Some of the things you wanted to know about uncertainty (and were too busy to ask)

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Me, myself and I

- 2005-2008: PhD student
  - Main topic: uncertainty modelling and treatment in nuclear safety
- 2009-2011: research engineer in agronomical research institute
  - Main topic: aid-decision in agronomical production chains
- 2011- ?: researcher at Centre National de la Recherche Scientifique (CNRS), in the Heudiasyc joint unit research
  - Main (current) topics: reliability analysis and machine learning

Only common point: the modelling and handling of uncertainty (including an imprecision component)

Tutorial goals and contents

What you will find in this tutorial

- Mostly practical considerations about uncertainty
- An overview of "mainstream" uncertainty theories
- Elements and illustrations of their use to
  - build or learn uncertainty representations
  - make inference (and decision)
- A "personal" view about those things

What you will not find in this tutorial

- A deep and exhaustive study of a particular topic
- Elements about other important problems (learning models, information fusion/revision)

Plan

Section goals: it's all about basics

- Introduce a basic framework
- Give basic ideas about uncertainty
- Introduce some basic problems

A generic framework

datum: ω | data (population): $\{\omega_1, \dots, \omega_n\}$ + model

singular information | generic information

- model describes a relation in data space
- singular information: concern a particular situation/individual
- generic information: describe a general relationship, the behaviour of a population, ...

Uncertainty origins

Uncertainty: inability to answer precisely a question about a quantity

Can concern both:

- Singular information
  - items in a data-base, values of some logical variables, time before failure of **a** component
- Generic information
  - parameter values of classifiers/regression models, time before failure of component**s**, truth of a logical sentence ("birds fly")

Main origins

- **Variability** of a population $\rightarrow$ only concerns generic information
- **Imprecision** due to a lack of information
- **Conflict** between different sources of information (data/expert)

Some examples

datum: ω

data (population): $\{\omega_1, \dots, \omega_n\}$ + model

singular information | generic information

Classification

- Data space=input features $\mathcal{X} \times$ (structured) classes $\mathcal{Y}$
- model: classifier with parameters
- Uncertainty: mostly about model parameters
- Common problem: predict classes of individuals (singular information)

Some examples

datum: ω

data (population): $\{\omega_1, \dots, \omega_n\}$

+ model

singular information

generic information

Risk and reliability analysis

- Data space=input variables $\mathcal{X} \times$ output variable(s) $\mathcal{Y}$
- Model: transfer/structure function $f : \mathcal{X} \rightarrow \mathcal{Y}$
- Uncertainty: very often about $\mathcal{X}$ (sometimes f parameters)
- Common problem: obtain information about $\mathcal{Y}$, either generic (failure of products) or singular (nuclear power plant)

Some examples

datum: ω

data (population): $\{\omega_1, \ldots, \omega_n\}$

+ model

singular information

generic information

Data mining/clustering

- Data space=data features
- Model: clusters, rules, ...
- Uncertainty: mostly about model parameters
- Common problem: obtain the model from data $\{\omega_1, \ldots, \omega_n\}$

Some examples

datum: ω	data (population): $\{\omega_1, \dots, \omega_n\}$	+	model

singular information | generic information

Data base querying

- Data space=data features
- Model: a query inducing preferences over observations
- Uncertainty: mostly about the query, sometimes data
- Common problem: retrieve and order interesting items in $\{\omega_1, \dots, \omega_n\}$

Some examples

datum: ω	data (population): $\{\omega_1, \dots, \omega_n\}$	+	model
singular information	generic information		

Propositional logic

- Data space=set of possible interpretations
- Model: set of sentences of the language
- Uncertainty: on sentences or on the state of some atoms
- Common problem: deduce the uncertainty about the truth of a sentence S from facts and knowledge

Handling uncertainty

datum: ω

data (population): $\{\omega_1, \ldots, \omega_n\}$

+

model

singular information

generic information

Common problems in one sentence

- **Learning**: use singular information to estimate generic information
- **Inference**: interrogate model and observations to deduce information on quantity of interest
- **Information fusion**: merge multiple information pieces about same quantity
- **Information revision**: merge new information with old one

Plan

Section goals

- Introduce main ideas of theories
- Provide elements about links between them
- Illustrate how to get uncertainty representations within each

Basic framework

Quantity S with possible **exclusive** states $\mathcal{S} = \{s_1, \ldots, s_n\}$

- ▷ $\mathcal{S}$: data feature, model parameter, ...

