Extended LK heuristics for the optimization of linear consecutive-k-out-of-n: F systems considering parametric uncertainty and model uncertainty

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Abstract

The optimization of consecutive-k-out-of-n system is to find the optimal assignment of components that maximizes the system reliability. Many efforts have been devoted to the optimal arrangement problems for such systems, however, uncertainties about the reliability data of components and system structure have never been deeply studied in optimal arrangement problems before. Due to the insufficiency of historical data and system complexities, uncertainties inevitably exist in many reliability problems. This paper extends the LK heuristic using a non-probabilistic graphical model called evidential network, and interval-valued importance measures to perform the optimization of linear consecutive-k-out-of-n: F systems under data uncertainty (related to the failure data of components) and model uncertainty (related to the system structure). Moreover, the extended LK heuristics are applied on an oil pipeline system.

Keywords: Reliability optimization, Consecutive-k-out-of-n systems, Uncertainty, Interval analysis, Evidential Network, Birnbaum importance.

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1. Introduction

According to the arrangement of components (linear or circular) and the functioning principle (failure (F) or good (G) system), a consecutive-k-out-of-n system (Con/k/n) can be classified into four categories: linear consecutive-k-out-of-n: F system (denoted as Lin/Con/k/n: F), linear consecutive-k-out-of-n: G system (denoted as Lin/Con/k/n: G), circular consecutive-k-out-of-n: F system (denoted as Cir/Con/k/n: F), circular consecutive-k-out-of-n: G system (denoted as Cir/Con/k/n: G). A Lin/Con/k/n: F (Cir/Con/k/n: F) system is a system consisting of n linearly (circularly) arranged components that fails if and only if at least k consecutive components fail. A Lin/Con/k/n: G (Cir/Con/k/n: G) system is a system consisting of n linearly (circularly) arranged components that works if and only if at least k consecutive components work. Con/k/n systems can model a wide range of engineering systems, such as microwave stations of a telecom network, oil pipeline systems, vacuum systems in accelerators, spacecraft relay stations, etc [1]. For example, an oil pipeline system transports oil through pipeline. The oil is moved through the pipeline and pressurized by pumps along the pipeline. There are n equally spaced pump stations along the pipeline. If one pump cannot work, its neighbour can pressurize the oil to keep it move through the pipeline. However, if two consecutive pumps cannot work, the oil transportation is interrupted. This is an example of the Lin/Con/2/n: F system.

Assume components are functionally interchangeable and need to be assigned to different positions of a system, the system reliability is influenced by the assignment of components. The component assignment problem (CAP) is to find the optimal assignment of components that maximizes the system reliability. Optimal assignments are classified into two types: invariant optimal assignment that means the optimal assignment depends only on the ordering of the numerical values of component reliabilities, and variant optimal assignment that depends on the numerical values of component reliabilities.

The CAP is an NP-hard problem, therefore, the only way to find the exact optimal assignment is the enumeration method [2]. It takes long time for the enumeration method to evaluate $n!$ assignments of components to find the exact optimal assignment, especially for large systems. Generally, heuristics are used to handle this kind of problems. Importance measures reflect the effect of the individual component reliability on the system reliability. Birnbaum [3] was the first to quantify the influence of components' reliability on binary coherent systems' reliability. The Birnbaum importance (BI) measure of a component can be defined as the rate at which the systems reliability improves as the reliability of component is improved. It measures the relative importance of a component to the system reliability. Two groups of BI-based heuristics were designed to handle CAPs: the local search methods and the genetic-algorithm based methods.

In the literature, some BI-based local search methods were designed to handle CAPs. Zuo and Kuo [4] summarized the results available for the invariant optimal designs of Con/k/n systems, and proposed two BI-based heuristic methods by pairwisely exchanging component allocations to match the BI ordering of positions to find at least suboptimal designs when invariant optimal designs do not exist. Hereinafter these two heuristics are referred to as ZKA and ZKB heuristics. Lin and Kuo [5] concluded that the optimal allocation is to assign components according to the BI ordering when there exists an invariant optimal allocation, and proposed a heuristic method called the LK heuristic with an analytically calculated error to find the near optimal allocation of components when invariant optimal allocations do not exist. The LK heuristic initializes the system with the least reliable component, and iteratively allocates unassigned most reliable component to the position with the largest BI. Based on LK, ZKA and ZKB heuristics, Yao *et al.* [6] proposed three new LK-type heuristics called LKB, LKC, LKD heuristics, and two ZK-type heuristics called ZKC, ZKD heuristics. Differences among these LK-type heuristics lie in the initializations and the assignment rules. Differences among ZK-type heuristics lie in the way of choosing components for exchange. Based on the comparison of results obtained by LK-type heuristics and ZK-type heuristics, Yao *et al.* [6] proposed a BI-based two-stage (BIT) approach that generates two initial arrangements by LKA and LKB heuristics, and uses either ZKB or ZKD heuristic to obtain two improved solutions and chooses the one with the higher system reliability as the final solution.

Genetic-algorithm based methods can break the local optimal limit and were also developed to solve CAPs. Yao *et al.* [2] proposed a BI-based genetic local search (BIGLS) algorithm which is a hybrid genetic algorithm with a BI-based three-way exchange method. Numerical tests showed the BIGLS algorithm improved almost all the non-optimal assignments generated by the BIT approach. Cai *et al.* [7] proposed a BI-based genetic algorithm (BIGA) to search the near global optimal solution for Lin/Con/k/n systems. The BIGA introduced the BI-based initial population generation method, BI-based chromosome update principle and comprehensive elitism strategy to improve the searching algorithm compared with the BIGLS. Numerical tests shows the BIGA gets better results than the BIGLS in large systems.

