distribution. Thus, $\overline{a}$ follows a Gamma(X+1,t) distribution ($\alpha_2 = X + 1$, $\beta_2 = 1/t$, and $\gamma_2 = 0$).

Applying Theorem 3, we obtain

$$f^{\mathscr{T}}(\underline{a},\overline{a}) = \frac{\underline{a}^{X-1}}{(1/t)^{\alpha_2^*}\prod_{i=1}^{2}\Gamma(\alpha_i)}.(\overline{a}-\underline{a})^{\alpha_1}.exp(-\frac{\overline{a}}{1/t}) \quad (30)$$

where $\alpha_2^* = \sum_{i=1}^{2}\alpha_i = 2X+1$.

Then using Definition 1, we can define a BBA over λ

$$\begin{aligned} m^{\mathscr{T}}([\underline{a},\overline{a}]) &= f^{\mathscr{T}}(\underline{a},\overline{a}) \\ &= \frac{t^{2X+1}}{X!(X-1)!}\underline{a}^{X-1}(\overline{a}-\underline{a})^{X}exp(-\overline{a}t) \quad (31) \end{aligned}$$

Similarly to the method presented in Section III, we present our proposed expressions for belief and plausibility functions as follows

$$Bel([\alpha,\beta]) = \int_{\underline{a}=\alpha}^{\underline{a}=\beta}\int_{\overline{a}=\underline{a}}^{\overline{a}=\beta}\frac{t^{2X+1}}{X!(X-1)!}\underline{a}^{X-1}(\overline{a}-\underline{a})^{X}exp(-\overline{a}t)d\underline{a}d\overline{a} \quad (32)$$

$$\begin{aligned} Pl([\alpha,\beta]) &= \int_{\underline{a}=-\infty}^{\underline{a}=\alpha}\int_{\overline{a}=\alpha}^{\overline{a}=+\infty}\frac{t^{2X+1}}{X!(X-1)!}\underline{a}^{X-1}(\overline{a}-\underline{a})^{X}exp(-\overline{a}t)d\underline{a}d\overline{a} \quad (33) \\ &+ \int_{\underline{a}=\alpha}^{\underline{a}=\beta}\int_{\overline{a}=\underline{a}}^{\overline{a}=+\infty}\frac{t^{2X+1}}{X!(X-1)!}\underline{a}^{X-1}(\overline{a}-\underline{a})^{X}exp(-\overline{a}t)d\underline{a}d\overline{a} \end{aligned}$$

Hence, from (27), we get formulas for upper and lower expectations of λ

$$\underline{E}[\lambda] = \frac{X}{t}, \qquad \overline{E}[\lambda] = \frac{X+1}{t} \quad (34)$$

Remark 4. The imprecision (epistemic uncertainty) of the parameter estimate λ is given by

$$\begin{aligned} \Delta E[\lambda] &= \overline{E}[\lambda]-\underline{E}[\lambda] \quad (35) \\ &= \frac{1}{t} \quad (36) \end{aligned}$$

As we can see, $\Delta E[\lambda]$ tends to 0 as $t \to \infty$.

Remark 5. Let us compare, the upper and lower belief expected values of λ, with estimations based on some non informative prior defined in the Bayesian approach.

The ML estimate $\hat{\lambda}_{ML} = \frac{X}{t}$ corresponds to the lower expected belief value of λ. The uniform prior ($\beta(1,1)$) produces the value $\hat{\lambda}_{Uniform\ prior} = \frac{X+1}{t}$ which corresponds to the upper expected belief value of λ.

The Jeffrey's prior ($\beta(1/2,1/2)$) produces the value $\hat{\lambda}_{Jeffrey\ prior} = \frac{X+1/2}{t}$. Thus

$$\begin{aligned} \hat{\lambda}_{Jeffrey\ prior} - \underline{E}[\lambda] &= \frac{1}{2t} > 0 \\ \overline{E}[\lambda] - \hat{\lambda}_{Jeffrey\ prior} &= \frac{1}{2t} > 0 \end{aligned}$$

All these three Bayesian estimations lie between upper and lower expected belief values.