Basic tools

A confidence degree $\mu : 2^{|\mathcal{S}|} \to [0, 1]$ is such that

- $\mu(A)$: confidence $S \in A$
- $\mu(\emptyset) = 0$, $\mu(\mathcal{S}) = 1$
- $A \subseteq B \Rightarrow \mu(A) \leq \mu(B)$

Uncertainty modelled by 2 degrees $\underline{\mu}, \overline{\mu} : 2^{|\mathcal{S}|} \to [0, 1]$:

- $\underline{\mu}(A) \leq \overline{\mu}(A)$ (monotonicity)
- $\underline{\mu}(A) = 1 - \overline{\mu}(A^c)$ (duality)

Probability

Basic tool

A probability distribution $p : \mathcal{S} \to [0, 1]$ from which

- $\underline{\mu}(A) = \overline{\mu}(A) = \mu(A) = \sum_{s \in A} p(s)$
- $\mu(A) = 1 - \mu(A^c)$: auto-dual

Main interpretations

- **Frequentist [3]:** $\mu(A)$= number of times A observed in a population
  - ▷ only applies when THERE IS a population
- **Subjectivist [1]:** $\mu(A)$= price for gamble giving 1 if A happens, 0 if not
  - ▷ applies to singular situation and populations

Probability and imprecision: short comment

- Probability often partially specified over $\mathcal{S}$
- Probability on rest of $\mathcal{S}$ usually imprecise

A small example

- $\mathcal{S} = \{s_1, s_2, s_3, s_4\}$
- $p(s_1) = 0.1, p(s_2) = 0.4$
- we deduce $p(s_i) \in [0, 0.5]$ for $i = 3, 4$

Probability and imprecision: short comment

- Probability often partially specified over $\mathcal{S}$
- Probability on rest of $\mathcal{S}$ usually imprecise

Another (logical) example

- q, r two propositional variables
- $P(\neg q \vee r) = \alpha$, $P(q) = \beta$
- we deduce $P(r) \in [\beta - 1 + \alpha, \alpha]$

Sets

Basic tool

A set $E \subseteq \mathcal{S}$ with true value $S \in E$ from which

- $E \subseteq A \rightarrow \underline{\mu}(A) = \overline{\mu}(A) = 1$ (certainty truth in A)
- $E \cap A \neq \emptyset, E \cap A^c \neq \emptyset \rightarrow \underline{\mu}(A) = 0, \overline{\mu}(A) = 1$ (ignorance)
- $E \cap A = \emptyset \rightarrow \underline{\mu}(A) = \overline{\mu}(A) = 0$ (truth cannot be in A)

$\underline{\mu}, \overline{\mu}$ are binary $\rightarrow$ limited expressiveness

Classical use of sets:

- Interval analysis [2] (E is a subset of $\mathbb{R}$)
- Propositional logic (E is the set of models of a KB)

Other cases: robust optimisation, decision under risk, ...

In summary

Probabilities . . .

- (+) very informative quantification (do we need it?)
- (-) need lots of information (do we have it?)
- (-) if not enough, requires a choice (do we want to do that?)
- use probabilistic calculus (convolution, stoch. independence, . . .)

Sets . . .

- (+) need very few information
- (-) very rough quantification of uncertainty (Is it sufficient for us?)
- use set calculus (interval analysis, Cartesian product, . . .)

$\rightarrow$ Need of **frameworks bridging these two**

Possibility theory

Basic tool

A distribution $\pi : \mathcal{S} \to [0,1]$, usually with s_i such that $\pi(s_i) = 1$, from which

- $\overline{\mu}(A) = \max_{s \in A} \pi(s)$ (Possibility measure)
- $\underline{\mu}(A) = 1 - \overline{\mu}(A^c) = \min_{s \in A^c}(1 - \pi(s))$ (Necessity measure)

Sets E captured by $\pi(s) = 1$ if $s \in E$, 0 otherwise

$[\underline{\mu}, \overline{\mu}]$ as

- confidence degrees of **possibility theory** [9]
- bounds of an **ill-known probability** $\mu \Rightarrow \underline{\mu} \leq \mu \leq \overline{\mu}$ [10]