However, in previous studies, uncertainties have never been deeply studied before. Uncertainty is an important topic in reliability and risk analysis [8, 9]. There are many kinds of classifications of uncertainties in the literature. The most common one is to divide uncertainties into aleatory and epistemic uncertainties [10]. Aleatory uncertainty is due to the randomness of natural phenomena, whereas epistemic uncertainty is due to the lack of knowledge. The distinction is important because a better understanding of knowledge cannot reduce the aleatory uncertainty, but it can reduce the epistemic uncertainty. Aleatory uncertainty is usually represented and quantified by probability models and frequentist probabilities. Some authors claim that classical probability is not appropriate to distinguish aleatory and epistemic uncertainties [11]. Several alternative frameworks based on non-probabilistic theories have been proposed to represent epistemic uncertainty, such as belief functions theory, fuzzy sets theory, imprecise probabilities, etc.

In real engineering systems, historical failure data of components may be insufficient so that interval-valued reliabilities of components are given instead of precise values. This kind of uncertainty is referred to as parametric uncertainty [12]. For some complex systems, analysts sometimes cannot give precise system structures. This kind of uncertainty is referred to as model uncertainty [12]. In this work, parametric uncertainty (related to the failure data of components), and model uncertainty (related to the system structure) due to randomness and the lack of knowledge are all taken into account in the proposed extended LK heuristics.

The goal of this paper is to propose extended LK heuristics to search the near global optimal assignment of components of Lin/Con/k/n: F systems when there exist uncertainties related to the failure data of components and the system structure using:

- Evidential network to represent systems and to compute interval-valued systems reliability.
- Interval-valued importance measures to identify components within the system that more significantly influence the systems behaviour with respect to reliability.
- Heuristic method to obtain optimal assignment of components based on decision maker's preference.

The remainder of this paper is organized as follows. Section 2 introduces briefly some basic notions of evidential network. Section 3 details the extended LK heuristics considering parametric uncertainty and model uncertainty. Section 4 applies the extended LK heuristics to the optimization of an oil pipeline system. Section 5 concludes this paper and give perspectives for future work.

2. Evidential network

Evidential network is an appropriate graphical tool for representing and managing uncertainty using Valuation-Based System (VBS) as a framework and belief functions theory as a tool to interpret and combine information [13, 14]. VBS was first proposed by Shenoy [15, 16] as a framework for knowledge representation and reasoning under all types of uncertainties. A set of variables and a set of valuations assigned to subsets of variables are used to represent uncertain knowledge. Combination and marginalisation operators are used to make inference with uncertain knowledge. VBS can represent different kinds of uncertain knowledge using different uncertainty theories, including probability, belief functions theory, fuzzy sets theory, imprecise probability, etc. Evidential network is the VBS that represents uncertainties using belief functions theory. It consists of a 5-tuple $\{X, \Omega_X, \mathcal{M}_X, \otimes, \downarrow\}$. Basic notions of the evidential network are detailed below [13, 12, 17].

2.1. *X - set of variables*

A real world problem can be described by a set of variables $X = \{x_1, x_2, \ldots, x_n\}$. For example, a system and all its components are modelled by variables.

2.2. Ω_X *- set of frames of discernment*

The frame of discernment of a variable is the set of all its possible values. For a binary variable x_i, its frame of discernment is $\Omega_i = \{0_i, 1_i\}$. The set of frames of discernment $\Omega_X = \times\{\Omega_i | x_i \in X\}$ is defined as the Cartesian product of the frames of discernment of variables contained in X. Given a set of two variables $X = \{x_1, x_2\}$ with their frames of discernment $\Omega_1 = \{0_1, 1_1\}$ and $\Omega_2 = \{0_2, 1_2\}$. The set of frames of discernment $\Omega_X = \{(0_1, 0_2), (0_1, 1_2), (1_1, 0_2), (1_1, 1_2)\}$.

2.3. $\mathcal{M}_X$ *- set of valuations*

A valuation m^{Ω_i} holds the knowledge about the possible value of the variable x_i. It can be represented by probability, basic belief assignment (bba), possibility, etc. In the framework of evidential network, belief functions theory is used to quantify and propagate uncertainties, therefore, the valuation is represented by bba.

A bba is a mapping function $m^\Omega : 2^\Omega \to [0, 1]$ that assigns values to subsets of Ω in the interval [0,1] such that $\sum_{A \subseteq \Omega} m^\Omega(A) = 1$. $m^\Omega(A)$ represents the degree of belief that the expert assigns to the subset A.

Given a subset of variables $L = \{x_i, x_{i+1}, ..., x_j\}$, a valuation m^{Ω_L} holds the knowledge about the relation among variables $x_i, x_{i+1}, ..., x_j$. $\mathcal{M}_X = \{m^{\Omega_L} : L \subseteq X\}$ is the set of valuations of all subsets included in X.

2.4. $\otimes$ - combination operator

Combination operation combines two bbas into a joint bba. Given two bbas $m_i^{\Omega_i}$ and $m_j^{\Omega_j}$ defined over Ω_i and Ω_j. Before the combination operation, these two bbas need to be extended to the same frame of discernment $\Omega_i\Omega_j$ as follows [18]

$$m_i^{\Omega_i\Omega_j}(B) = m_i^{\Omega_i \uparrow \Omega_i\Omega_j}(B) = \begin{cases} m_i^{\Omega_i}(A) & \text{if } B = A \times \Omega_j \\ 0 & \text{otherwise.} \end{cases} \quad \forall A \subseteq \Omega_i \tag{1}$$

$$m_j^{\Omega_i\Omega_j}(B) = m_j^{\Omega_j \uparrow \Omega_i\Omega_j}(B) = \begin{cases} m_j^{\Omega_j}(A) & \text{if } B = A \times \Omega_i \\ 0 & \text{otherwise.} \end{cases} \quad \forall A \subseteq \Omega_j \tag{2}$$

Then, $m_i^{\Omega_i\Omega_j}$ and $m_j^{\Omega_i\Omega_j}$ are combined using Dempster's rule of combination as follows [18]

$$\begin{cases} m_{i\otimes j}^{\Omega_i\Omega_j}(H) = \frac{\sum_{A\cap B=H} m_i^{\Omega_i\Omega_j}(A) m_j^{\Omega_i\Omega_j}(B)}{1-k}, \forall A, B \subseteq \Omega_i\Omega_j, H \neq \emptyset \\ m_{i\otimes j}^{\Omega_i\Omega_j}(\emptyset) = 0 \end{cases} \tag{3}$$

where $k = \sum_{A\cap B=\emptyset, \forall A,B\subseteq\Omega_i\Omega_j} m_i^{\Omega_i\Omega_j}(A) m_j^{\Omega_i\Omega_j}(B)$ is the conflict factor between combined bbas. This rule assumes that all bbas come from independent sources (i.e. the experts' opinions are not based on overlapping experiences). In the presences of conflictual or dependent sources, there are other rules such as Denoeux's rule for the dependent case [19] and Yager's rule for the high conflict case [20].