When obtaining $\lambda \in [\underline{\lambda},\overline{\lambda}]$, if T is the amount of time when the component must function for the system to succeed and let w be the lifetime of component. The variable w follows an exponential distribution with scale parameter $\frac{1}{\lambda}$. Then, the variable $v = \lambda w$. The component will fail during operation

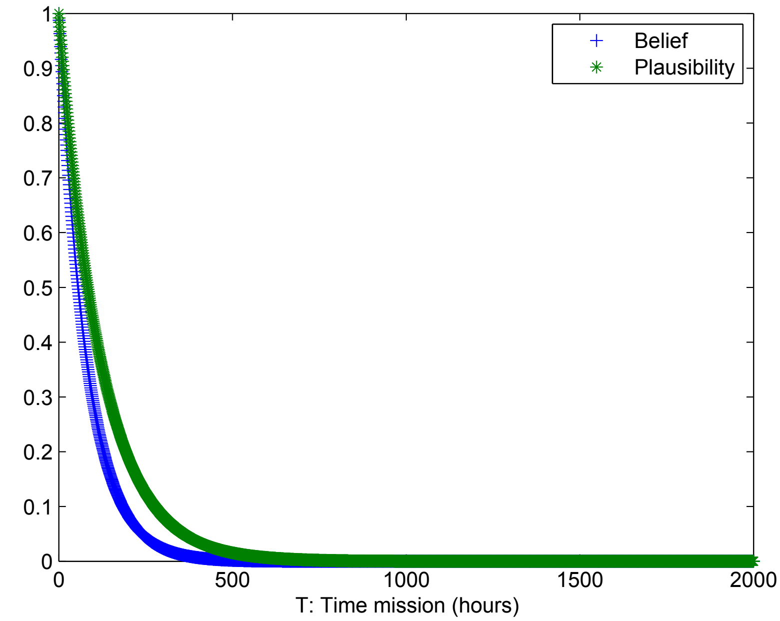

Fig. 9: Belief and Plausibility of the reliability of the aireborne control system vs time

when $w \leq t$ or $\lambda \geq v/t$. Let Y be an indicator variable of component failure, if $v/T \leq \underline{\lambda}$ the component will certainly fail, i.e.

$$\begin{aligned} Bel(Y=1)(T) &= 1 - exp(-\underline{\lambda} T) \\ &= 1 - exp(-\frac{X}{t} T) \end{aligned} \tag{37}$$

Similarly, if $v/T \leq \overline{\lambda}$, the component may fail, i.e.

$$\begin{aligned} Pl(Y=1)(T) &= 1 - exp(-\overline{\lambda} T) \\ &= 1 - exp(-\frac{X+1}{t} T) \end{aligned} \tag{38}$$

Example 4. Consider an airborne fire control system observed only for $t = 10$ days and about which $X = 2$ failures are observed. The failure of the airborne has an exponential distribution with an unknown failure rate λ. What are the belief and plausibility that it will not fail during a 25 days mission?

Using (37), (38), and (5), we obtain

$$\begin{aligned} Bel(Y=0)(T) &= exp(-\frac{X+1}{t} T) \\ &= exp(-\frac{3}{10*24} T) \end{aligned} \tag{39}$$

$$\begin{aligned} Pl(Y=0)(T) &= exp(-\frac{X}{t} T) \\ &= exp(-\frac{2}{10*24} T) \end{aligned} \tag{40}$$

Finally, the belief interval that the airborne control system will not fail during a 25 days mission is $[Bel(Y=0)(600h), pl(Y=0)(600h)] = [7.2\ 10^{-9}, 3.7\ 10^{-6}]$.