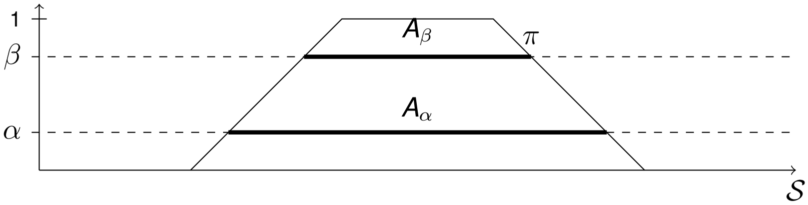
A nice characteristic: Alpha-cut [5]

Definition

$$A_\alpha = \{s \in \mathcal{S} | \pi(s) \geq \alpha\}$$

- $\underline{\mu}(A_\alpha) = 1 - \alpha$
- If $\beta \leq \alpha$, $A_\alpha \subseteq A_\beta$

Simulation: draw $\alpha \in [0, 1]$ and associate A_α



$\Rightarrow$ Possibilistic approach ideal to model **nested structures**

A basic distribution: simple support

A set E of most plausible values

A confidence degree $\alpha = \underline{\mu}(E)$

Two interesting cases:

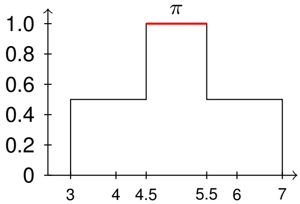
- Expert providing most plausible values E
- E set of models of a formula ϕ

Both cases extend to multiple sets $E_1, \ldots, E_p$:

- confidence degrees over nested sets [36]
- hierarchical knowledge bases [33]

pH value $\in [4.5, 5.5]$ with

$\alpha = 0.5$ ($\sim$ "more probable than")



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variables p, q

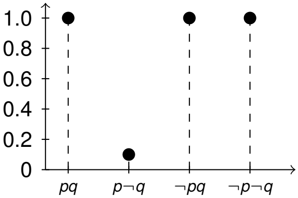
$$\Omega = \{pq, \neg pq, p\neg q, \neg p\neg q\}$$

$$\underline{\mu}(p \Rightarrow q) = 0.9$$

($\sim$ "almost certain")

$$E = \{pq, p\neg q, \neg p\neg q\}$$

- $\pi(pq) = \pi(p\neg q) = \pi(\neg p\neg q) = 1$
- $\pi(\neg pq) = 0.1$

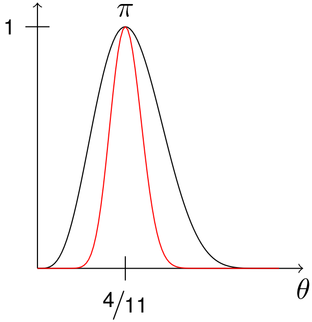


Normalized likelihood as possibilities [8] [26]

$$\pi(\theta) = {}^{\mathcal{L}(\theta|x)}/_{\max_{\theta \in \Theta} \mathcal{L}(\theta|x)}$$

Binomial situation:

- $\theta =$ success probability
- x number of observed successes
- x= 4 succ. out of 11
- x= 20 succ. out of 55



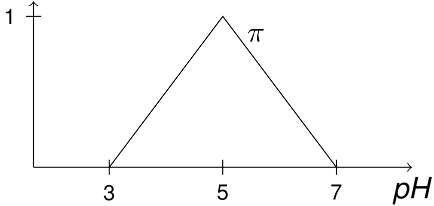
Partially specified probabilities [25] [32]

Triangular distribution: $[\underline{\mu}, \overline{\mu}]$ encompass all probabilities with

- mode/reference value M
- support domain $[a, b]$.