2.5. $\downarrow$ - marginalisation operator

The marginalization operator is used to focus the information contained by a valuation onto a smaller domain. Given a joint bba $m^{\Omega_i\Omega_j}$ defined over $\Omega_i\Omega_j$.

The marginal bbas defined over Ω_i and Ω_j are computed as follows [18]

$$m^{\Omega_i\Omega_j\downarrow\Omega_i}(A) = \sum_{B\subseteq\Omega_j} m^{\Omega_i\Omega_j}(A\times B), \forall A\subseteq\Omega_i \tag{4}$$

$$m^{\Omega_i\Omega_j\downarrow\Omega_j}(B) = \sum_{A\subseteq\Omega_i} m^{\Omega_i\Omega_j}(A\times B), \forall B\subseteq\Omega_j \tag{5}$$

2.6. Fusion algorithm

The idea of the fusion algorithm is to combine all bbas and to compute the marginal bba over the frame of discernment of the variable of interest. The obtained marginal bba is the solution of the reasoning that helps the decision maker to make a decision. The fusion algorithm is based on local computations propagated through a binary join tree (BJT).

A BJT is a structured tree composed of nodes and edges. Each node holds a bba and is connected with at most one parent and two children, and no node has more than three neighbours. Bbas of two children are combined and then stored in their parent. A node without any child is called a leaf. Bbas that represent the knowledge about the model are stored in the leaf nodes. Non-leaf nodes hold the intermediary bbas obtained by combining the bbas of their corresponding children. A node without any parent is called a root. The root holds the bba of the variable of interest with which the decision is made. The structure of the BJT reflects the computational complexity of the fusion algorithm. The deletion sequences do not influence the final result, but the computational complexity of different deletion sequences is different. Finding an optimal deletion sequence is an NP-complete problem [15].

2.7. Belief and plausibility measures

Belief functions theory is a kind of uncertainty theory that can represent and distinguish aleatory and epistemic uncertainties. It was first proposed in the work of Dempster [18] on upper and lower probabilities and then developed by Shafer [21] as the theory of evidence. There are three main functions: mass (bba), belief (bel) and plausibility (pl) functions.

Mass function is also called bba. $m(A)$ represents the degree of belief assigned to the hypothesis that the truth lies in the subset A without further dividing this belief to a strict subset of A.

Belief function quantifies the amount of justified specific support to be given to A [22]. $bel(A)$ is defined to be the sum of all bbas of subsets contained in A [21]

$$bel(A) = \sum_{B \subseteq A} m^{\Omega}(B) \qquad A, B \subseteq \Omega \tag{6}$$

Plausibility function quantifies the maximum amount of potential specific support that could be given to A [22]. $pl(A)$ is defined to be the sum of all bbas of subsets that having non-empty intersection with A [21]

$$pl(A) = \sum_{B \cap A \neq \emptyset} m^{\Omega}(B) \qquad A, B \subseteq \Omega \tag{7}$$

The probability of the subset A is bounded by the interval $[bel(A), pl(A)]$. $pl(A) - bel(A)$ measures the uncertainty about the subset A.

Take a binary component C as an example. Its frame of discernment is $\Omega_C = \{0_C, 1_C\}$. 1_C and 0_C denote the working state and the failed state of component C. An expert gives the following bbas to represent his/her degree of belief to the states of the component: $m^{\Omega_C}(\{1_C\}) = 0.9, m^{\Omega_C}(\{0_C\}) = 0.03, m^{\Omega_C}(\{0_C, 1_C\}) = 0.07$. The probability of component C being working is bounded by the interval $[bel(1_C), pl(1_C)] = [m^{\Omega_C}(\{1_C\}), m^{\Omega_C}(\{1_C\}) + m^{\Omega_C}(\{0_C, 1_C\})] = [0.9, 0.97]$. The length of the interval represents the expert's uncertainty.

3. Extended LK heuristics under uncertainty

In this section, first, the Lin/Con/k/n: F system is modelled in evidential network, and the LK heuristic is applied to find the near optimal arrangement of components. Next, extended LK heuristics are proposed successively to consider parametric uncertainty and model uncertainty, and to find the near optimal arrangement of components under uncertainty.

3.1. LK heuristic in evidential network

Lin and Kuo [5] proposed the LK heuristic that initializes the system with the least reliable component, and iteratively allocates unassigned most reliable component to the position with the largest BI.

Steps of the LK heuristic in evidential network are detailed below.

Step 1: Develop the evidential network of the studied system;

A Lin/Con/k/n: F system consists of n positions and n binary components with reliabilities $r_1 \leq r_2 \leq ... \leq r_n$. Its graphical representation is shown in Fig. 1.

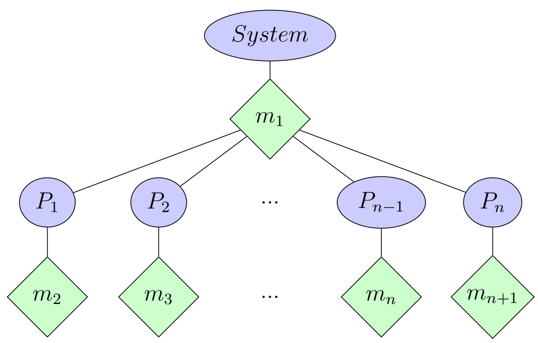

Figure 1: Evidential network of the Lin/Con/k/n: F system.