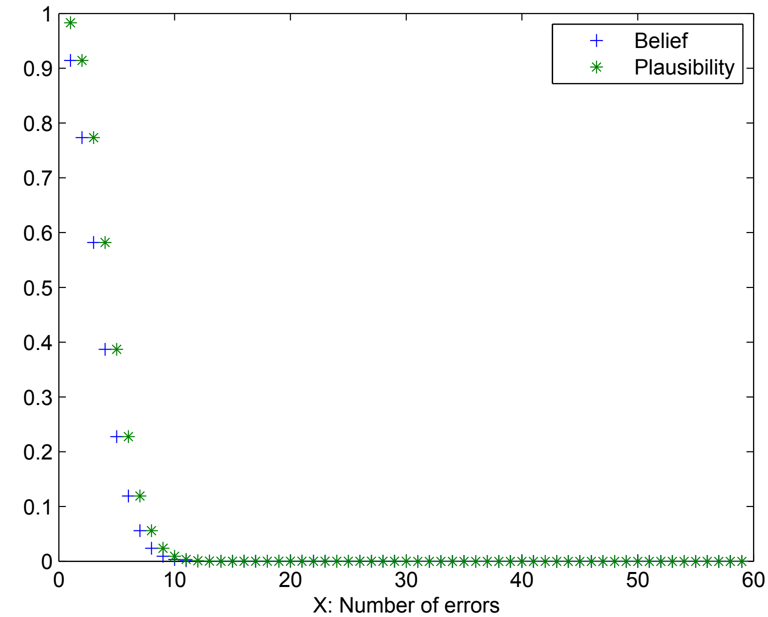

Fig. 10: Belief and Plausibility over the failure rate of the computer vs Number of errors

Example 5. A computer has a constant error rate of $X = 2$ errors every 17 days of continuous operation. What are the plausibility and belief that the failure rate λ of the computer is no more than 0.01/h. Using Eq. 32 and 33 , the obtained belief interval is $[Bel([0, 0.01]), Pl([0, 0.01])] = [0.7734, 0.9141]$. (cf. Figure 10).

V. Reliability evaluation of systems using belief function theory

Let us consider that a component i has two possible states defined over the frame of discernment $\Omega = \{F_i, W_i\}$. F_i and W_i represent, respectively, the failure and the working state of the i^{th} component. The BBA m^{Ω_i} of each component maps the power set $2^{\Omega_i} = \{(F_i), (W_i), (F_i, W_i)\}$ to the interval $[0, 1]$. The masses m^{Ω} given to F_i and W_i represent, respectively, the probability that the component is in a Failure state and in a working state. The mass given to (F_i, W_i) represents the epistemic uncertainty that is the imprecision about the components state. The structure of the reliability mass is as follows:

$$\begin{array}{rcl} m^{\Omega_i}(\{F_i\}) & = & f_i \\ m^{\Omega_i}(\{W_i\}) & = & w_i \\ m^{\Omega_i}(\{W_i, F_i\}) & = & 1 - w_i - f_i \\ i & = & 1, 2 \ldots n \end{array} \tag{41}$$

For example, if the failure of a component i follows an exponential distribution with an imprecise failure rate λ_i. Then, using (37) and (38), we obtain

$$\begin{cases} Bel(F_i)(T) & = 1 - exp(-\underline{\lambda}_i T) \\ Pl(F_i)(T) & = 1 - exp(-\overline{\lambda}_i T) \end{cases}$$

And

$$\begin{cases} Bel(W_i)(T) & = exp(-\overline{\lambda}_i T) \\ Pl(W_i)(T) & = exp(-\underline{\lambda}_i T) \end{cases}$$

x_1	x_2	x_3	x_{S1}
0_1	0_2	0_3	0_{S1}
0_1	0_2	1_3	0_{S1}
0_1	1_2	0_3	0_{S1}
0_1	1_2	1_3	0_{S1}
1_1	0_2	0_3	0_{S1}
1_1	0_2	1_3	1_{S1}
1_1	1_2	0_3	1_{S1}
1_1	1_2	1_3	1_{S1}

TABLE II: Truth table of $S1$

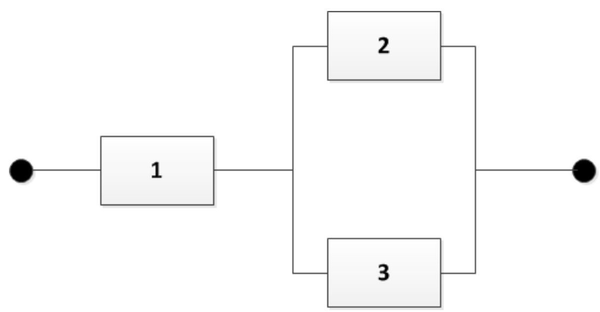

Fig. 11: Reliability block diagram of $S1$.