Getting back to pH

- $M = 5$
- $[a, b] = [3, 7]$



Other examples

- Statistical inequalities (e.g., Chebyshev inequality) [32]
- Linguistic information (fuzzy sets) [28]
- Approaches based on nested models

Possibility: limitations

$$\underline{\mu}(A) > 0 \Rightarrow \overline{\mu}(A) = 1$$

$$\overline{\mu}(A) < 1 \Rightarrow \underline{\mu}(A) = 0$$

$\Rightarrow$ interval $[\underline{\mu}(A), \overline{\mu}(A)]$ with one trivial bound

Does not include probabilities as special case:

$\Rightarrow$ possibility and probability at odds

$\Rightarrow$ respective calculus hard (sometimes impossible?) to reconcile

Going beyond

Extend the theory

- $\Rightarrow$ by complementing π with a lower distribution δ ($\delta \leq \pi$) [11], [31]
- $\Rightarrow$ by working with interval-valued possibility/necessity degrees [4]
- $\Rightarrow$ by working with sets of possibility measures [7]

Use a more general model

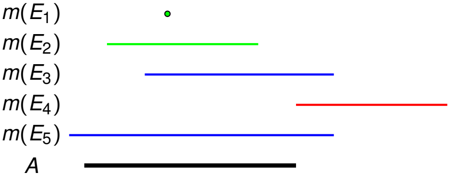
$\Rightarrow$ Random sets and belief functions

Random sets and belief functions

Basic tool

A positive distribution $m : 2^{|\mathcal{S}|} \to [0,1]$, with $\sum_E m(E) = 1$ and usually $m(\emptyset = 0)$, from which

- $\overline{\mu}(A) = \sum_{E \cap A \neq \emptyset} m(E)$ (Plausibility measure)
- $\underline{\mu}(A) = \sum_{E \subseteq A} m(E) = 1 - \overline{\mu}(A^c)$ (Belief measure)



$\underline{\mu}(A) = m(E_1) + m(E_2)$

$\overline{\mu}(A) = m(E_1) + m(E_2) + m(E_3) + m(E_5)$

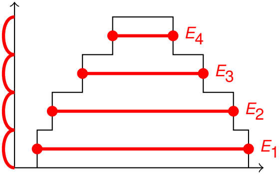
$[\underline{\mu}, \overline{\mu}]$ as

- confidence degrees of **evidence theory** [16], [17]
- bounds of an **ill-known probability** $\mu \Rightarrow \underline{\mu} \leq \mu \leq \overline{\mu}$ [14]

special cases

Measures $[\underline{\mu}, \overline{\mu}]$ include:

- Probability distributions: mass on atoms/singletons
- Possibility distributions: mass on nested sets



Frequencies of imprecise observations

Imprecise poll: "Who will win the next Wimbledon tournament?"

∘ N(adal) ∘ F(ederer) ∘ D(jokovic) ∘ M(urray) ∘ O(ther)

60 % replied $\{N, F, D\} \rightarrow m(\{N, F, D\}) = 0.6$

15 % replied "I do not know" $\{N, F, D, M, O\} \rightarrow m(\mathcal{S}) = 0.15$

10 % replied Murray $\{M\} \rightarrow m(\{M\}) = 0.1$

5 % replied others $\{O\} \rightarrow m(\{O\}) = 0.05$

...

P-box [34]

A pair $[\underline{F}, \overline{F}]$ of cumulative distributions

Bounds over events $[-\infty, x]$

- Percentiles by experts;
- Kolmogorov-Smirnov bounds;

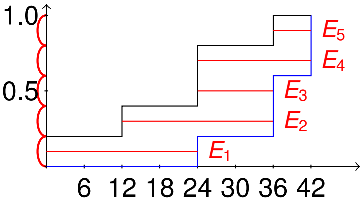
Can be extended to any pre-ordered space [30], [37] $\Rightarrow$ multivariate spaces!

Expert providing percentiles

$$0 \leq P([-\infty, 12]) \leq 0.2$$

$$0.2 \leq P([-\infty, 24]) \leq 0.4$$

$$0.6 \leq P([-\infty, 36]) \leq 0.8$$

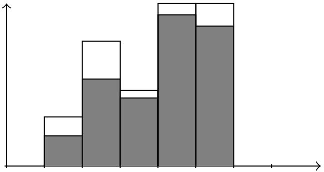


Other means to get random sets/belief functions

- Extending modal logic: probability of provability [18]
- Parameter estimation using pivotal quantities [15]
- Statistical confidence regions [29]
- Modify source information by its reliability [35]
- ...