There are $n+1$ variables representing the system and its components, and $n+1$ bbas representing knowledge about variables and system structure. All variables are binary, and their frames of discernment are $\Omega = \{0, 1\}$, where 0 and 1 denote the failed state and the working state.

m_1 is the bba representing the knowledge about the system structure which is expressed by the truth table in Table 1.

m_{i+1} is the bba representing the knowledge about component Ci. r_i is the reliability of component Ci, so we have

$$
\begin{aligned}
m_{i+1}^{\Omega_i}(\{1_i\}) &= r_i \\
m_{i+1}^{\Omega_i}(\{0_i\}) &= 1 - r_i
\end{aligned}
$$

Table 1: Truth table of the Lin/Con/k/n: F system.

P_1	P_2	P_3	...	P_k	P_{k+1}	P_{k+2}	P_{k+3}	...	P_{n-k-2}	P_{n-k-1}	P_{n-k}	P_{n-k+1}	...	P_{n-2}	P_{n-1}	P_n	System
0	0	0	...	0	{0,1}	{0,1}	{0,1}	...	{0,1}	{0,1}	{0,1}	{0,1}	...	{0,1}	{0,1}	{0,1}	0
{0,1}	0	0	...	0	0	{0,1}	{0,1}	...	{0,1}	{0,1}	{0,1}	{0,1}	...	{0,1}	{0,1}	{0,1}	0
{0,1}	{0,1}	0	...	0	0	0	{0,1}	...	{0,1}	{0,1}	{0,1}	{0,1}	...	{0,1}	{0,1}	{0,1}	0
...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	0
{0,1}	{0,1}	{0,1}	...	{0,1}	{0,1}	{0,1}	{0,1}	...	{0,1}	0	0	0	...	0	{0,1}	{0,1}	0
{0,1}	{0,1}	{0,1}	...	{0,1}	{0,1}	{0,1}	{0,1}	...	{0,1}	{0,1}	0	0	...	0	0	{0,1}	0
{0,1}	{0,1}	{0,1}	...	{0,1}	{0,1}	{0,1}	{0,1}	...	{0,1}	{0,1}	{0,1}	0	...	0	0	0	0
All other combinations																	1

The system reliability is computed by the equation $(m_2^{\Omega_1} \otimes m_3^{\Omega_2} \otimes ... \otimes m_{n+1}^{\Omega_n} \otimes m_1^{\Omega_1 \Omega_2 ... \Omega_n \Omega_S})^{\downarrow \Omega_S}$.

Step 2: Initialize all positions of the system with the least reliable component;

Because the components' reliabilities are given in a non-decreasing order, all positions of the system are initialized with component $C1$.

Step 3: Compute BIs of all unassigned positions;

The BI of position Pi is defined as the probability that the failure of the component in position Pi results in system failure and can be calculated as [3]

$$\begin{aligned} BI_i = & \frac{\partial R_S(p_1, p_2, ..., p_n)}{\partial p_i} \\ = & R_S(p_1, ..., p_{i-1}, 1, p_{i+1}, ..., p_n) - R_S(p_1, ..., p_{i-1}, 0, p_{i+1}, ..., p_n) \end{aligned} \tag{8}$$

where p_i represents the reliability of the component in position Pi and $R_S(p_1, p_2, ..., p_n)$ is the system reliability. BI measures the relative importance of a component to the system reliability.

Step 4: Allocate the unassigned most reliable component to the unassigned position with the largest BI;

Step 5: If all components are assigned, stop the process and output the assignment of components and the system reliability. Otherwise, go back to Step 3.

The LK heuristic was designed for CAPs with precise components reliabilities and precise system structures. However, due to the insufficiency of historical data or the system complexity, parametric uncertainty and model uncertainty may exist in CAPs. Next, extended LK heuristics are proposed to represent and propagate these two kinds of uncertainty and to search the optimal arrangement of components under uncertainty.

3.2. Extended LK heuristic under parametric uncertainty

In this subsection, parametric uncertainty related to reliabilities of components is taken into account. Due to the insufficiency of historical data, sometimes, interval-valued reliabilities of components are given instead of precise values. Therefore, in this subsection, reliabilities of components are all interval-valued numbers. For example, the reliability of component Ci is $[r_i] = [\underline{r}_i, \overline{r}_i]$.

Steps of the extended LK heuristic under parametric uncertainty are detailed below.

Step 1: Develop the evidential network of the studied system;

The graphical representation of the studied system, and the bba m_1 are the same as the case without uncertainty. Due to the parametric uncertainty, bbas assigned to components are changed as follows

$$
\begin{aligned}
m_{i+1}^{\Omega_i}(\{1_i\}) &= \underline{r}_i \\
m_{i+1}^{\Omega_i}(\{0_i\}) &= 1 - \overline{r}_i \\
m_{i+1}^{\Omega_i}(\{0_i, 1_i\}) &= \overline{r}_i - \underline{r}_i
\end{aligned}
$$

The system reliability is computed by the equation $(m_2^{\Omega_1} \otimes m_3^{\Omega_2} \otimes ... \otimes m_{n+1}^{\Omega_n} \otimes m_1^{\Omega_1\Omega_2...\Omega_n\Omega_S})^{\downarrow\Omega_S}$. Because reliabilities of components are interval-valued numbers, the obtained system reliability is also bounded by an interval $[R_S] = [\underline{R}_S, \overline{R}_S]$.

Step 2: Initialize all positions of the system with the least reliable component;

Because reliabilities of components are interval-valued numbers, here we have to sort these interval-valued numbers. In the existing literature, some interval order relations were already proposed. In this work, the interval order relations proposed by Wang *et al.* [23] are used to compare interval-valued numbers.