The obtained structure of the reliability mass is as follows

$$\begin{aligned} m^{\Omega_i}(\{F_i\})(T) &= Bel(F_i)(T) \\ m^{\Omega_i}(\{W_i\})(T) &= Bel(W_i)(T) \\ m^{\Omega_i}(\{W_i,F_i\})(T) &= 1-Bel(F_i)(T)-Bel(W_i)(T) \end{aligned} \tag{42}$$

Note that if $m^{\Omega_i}(\{W_i,F_i\})=0$, the mass is called Bayesian mass function and if this holds for every component i, then, the belief functions model will yield the same results as the classical probability reliability analysis. Similarly, we obtain BBA of all the system components.

Then, we need to compute the BBA which represents the configuration of the system. For example, if we consider the system S1 presented in Fig. 11. Using the truth table of the system $S1$ presented in Table II (x_i denotes the binary state of component i), the BBA of the configuration is given by

$$\begin{aligned} m_{config}^{\Omega_1\times\Omega_2\times\Omega_3\times\Omega_S}(\{&(0_1,0_2,0_3,0_{S1}),(0_1,0_2,1_3,0_{S1}),(0_1,1_2,0_3,0_{S1}),(0_1,1_2,1_3,0_S1), \\ &(1_1,0_2,0_3,0_{S1}),(1_1,0_2,1_3,1_{S1}),(1_1,1_2,0_3,1_{S1}),(1_1,1_2,1_3,1_{S1})\})=1 \end{aligned} \tag{43}$$

Then, the basic and primitive approach to compute the reliability of system with n components are the following

1) The first step consists in doing a vacuous extension (cf. Appendix) of the reliability masses to the product space $\Omega_1\times\Omega_2\times\ldots\times\Omega_n\times\Omega_S$. It is required that mass functions are expressed in the same space if they are going to be combined.
2) Afterwards, these masses are combined using the conjunctive rule of combination (cf. Appendix).

Components	Total observations periods T_i (hours)	Number of failures N_{fi} during T_i
1	720	2
2	1440	4
3	720	1
4	1440	1
5	1000	5
6	720	1
7	1000	2
8	720	4
9	720	3
10	1440	5
11	1000	2
12	1440	3

TABLE III: Components failure rate data

This combination needs the assumption that the components are independents.

3) Following that, the resulting mass is combined with the mass of configuration $m_{config}^{\Omega_1 \times \Omega_2 \times \ldots \Omega_n \times \Omega_S}$. This step acts as a filter that removes all of the impossible events for the given system, e.g., the event $(0_1, 0_2, ..., 0_n, 1_S)$ is impossible for all coherent systems.
4) Finally, the obtained mass is marginalized to Ω_S (the domain of the variable of interest) and with the Mobius Transform (cf. Appendix) we obtain the bounding interval of reliability $R_S \in [Bel(W_S), Pl(W_S)]$.

This is considered a brute force approach as it implies that all of the masses must be expressed in the product space. This grows very quickly with the size of the system. That's why several authors developed some algorithms to reduce the computational complexity of the combination rules. However, in this work, we are only concerned with the construction of reliability parameters of components from reliability data.

VI. Numerical example

Let us consider a system S composed of 12 components: $\{1, 2, ..., 12\}$. The parallel-series system configuration is depicted in Fig. 12. The collected data on total observation periods T_i in hours and the number of failures during the periods T_i of each component are presented in Table III. The failures of components are assumed to be independent and follow an exponential distribution. We aim to compare the results of reliability evaluation of system S using belief approach and probabilistic approach based on Bayesian estimation of failure rates.

