Reliability analysis with ill-known probabilities and dependencies

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Introduction

- System reliability is computed by using the reliabilities of its components and a structure function linking the states of components to the system states.
- First assumption : Systems and components are supposed binary : either working or failing.
- Second assumption : Probabilities of component failures are precisely known.
- Third assumption : Components failures are stochastically independent.

Introduction

- Second assumption : is quite strong (few or no data are available, modelling expert opinion).
- The use of precise probabilities means adding some assumption not supported by available evidence (e.g., using maximum entropy principle).
- An alternative is to include the imprecision by considering probability bounds.
- The third assumption : is in general more likely to hold.
- We have the case where the possible dependencies between components are unknown or only partially known.

Introduction

- Both issues have been investigated, in general settings, by imprecise probability theories (Walley, Couso et al.).
- However, the specific problem of assessing a system reliability under such conservative assumptions has only been explored in a very few works (Utkin, Berleant, Pedroni, Fetz).
- The case of partially specified independence in even less (Hill, Troffaes).
- In this presentation, we recall some of the main results of these previous works, setting them in a general framework.
- We also provide some preliminary results about consecutive k-out-of-n systems, that have not been studied yet within an imprecise probabilistic framework.

Preliminaries

- A set of components $X_1, \dots, X_N$, whose values are described by domain $\mathscr{X} = \{1, 0\}$ (1 for working and 0 for not working).
- A set of all possible system states $\mathscr{X}^N = \times_{i=1}^N \mathscr{X}$.
- A state of the system $\mathbf{x} = (x_1, \dots, x_N) \in \mathscr{X}^N$.
- The uncertainty about X_i is described by two bounds

$$\underline{p}_i = \underline{p}(X_i = 1) \text{ and } \overline{p}_i = \overline{p}(X_i = 1)$$

- The assessment "component X_i has a probability of working that is between 0.8 and 0.9" corresponds to $\underline{p}_i = 0.8$, $\overline{p}_i = 0.9$.
- the structure function $\phi : \mathscr{X}^N \to \{0, 1\}$ maps each system state $\mathbf{x} \in \mathscr{X}^N$ to 1 if the system works in this state, and 0 if the system fails in this state.
- $\phi^{-1}(0)$ and $\phi^{-1}(1) \subseteq \mathscr{X}^N$ respectively denote the set of states for which the system fails and the set of states for which it works.
- The system is coherent : if $\mathbf{x} \geq \mathbf{x}'$ then $\phi(\mathbf{x}) \geq \phi(\mathbf{x}')$.

Problem formulation

- Estimation of the uncertainty bounds of $\phi^{-1}(1)$: $\underline{p}(\phi^{-1}(1))$ and $\overline{p}(\phi^{-1}(1))$, given our knowledge about the component uncertainties.
- The problem of estimation of $\underline{p}(\phi^{-1}(1))$ can be expressed as :

$$\min_{p} \sum_{\mathbf{x}\in\mathscr{X}^N, \phi(\mathbf{x})=1} p(\mathbf{x}) \quad (1)$$

under the constraints

$$\underline{p}_i \leq \sum_{\mathbf{x}\in\mathscr{X}^N, x_i=1} p(\mathbf{x}) \leq \overline{p}_i, \forall i \in [1, N] \quad (2)$$

$$\sum_{\mathbf{x}\in\mathscr{X}_{1:N}} p(\mathbf{x}) = 1, p(\mathbf{x}) \geq 0 \ \ \forall \mathbf{x} \in \mathscr{X}^N.$$

- It is a NP-hard problem.

Problem formulation : Simplification

- We consider components with identical uncertainty : $\underline{p}_i = \underline{p}_w$ and $\overline{p}_i = \overline{p}_w$ for all i.
- The system is coherent : the minimum in (1) is obtained by considering $p_i = \underline{p}_i$ for every i.
- We can replace (2) by

$$\underline{p}_i = \sum_{\mathbf{x} \in \mathscr{X}_{1:N}, x_i = 1} p(\mathbf{x})$$

for every i.

Case of independent components

$$p(\mathbf{x}) = \prod_{i, x_i = 1} p_i \prod_{i, x_i = 0} (1 - p_i), \qquad (3)$$

- Obtaining $\underline{p}(\phi^{-1}(1)), \overline{p}(\phi^{-1}(1))$ simply consists in replacing the probability that a component will be working by the appropriate bound in Eq. (3).
- For instance take $p_i = \underline{p}_i$ to compute $\underline{p}(\phi^{-1}(1))$.