For an interval whose bounds are both lower than the bounds of another interval, it is easy to decide the order of these two intervals. For an interval that is included in another interval, it may be difficult to decide the order of these two intervals. In this case, the preference of the decision maker needs to be considered.

ρ_0 denotes the level of the decision maker's preference. Given two intervals $[a] = [\underline{a}, \overline{a}]$ and $[b] = [\underline{b}, \overline{b}]$.

For minimization problems, the order relation $\leqslant_{min}$ between two intervals $[a]$ and $[b]$ are defined as follows [23]:

- If $\underline{a} \leqslant \underline{b}$ and $\overline{a} \leqslant \overline{b}$, then $[a] \leqslant_{min} [b]$;
- If $\underline{b} < \underline{a} \leqslant \overline{a} < \overline{b}$, then $[a] \leqslant_{min} [b]$ if $\rho > \rho_0$, where $\rho = \frac{\overline{b}-\overline{a}}{\underline{a}-\underline{b}}$.

For maximization problems, the order relation $\geqslant_{max}$ between two intervals $[a]$ and $[b]$ are defined as follows [23]:

- If $\underline{a} \geqslant \underline{b}$ and $\overline{a} \geqslant \overline{b}$, then $[a] \geqslant_{max} [b]$;
- If $\underline{b} < \underline{a} \leqslant \overline{a} < \overline{b}$, then $[a] \geqslant_{max} [b]$ if $\rho > \rho_0$, where $\rho = \frac{\underline{a}-\underline{b}}{\overline{b}-\overline{a}}$.

These two order relation operators can be used to find the least reliable component with interval-valued reliability.

Step 3: Compute interval-valued BIs of all unassigned positions;

Interval-valued BI of position Pi is computed as follows

$$
\begin{aligned}
[BI_i] =& [R_S(p_1, ..., p_{i-1}, 1, p_{i+1}, ..., p_n)] - [R_S(p_1, ..., p_{i-1}, 0, p_{i+1}, ..., p_n)] \\
=& [\underline{R}_S(p_1, ..., p_{i-1}, 1, p_{i+1}, ..., p_n) - \overline{R}_S(p_1, ..., p_{i-1}, 0, p_{i+1}, ..., p_n), \\
& \overline{R}_S(p_1, ..., p_{i-1}, 1, p_{i+1}, ..., p_n) - \underline{R}_S(p_1, ..., p_{i-1}, 0, p_{i+1}, ..., p_n)]
\end{aligned} \tag{9}
$$

Step 4: Allocate the unassigned most reliable component to the unassigned position with the largest interval-valued BI;

The order relation $\geqslant_{max}$ is used to find the unassigned most reliable component and the unassigned position with the largest interval-valued BI.

Step 5: If all components are assigned, stop the process and output the assignment of components and the system reliability. Otherwise, go back to Step 3.

3.3. Extended LK heuristic under model uncertainty

In this subsection, model uncertainty related to the system structure is taken into account. Reliabilities of components are precise values (parametric uncertainty is not considered in this subsection). Due to the lack of knowledge, the expert sometimes cannot give the precise structure of the studied system. Therefore, in this subsection, suppose the expert considers that the studied system is a Lin/Con/k/n: F system or a Lin/Con/m/n: F system ($k < m < n$).

Steps of the extended LK heuristic under model uncertainty are detailed below.

Step 1: Develop the evidential network of the studied system;

The graphical representation of the studied system, and bbas of components are the same as the case without uncertainty. m_1 is the bba representing the knowledge about the system structure. Considering the model uncertainty, the truth table of the Lin/Con/k or m/n: F system ($k < m < n$) is shown in Table 2. Items in bold represent the model uncertainty. The system fails if and only if k or m consecutive components fail.

The system reliability is computed by the equation $(m_2^{\Omega_1} \otimes m_3^{\Omega_2} \otimes ... \otimes m_{n+1}^{\Omega_n} \otimes m_1^{\Omega_1 \Omega_2 ... \Omega_n \Omega_S})^{\downarrow \Omega_S}$. Because of the model uncertainty, the obtained system reliability is bounded by an interval.

Step 2: Initialize all positions of the system with the least reliable component;

Table 2: Truth table of the Lin/Con/k or m/n: F system.

P_1	P_2	P_3	...	P_k	P_{k+1}	P_{k+2}	P_{k+3}	...	P_m	P_{m+1}	P_{m+2}	P_{m+3}	...
0	0	0	...	0	{0,1}	{0,1}	{0,1}	...	{0,1}	{0,1}	{0,1}	{0,1}	...
0	0	0	...	0	0	0	0	...	0	{0,1}	{0,1}	{0,1}	...
{0,1}	0	0	...	0	0	{0,1}	{0,1}	...	{0,1}	{0,1}	{0,1}	{0,1}	...
{0,1}	0	0	...	0	0	0	0	...	0	0	{0,1}	{0,1}	...
{0,1}	{0,1}	0	...	0	0	0	{0,1}	...	{0,1}	{0,1}	{0,1}	{0,1}	...
{0,1}	{0,1}	0	...	0	0	0	0	...	0	0	0	{0,1}	...
...	...	...	...	...	...	...	...	...	...	...	...	...	...
{0,1}	{0,1}	{0,1}	...	{0,1}	{0,1}	{0,1}	{0,1}	...	{0,1}	{0,1}	{0,1}	{0,1}	...
{0,1}	{0,1}	{0,1}	...	{0,1}	{0,1}	{0,1}	{0,1}	...	{0,1}	{0,1}	{0,1}	{0,1}	...
{0,1}	{0,1}	{0,1}	...	{0,1}	{0,1}	{0,1}	{0,1}	...	{0,1}	{0,1}	{0,1}	{0,1}	...
{0,1}	{0,1}	{0,1}	...	{0,1}	{0,1}	{0,1}	{0,1}	...	{0,1}	{0,1}	{0,1}	{0,1}	...
{0,1}	{0,1}	{0,1}	...	{0,1}	{0,1}	{0,1}	{0,1}	...	{0,1}	{0,1}	{0,1}	{0,1}	...
All other combinations													