In the probabilistic approach, three estimations of failure rates are used:

- ML estimation: $\hat{\lambda}_{iML} = \frac{N_{fi}}{T_i}$
- Estimation based on uniform prior: $\hat{\lambda}_{i\ Uniform\ prior} = \frac{N_{fi}+1}{T_i}$
- Estimation based on Jeffrey's prior: $\hat{\lambda}_{i\ Jeffrey\ prior} = \frac{N_{fi}+1/2}{T_i}$

The minimal paths of system S are:

Components	Belief approach	Probabilistic approach λ ML	λ Uniform prior	λ Jeffrey's prior
1	[0.0028,0.0042]	0.0028	0.0035	0.0042
2	[0.0028,0.0035]	0.0028	0.0031	0.0035
3	[0.0014,0.0028]	0.0014	0.0021	0.0028
4	[0.0007,0.0014]	0.0007	0.0010	0.0014
5	[0.0050,0.0060]	0.0050	0.0055	0.0060
6	[0.0014,0.0028]	0.0014	0.0021	0.0028
7	[0.0020,0.0030]	0.0020	0.0025	0.0030
8	[0.0056,0.0069]	0.0056	0.0063	0.0069
9	[0.0042,0.0056]	0.0042	0.0049	0.0056
10	[0.0035,0.0042]	0.0035	0.0038	0.0042
11	[0.0020,0.0030]	0.0020	0.0025	0.0030
12	[0.0021,0.0028]	0.0021	0.0024	0.0028

TABLE IV: Interval estimates of components failure rates (h^{-1})

$\mathscr{P}_1 = \{1,2,3,4,12\}$
$\mathscr{P}_2 = \{1,2,5,6,12\}$
$\mathscr{P}_3 = \{1,7,9,10,12\}$
$\mathscr{P}_4 = \{1,7,9,11,12\}$
$\mathscr{P}_5 = \{1,8,9,10,12\}$
$\mathscr{P}_6 = \{1,8,9,11,12\}$

The values of estimated failure rates of components are presented in Table IV. Then, we estimate the reliability of system S based on minimal paths and exponential distributions of components failure. Finally, the system's reliability vs. time plot for each estimated λ are presented in Fig. 13, Fig. 14, and Fig. 15.

In the belief function approach, the estimations of failure rates λ_i are obtained using the intervals $[\underline{E}[\lambda], \overline{E}[\lambda]] = [\frac{N_{fi}}{T_i}, \frac{N_{fi}+1}{T_i}]$ which are presented in Table IV. Then, we use (37) and (38) to compute the BBAs of the components' reliability at each time t. Finally, based on the algorithm presented in the last section, we compute the reliability of system S from the reliabilities of its components at time t. The system's reliability vs. time plot using belief approach is presented in Fig. 13, Fig. 14, and Fig. 15.

As we can see, the system's reliability obtained in the probabilistic approach (based on the ML, Uniform prior and Jeffrey's prior estimations of λ) are between the belief and plausibility functions of system's reliability obtained using belief function theory. In this example, The belief function theory is more conservative than the probabilistic approach based on the latter three estimations of λ. Note that in this example, a parallel series system is considered. However, our proposed approach can be applied easily to any complex system configuration.

VII. Conclusion and future works

Elicitation of components' reliability parameters from statistical data about reliability is a key-point to reliability analysis of complex systems when considering epistemic uncertainties. This paper proposes different methods to tackle this problem using belief function theory under epistemic uncertainties. A