$k/n:F$	$\overline{p}(\phi^{-1}(1)) =$	$\sum_{i=0}^{k-1}\binom{n}{i}(\overline{p}_w)^{n-i}(1-\overline{p}_w)^i$
	$\underline{p}(\phi^{-1}(1)) =$	$\sum_{i=0}^{k-1}\binom{n}{i}(\underline{p}_w)^{n-i}(1-\underline{p}_w)^i$
$L:k/n:F$	$\overline{p}(\phi^{-1}(1)) =$	$\sum_{i=0}^{n}\binom{n-i\cdot k}{i}(-1)^i(\overline{p}_w(1-\overline{p}_w)^k)^i$ $-(1-\overline{p}_w)^k\sum_{i=0}^{k-1}\binom{n-k(i+1)}{i}(-1)^i(\overline{p}_w(1-\overline{p}_w)^k)^i$
	$\underline{p}(\phi^{-1}(1)) =$	$\sum_{i=0}^{n}\binom{n-i\cdot k}{i}(-1)^i(\underline{p}_w(1-\underline{p}_w)^k)^i$ $-(1-\underline{p}_w)^k\sum_{i=0}^{k-1}\binom{n-k(i+1)}{i}(-1)^i(\underline{p}_w(1-\underline{p}_w)^k)^i$
$C:k/n:F$	$\overline{p}(\phi^{-1}(1)) =$	$\sum_{i=0}^{n}\binom{n-i\cdot k}{i}(-1)^i(\overline{p}_w(1-\overline{p}_w)^k)^i$ $+k\sum_{i=0}^{k-1}\binom{n-k(i+1)-1}{i}(-1)^i(\overline{p}_w(1-\underline{p}_w)^k)^{i+1}-(1-\overline{p}_w)^n$
	$\underline{p}(\phi^{-1}(1)) =$	$\sum_{i=0}^{n}\binom{n-i\cdot k}{i}(-1)^i(\underline{p}_w(1-\underline{p}_w)^k)^i$ $+k\sum_{i=0}^{k-1}\binom{n-k(i+1)-1}{i}(-1)^i(\underline{p}_w(1-\underline{p}_w)^k)^{i+1}-(1-\underline{p}_w)^n$

TABLE: Reliability bound formulas in the independent case

Case of independent components

- Consider components such that $[\underline{p}_w, \overline{p}_w] = [0.95, 0.99]$ and 2/4 systems.
- Using the formulas of Table 1, we obtain :

$$2/4 : F \quad [\underline{p}(\phi^{-1}(1)), \overline{p}(\phi^{-1}(1))] = [0.9859, 0.9993];$$

$$C : 2/4 : F \quad [\underline{p}(\phi^{-1}(1)), \overline{p}(\phi^{-1}(1))] = [0.9905, 0.9996].$$

$$L : 2/4 : F \quad [\underline{p}(\phi^{-1}(1)), \overline{p}(\phi^{-1}(1))] = [0.9927, 0.9997];$$

Case of unknown independence

- For the $k/n:F$ systems, we recall that Utkin indicates that

$$\underline{p}(\phi^{-1}(1)) = \max(0, (n-k+1)\underline{p}_w + k - n);$$

$$\overline{p}(\phi^{-1}(1)) = \min(1, k\overline{p}_w).$$

- In the case of series and parallel systems : we retrieve the Frechet bounds.
- The cases of $L:k/n:F$ and $C:k/n:F$ systems have not been investigated up to now.
- Obtaining bounds under an assumption of unknown independence for such systems is harder than for the assumption of independence.

Case of unknown independence

Proposition

Given component uncertainty $\underline{p}_w, \overline{p}_w$ and unknown independence, the lower and upper bounds $\underline{p}(\phi^{-1}(1)), \overline{p}(\phi^{-1}(1))$ for a $L:2/3:F$ system are

$$\underline{p}(\phi^{-1}(1)) = \underline{p}_w$$

$$\overline{p}(\phi^{-1}(1)) = \min(1, 2\overline{p}_w)$$

Consider components such that $[\underline{p}_w, \overline{p}_w] = [0.95, 0.99]$:

$$2/3:F \quad [\underline{p}(\phi^{-1}(1)), \overline{p}(\phi^{-1}(1))] = [0.9, 1];$$

$$L:2/3:F \quad [\underline{p}(\phi^{-1}(1)), \overline{p}(\phi^{-1}(1))] = [0.95, 1].$$

The lower bound of the $L:2/3:F$ is slightly higher than the bound of the $2/3:F$ system.

Case of unknown independence

Proposition

Given component uncertainty $\underline{p}_w, \overline{p}_w$ and unknown independence, the lower and upper bounds $\underline{p}(\phi^{-1}(1)), \overline{p}(\phi^{-1}(1))$ for a $C:2/4:F$ system are

$$\underline{p}(\phi^{-1}(1)) = \max(0, 2\underline{p}_w - 1)$$

$$\overline{p}(\phi^{-1}(1)) = \min(1, 2\overline{p}_w)$$

Consider components such that $[\underline{p}_w, \overline{p}_w] = [0.95, 0.99]$:

$$2/4:F \quad [\underline{p}(\phi^{-1}(1)), \overline{p}(\phi^{-1}(1))] = [0.85, 1];$$

$$C:2/4:F \quad [\underline{p}(\phi^{-1}(1)), \overline{p}(\phi^{-1}(1))] = [0.9, 1].$$

The lower bound of the $C:2/4:F$ is slightly higher than the bound of the $2/4:F$ system.

Discussion and conclusions

- We have recalled results regarding the evaluation of lower and upper reliabilities of systems.
- We have settled them as a generic optimization problem.
- We have proposed closed formulas (particularly consecutive k-out-of-n :F systems) for the evaluation of lower and upper reliabilities of systems in the independent case.
- We have started to investigate the case of unknown independence and give closed formulas for some particular configurations.
- We intend to study how to integrate some known dependency information in the constrained problem.
- We intend to study other aspects of consecutive k-out-of-n systems when probabilities or dependencies are ill-known (importance measures, multi-state systems, design optimization).

Thank you for your attention !

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