P_{n-m-2}	P_{n-m-1}	P_{n-m}	P_{n-m+1}	...	P_{n-k-2}	P_{n-k-1}	P_{n-k}	P_{n-k+1}	...	P_{n-2}	P_{n-1}	P_n	System
{0,1}	{0,1}	{0,1}	{0,1}	...	{0,1}	{0,1}	{0,1}	{0,1}	...	{0,1}	{0,1}	{0,1}	**{0,1}**
{0,1}	{0,1}	{0,1}	{0,1}	...	{0,1}	{0,1}	{0,1}	{0,1}	...	{0,1}	{0,1}	{0,1}	0
{0,1}	{0,1}	{0,1}	{0,1}	...	{0,1}	{0,1}	{0,1}	{0,1}	...	{0,1}	{0,1}	{0,1}	**{0,1}**
{0,1}	{0,1}	{0,1}	{0,1}	...	{0,1}	{0,1}	{0,1}	{0,1}	...	{0,1}	{0,1}	{0,1}	0
{0,1}	{0,1}	{0,1}	{0,1}	...	{0,1}	{0,1}	{0,1}	{0,1}	...	{0,1}	{0,1}	{0,1}	**{0,1}**
{0,1}	{0,1}	{0,1}	{0,1}	...	{0,1}	{0,1}	{0,1}	{0,1}	...	{0,1}	{0,1}	{0,1}	0
...	...	...	...	...	...	...	...	...	...	...	...	...	...
{0,1}	{0,1}	{0,1}	{0,1}	...	{0,1}	0	0	0	...	0	{0,1}	{0,1}	**{0,1}**
{0,1}	0	0	0	...	0	0	0	0	...	0	{0,1}	{0,1}	0
{0,1}	{0,1}	{0,1}	{0,1}	...	{0,1}	{0,1}	0	0	...	0	0	{0,1}	**{0,1}**
{0,1}	{0,1}	0	0	...	0	0	0	0	...	0	0	{0,1}	0
{0,1}	{0,1}	{0,1}	{0,1}	...	{0,1}	{0,1}	{0,1}	0	...	0	0	0	**{0,1}**
{0,1}	{0,1}	{0,1}	0	...	0	0	0	0	...	0	0	0	0
All other combinations													1

Because the component reliabilities are given in a non-decreasing order, component $C1$ is assigned to all positions of the system. The obtained system reliability is an interval-valued number.

Step 3: Compute interval-valued BIs of all unassigned positions;

Interval-valued BI of position Pi is computed using Eq. 9.

Step 4: Allocate the unassigned most reliable component to the unassigned position with the largest interval-valued BI;

The order relation $\geqslant_{max}$ is used to find the unassigned position with the largest interval-valued BI.

Step 5: If all components are assigned, stop the process and output the assignment of components and the system reliability. Otherwise, go back to Step 3.

4. Case study

In this section, the proposed extended LK heuristics under parametric uncertainty and model uncertainty are applied to an oil pipeline system to find the optimal arrangement of pumps under uncertainty.

4.1. System description

Consider an oil pipeline system that transports oil through pipeline as shown in Fig. 2 [7]. There are 10 equally spaced pump stations along the pipeline. If the distance between two pump stations is 100 km, and a pump can pressurize the oil to move through the pipeline for up to 200 km. Thus, the failure of two consecutive pumps causes the interruption of the oil transportation. This is a Lin/Con/2/10: F system.

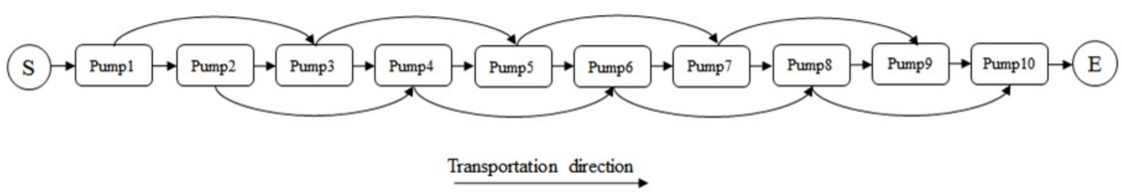

Figure 2: Illustration of the oil pipeline system.

Initial reliabilities of pumps are given in a non-decreasing order as follows

$$r = \begin{Bmatrix} 0.949106, 0.949243, 0.949787, 0.949842, 0.949881 \\ 0.949930, 0.949969, 0.950079, 0.950191, 0.950208 \end{Bmatrix}$$

4.2. Optimization without considering uncertainty

This section models the oil pipeline system in evidential network and applies the LK heuristic to search its optimal arrangement of pumps without considering uncertainty.

The graphical representation of the oil pipeline system is shown in Fig. 3. There are 11 variables representing the system and all pumps, and 11 bbas representing knowledge about variables and system structure. All variables are binary, and their frames of discernment are $\Omega = \{0, 1\}$, where 0 and 1 denote the failed state and the working state.

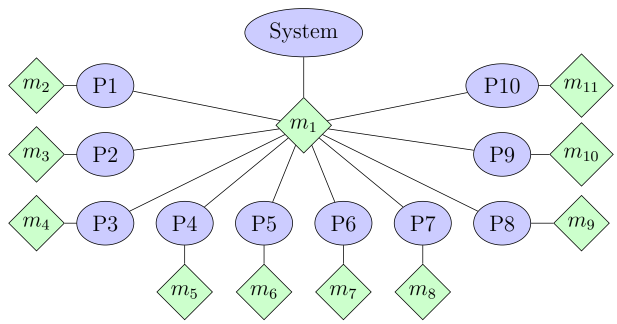

Figure 3: Evidential network of the oil pipeline system.

m_1 corresponds to the knowledge about the system structure which is described by the truth table in Table 3.