Limits of random sets

- Not yet satisfactory extension of Bayesian/subjective approach
- Still some items of information it cannot model in a simple way, e.g.,
  - probabilistic bounds over atoms s_i (imprecise histograms, ...) [27];
  - comparative assessments such as $2P(B) \leq P(A)$



Imprecise probabilities

Basic tool

A set $\mathcal{P}$ of probabilities on $\mathcal{S}$ or an equivalent representation

- $\overline{\mu}(A) = \sup_{P \in \mathcal{P}} P(A)$ (Upper probability)
- $\underline{\mu}(A) = \inf_{P \in \mathcal{P}} P(A) = 1 - \overline{\mu}(A^c)$ (Lower probability)

$[\underline{\mu}, \overline{\mu}]$ as

- subjective lower and upper betting rates [23]
- bounds of an **ill-known probability measure** $\mu \Rightarrow \underline{\mu} \leq \mu \leq \overline{\mu}$ [19] [24]

Means to get Imprecise probabilistic models

- Include all representations seen so far . . .
- . . . and a couple of others
  - probabilistic comparisons
  - density ratio-class
  - expectation bounds
  - . . .
- fully coherent extension of Bayesian approach

$$\mathcal{P}(\theta|x) = L(\theta|x)\mathcal{P}(\theta)$$

  $\rightarrow$ often easy for "conjugate prior" [22]
- make probabilistic logic approaches imprecise [21, 20]

A crude summary

Possibility distributions

- **+**: very simple, natural in many situations (nestedness), extend set-based approach
- **-**: at odds with probability theory, limited expressiveness

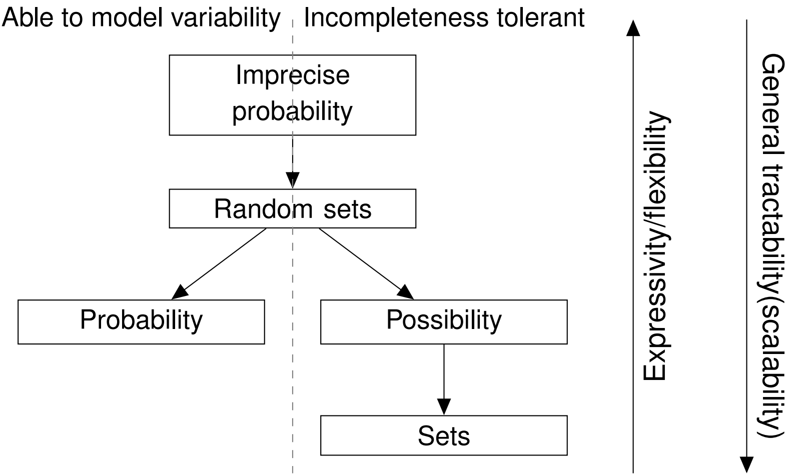
Random sets

- **+**: include probabilities and possibilities, include many models used in practice
- **-**: general models can be intractable, limited expressiveness

Imprecise probabilities

- **+**: most consistent extension of probabilistic approach, very flexible
- **-**: general models can be intractable

A not completely accurate but useful picture



Plan

Section goals

- Introduce the inference problem
- Introduce the notion of joint models
- Introduce how (basic) decision can be done
- Give some basic illustrations, mainly from regression/classification/reliability

The problem

datum: ω

data (population): $\{\omega_1, \ldots, \omega_n\}$ + model

singular information

generic information

- **uncertain Input**: marginal pieces of information on a part of the data space and the model
- Step 1: build a joint model from marginal information
- Step 2: deduce information (by propagation, conditioning, . . .) on data

Closeness requirement

datum: ω

data (population): $\{\omega_1, \ldots, \omega_n\}$ + model

singular information | generic information

- partial/marginal pieces of information are x
- joint model is x
- deduced information is x

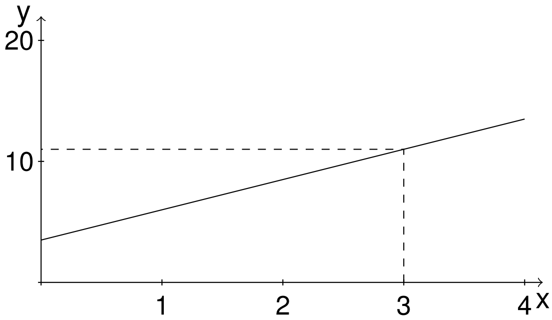
where $x \in \{$Prob. distribution, Poss. distribution, Belief function, Prob. set$\}$