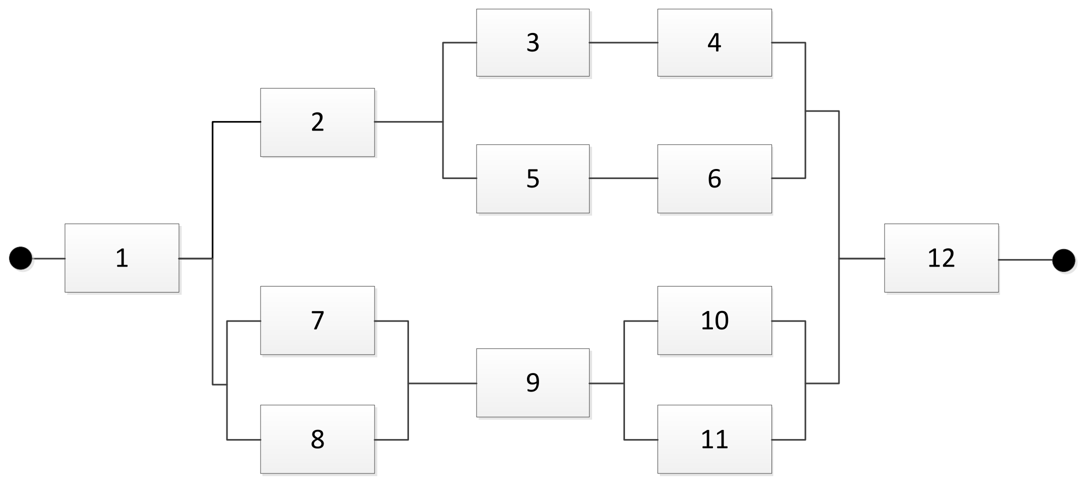

Fig. 12: Reliability block diagram of *S*.

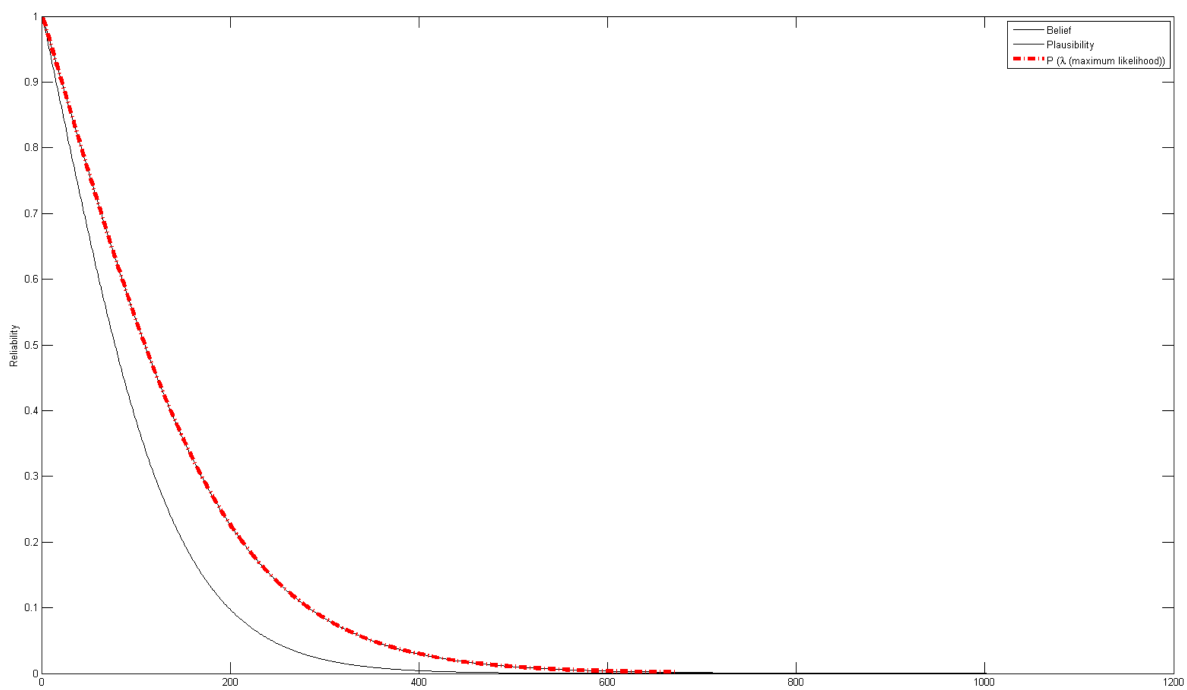

Fig. 13: Reliability vs. time plot for *S* (λ ML).