Table 3: Truth table of the Lin/Con/2/10: F system.

P1	P2	P3	P4	P5	P6	P7	P8	P9	P10	System
0	0	{0,1}	{0,1}	{0,1}	{0,1}	{0,1}	{0,1}	{0,1}	{0,1}	0
{0,1}	0	0	{0,1}	{0,1}	{0,1}	{0,1}	{0,1}	{0,1}	{0,1}	0
{0,1}	{0,1}	0	0	{0,1}	{0,1}	{0,1}	{0,1}	{0,1}	{0,1}	0
{0,1}	{0,1}	{0,1}	0	0	{0,1}	{0,1}	{0,1}	{0,1}	{0,1}	0
{0,1}	{0,1}	{0,1}	{0,1}	0	0	{0,1}	{0,1}	{0,1}	{0,1}	0
{0,1}	{0,1}	{0,1}	{0,1}	{0,1}	0	0	{0,1}	{0,1}	{0,1}	0
{0,1}	{0,1}	{0,1}	{0,1}	{0,1}	{0,1}	0	0	{0,1}	{0,1}	0
{0,1}	{0,1}	{0,1}	{0,1}	{0,1}	{0,1}	{0,1}	0	0	{0,1}	0
{0,1}	{0,1}	{0,1}	{0,1}	{0,1}	{0,1}	{0,1}	{0,1}	0	0	0
All other combinations										1

m_2, m_3, ..., m_{11} represent the knowledge about the reliability data of pumps in positions $P1$, $P2$, ..., $P10$.

Because the reliabilities of pumps are given in a non-decreasing order, all positions of the system are initialized with the least reliable pump $C1$, so we have

- $m_2^{\Omega_{P1}}(\{0\}) = ... = m_{11}^{\Omega_{P10}}(\{0\}) = 0.050894$

- $m_2^{\Omega_{P1}}(\{1\}) = ... = m_{11}^{\Omega_{P10}}(\{1\}) = 0.949106$

The initial system reliability R_S is 0.977872.

Because all positions are unassigned yet, BIs of all positions are calculated using Eq. 8 as follows

$$BI = \begin{Bmatrix} 0.047587, 0.095291, 0.092976, 0.093088, 0.093083 \\ 0.093083, 0.093088, 0.092976, 0.095291, 0.047587 \end{Bmatrix}$$

The Lin/Con/k/n: F system is symmetrical, so position $P2$ and position $P9$ have the same largest BI. Therefore, the unassigned most reliable pump $C10$ is assigned to position $P2$. After this assignment, there are still unassigned pumps, therefore, it needs to calculate BIs of the rest unassigned positions and assign unassigned pumps to those positions.

After nine iterations, all pumps are assigned. The obtained optimal assignment of pumps is [C1 C10 C3 C8 C5 C6 C7 C4 C9 C2] and the system reliability is $R_S = 0.978534$.

4.3. Optimization under parametric uncertainty

Parametric uncertainty related to reliability data of pumps is taken into account. Interval-valued reliability data of pumps are given as follows

$$\begin{aligned} r_1 &= [0.932523, 0.963177] & r_2 &= [0.917485, 0.978546] \\ r_3 &= [0.929536, 0.961255] & r_4 &= [0.931565, 0.989156] \\ r_5 &= [0.925156, 0.968215] & r_6 &= [0.911284, 0.971602] \\ r_7 &= [0.939265, 0.986542] & r_8 &= [0.936456, 0.976232] \\ r_9 &= [0.914101, 0.969165] & r_{10} &= [0.941564, 0.970556] \end{aligned}$$

The graphical representation of the Lin/Con/2/10: F system and the bba m_1 are the same as the case without uncertainty. Due to the parametric uncertainty, bbas assigned to components are changed to be

$$\begin{aligned} m_{i+1}^{\Omega_i}(\{1_i\}) &= \underline{r}_i \\ m_{i+1}^{\Omega_i}(\{0_i\}) &= 1-\overline{r}_i \\ m_{i+1}^{\Omega_i}(\{0_i,1_i\}) &= \overline{r}_i-\underline{r}_i \end{aligned}$$

Set the level of the decision maker's preference ρ_0 to be 2. Pumps are sorted in a non-decreasing order using the order relation operators and the obtained ordering of pumps is [C3 C9 C6 C1 C5 C2 C10 C8 C4 C7].

All positions are initialized with the least reliable pump $C3$, so we have

- $m_2^{\Omega_{P1}}(\{0\}) = ... = m_{11}^{\Omega_{P10}}(\{0\}) = 0.929536$
- $m_2^{\Omega_{P1}}(\{1\}) = ... = m_{11}^{\Omega_{P10}}(\{1\}) = 0.038745$
- $m_2^{\Omega_{P1}}(\{0,1\}) = ... = m_{11}^{\Omega_{P10}}(\{0,1\}) = 0.031719$

The initial system reliability R_S is [0.958568,0.986999].

All positions are unassigned yet, therefore, interval-valued BIs of all positions are calculated using Eq. 9 as follows

$$\begin{aligned} BI_1 &= [0.011545, 0.089045] & BI_2 &= [0.051593, 0.149939] \\ BI_3 &= [0.049968, 0.145955] & BI_4 &= [0.050037, 0.146216] \\ BI_5 &= [0.050034, 0.146199] & BI_6 &= [0.050034, 0.146199] \\ BI_7 &= [0.050037, 0.146216] & BI_8 &= [0.049968, 0.145955] \\ BI_9 &= [0.051593, 0.149939] & BI_{10} &= [0.011545, 0.089045] \end{aligned}$$

The Lin/Con/k/n: F system is symmetrical, so position $P2$ and position $P9$ have the same largest interval-valued BI. Therefore, the unassigned most reliable pump $C7$ is assigned to position $P2$. After this assignment, there are still unassigned pumps, therefore, it needs to calculate interval-valued BIs of the rest unassigned positions and assign unassigned pumps to those positions.