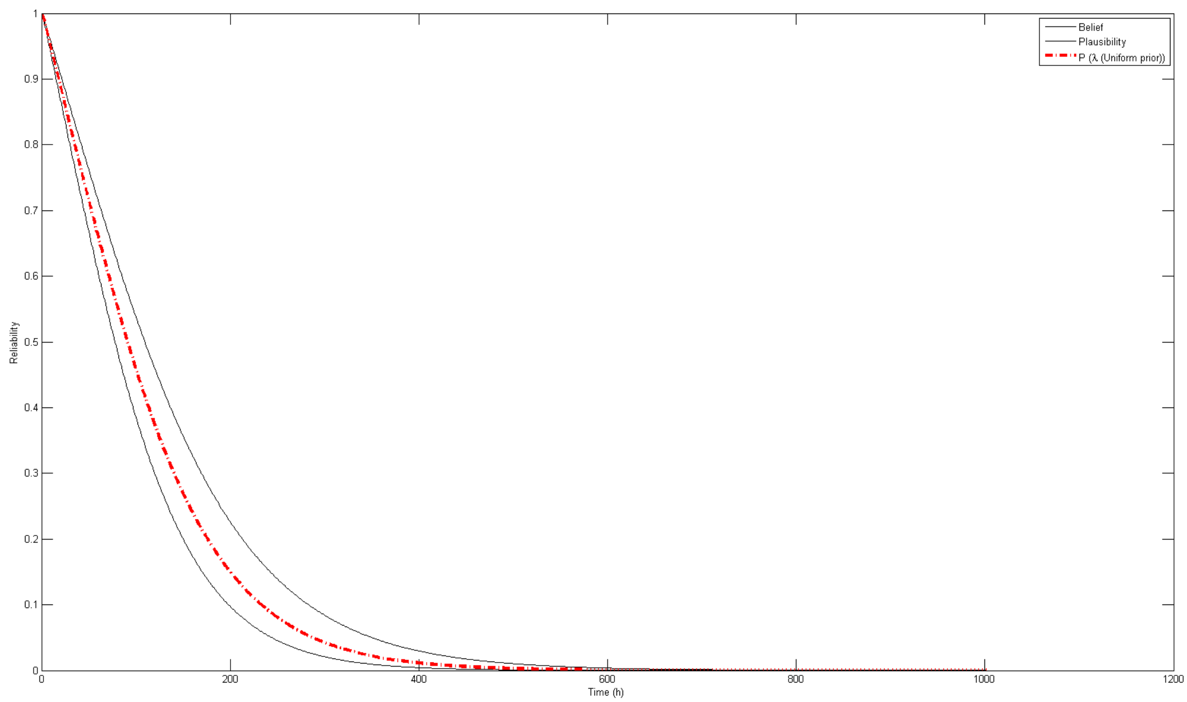

Fig. 14: Reliability vs. time plot for S (λ Uniform prior).

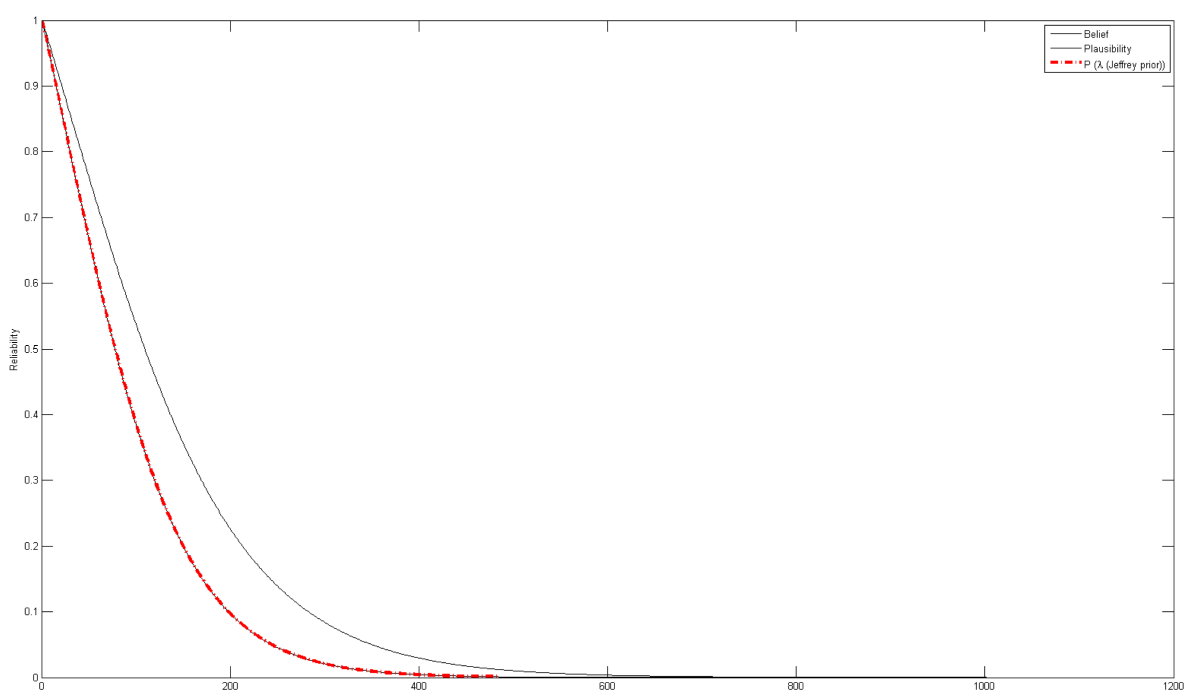

Fig. 15: Reliability vs. time plot for system S (λ Jeffrey's prior)

major conclusion is that belief function theory seems to be a promising theory for reliability assessments of systems. However, much work must be done to reduce the computational complexity of the operations involved in the belief function theory.

APPENDIX

A. Formula of marginalization

Consider a BBA $m^{\Omega_x\Omega_y}$ defined on the Cartesian product $\Omega_x\Omega_y$. The marginal BBA $m^{\Omega_x\Omega_y\downarrow\Omega_x}$ on Ω_x is defined by:

$$m^{\Omega_x\Omega_y\downarrow\Omega_x}(A) = \sum_{B\subseteq\Omega_x\Omega_y/Proj(B\downarrow\Omega_x)=A} m^{\Omega_x\Omega_y}(B) \qquad \forall A\subseteq\Omega_x \tag{44}$$

Where $Proj(B\downarrow\Omega_x) = \{x\in\Omega_x/\exists y\in\Omega_y,(x,y)\in B\}$.

B. Formula of vacuous extension

Consider a BBA m^{Ω_x} defined on Ω_x. Its vacuous extension on $\Omega_x\Omega_y$ is defined by:

$$m^{\Omega_x\uparrow\Omega_x\Omega_y}(B) = \begin{cases} m^{\Omega_x}(A) & \text{if} \quad B = A\times\Omega_y \\ 0 & \text{otherwise.}\end{cases} \qquad \forall A\subseteq\Omega_x \tag{45}$$

C. Formulas of combination rules

The Conjunctive $\bigcirc\!\!\!\!\cap$ and Disjunctive $\bigcirc\!\!\!\!\cup$ rules are defined by:

$$m^{\Omega}_{i\bigcirc\!\!\!\!\cap j}(H) = \sum_{A\cap B=H,\forall A,B\subseteq\Omega} m^{\Omega}_i(A)m^{\Omega}_j(B), \forall H\subseteq\Omega \tag{46}$$

$$m^{\Omega}_{i\bigcirc\!\!\!\!\cup j}(H) = \sum_{A\cup B=H,\forall A,B\subseteq\Omega} m^{\Omega}_i(A)m^{\Omega}_j(B), \forall H\subseteq\Omega \tag{47}$$

The Dempster's rule is given by:

$$m^X_{i\oplus j}(H) = \frac{\sum_{A\cap B=H,\forall A,B\subseteq X} m^X_i(A)m^X_j(B)}{1-\sum_{A\cap B=\emptyset,\forall A,B\subseteq X} m^X_i(A)m^X_j(B)} \tag{48}$$

D. Formula of Mobius Transform

The Mobius transform is given by

$$m^{\Omega}(A) = \sum_{B\subset A}(-1)^{|A-B|}Bel^{\Omega}(B) \tag{49}$$

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