After nine iterations, all pumps are assigned. The optimal assignment of pumps under parametric uncertainty obtained by the extended LK heuristic is [C3 C7 C6 C8 C1 C10 C2 C5 C4 C9], and the system reliability $R_S = [0.957840, 0.994612]$.

4.4. Optimization under model uncertainty

Model uncertainty related to the system structure is taken into account. Due to the limited knowledge about the pump, it is not sure that a pump can pressurize the oil up to 200 km or 300 km. Therefore, the studied system becomes a Lin/Con/2or3/10: F system.

The graphical representation of the Lin/Con/2or3/10: F system and the bbas assigned to components are the same as the case without uncertainty. Because of the model uncertainty, the truth table of m_1 is changed as shown in Table 4. Items in bold represent the model uncertainty.

Table 4: Truth table of the Lin/Con/2or3/10: F system.

P1	P2	P3	P4	P5	P6	P7	P8	P9	P10	System
0	0	0	{0,1}	{0,1}	{0,1}	{0,1}	{0,1}	{0,1}	{0,1}	0
{0,1}	0	0	0	{0,1}	{0,1}	{0,1}	{0,1}	{0,1}	{0,1}	0
{0,1}	{0,1}	0	0	0	{0,1}	{0,1}	{0,1}	{0,1}	{0,1}	0
{0,1}	{0,1}	{0,1}	0	0	0	{0,1}	{0,1}	{0,1}	{0,1}	0
{0,1}	{0,1}	{0,1}	{0,1}	0	0	0	{0,1}	{0,1}	{0,1}	0
{0,1}	{0,1}	{0,1}	{0,1}	{0,1}	0	0	0	{0,1}	{0,1}	0
{0,1}	{0,1}	{0,1}	{0,1}	{0,1}	{0,1}	0	0	0	{0,1}	0
{0,1}	{0,1}	{0,1}	{0,1}	{0,1}	{0,1}	{0,1}	0	0	0	0
0	0	1	{0,1}	{0,1}	{0,1}	{0,1}	{0,1}	{0,1}	{0,1}	**{0,1}**
1	0	0	1	{0,1}	{0,1}	{0,1}	{0,1}	{0,1}	{0,1}	**{0,1}**
{0,1}	1	0	0	1	{0,1}	{0,1}	{0,1}	{0,1}	{0,1}	**{0,1}**
{0,1}	{0,1}	1	0	0	1	{0,1}	{0,1}	{0,1}	{0,1}	**{0,1}**
{0,1}	{0,1}	{0,1}	1	0	0	1	{0,1}	{0,1}	{0,1}	**{0,1}**
{0,1}	{0,1}	{0,1}	{0,1}	1	0	0	1	{0,1}	{0,1}	**{0,1}**
{0,1}	{0,1}	{0,1}	{0,1}	{0,1}	1	0	0	1	{0,1}	**{0,1}**
{0,1}	{0,1}	{0,1}	{0,1}	{0,1}	{0,1}	1	0	0	1	**{0,1}**
{0,1}	{0,1}	{0,1}	{0,1}	{0,1}	{0,1}	{0,1}	1	0	0	**{0,1}**
All other combinations										1

All positions are initialized with the least reliable pump $C1$, so we have

- $m_2^{\Omega_{P1}}(\{0\}) = ... = m_{11}^{\Omega_{P10}}(\{0\}) = 0.050894$

- $m_2^{\Omega_{P1}}(\{1\}) = ... = m_{11}^{\Omega_{P10}}(\{1\}) = 0.949106$

The initial system reliability R_S is 0.977872.

All positions are unassigned yet, therefore, interval-valued BIs of all positions are calculated using Eq. 9 as follows

$$
\begin{aligned}
BI_1 &= [-0.0163670, 0.0664109] & BI_2 &= [-0.0116068, 0.1118127] \\
BI_3 &= [-0.0093920, 0.1097399] & BI_4 &= [-0.0095105, 0.1098399] \\
BI_5 &= [-0.0095102, 0.1098348] & BI_6 &= [-0.0095102, 0.1098348] \\
BI_7 &= [-0.0095105, 0.1098399] & BI_8 &= [-0.0093920, 0.1097399] \\
BI_9 &= [-0.0116068, 0.1118127] & BI_{10} &= [-0.0163670, 0.0664109]
\end{aligned}
$$

The Lin/Con/k/n: F system is symmetrical, so position $P2$ and position $P9$ have the same largest interval-valued BI. Therefore, the unassigned most reliable pump $C10$ is assigned to position $P2$. After this assignment, there are still unassigned pumps, therefore, it needs to calculate interval-valued BIs of unassigned positions and assign unassigned pumps to those positions.

After nine iterations, all pumps are assigned. The optimal assignment under model uncertainty obtained by the extended LK heuristic is [C1 C10 C3 C8 C5 C6 C7 C4 C9 C2], and the system reliability $R_S = [0.978534, 0.999038]$.

5. Conclusion

In this paper, we propose extended LK heuristics in evidential network to find the optimal arrangement of components of Lin/Con/k/n: F systems under parametric uncertainty (related to the failure data of components) and model uncertainty (related to the system structure). Because the obtained system reliability and BIs of positions are interval-valued numbers when uncertainties are considered, interval analysis is integrated into the extended LK heuristics to compare intervals.

In this work, interval-valued numbers are used to represent the parametric uncertainty. In the future, other representations of uncertain parameters will be analysed, such as probability distributions or fuzzy numbers. In this case, other kinds of uncertainty theories and decision criterion will be integrated into the extended LK heuristics.

Acknowledgement

This work was sponsored by Shanghai Pujiang Program under grant number 16PJ1404500, and French National Research Agency under grant numbers ANR-13-JS03-0007, ANR-11-IDEX-0004-02. The authors are thankful to the editor and anonymous reviewers for their valuable comments that help us improve the quality of the paper.

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