Straight ahead

$$y = ax + b$$

No uncertainty: $a = 2.5, b = 3.5 \rightarrow$ joint: 2.5×3.5

Infer y when $x = 3$

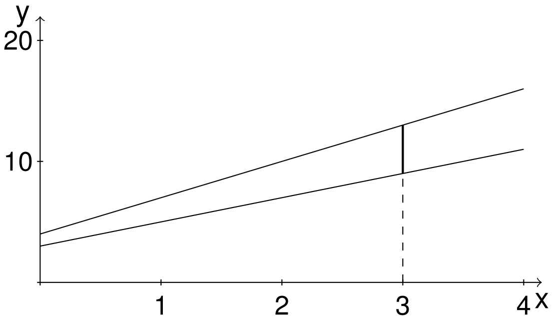


Straight ahead

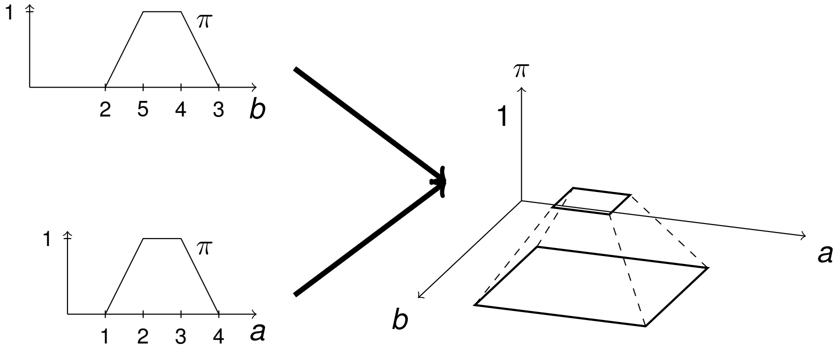
$$y = ax + b$$

Imprecision: $a \in [2,3], b \in [3,4] \rightarrow$ joint: $[2,3] \times [3,4]$

Infer y when $x = 3$



Joint models: possibilistic illustration

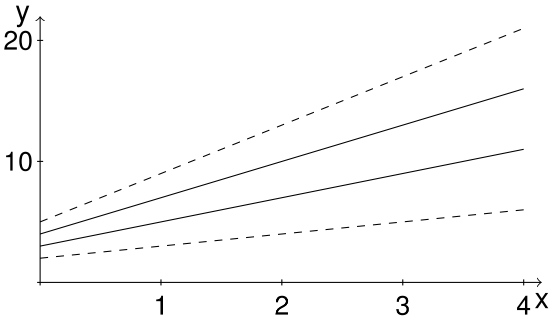


Fuzzy straight ahead

$$y = ax + b$$

Possibilistic uncertainty on a and b

Infer y when $x = 3$



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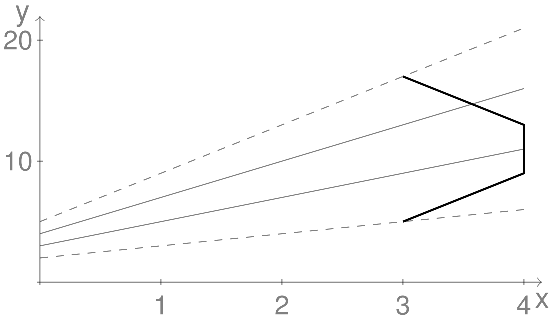


Reliable or not?

Model: structure function $\phi : C_1 \times C_2 \to S$

p(0)=0.1
p(1)=0.9

$\boxed{C_1 : \{0, 1\}}$

$$
\begin{aligned}
p(0 \times 0) &= 0.01\\
p(0 \times 1) &= 0.09\\
p(1 \times 0) &= 0.09\\
p(1 \times 1) &= 0.81
\end{aligned}
$$

$\boxed{S : \{0, 1\}}$

$p(0) = 0.01$
$p(1) = 0.99$

p(0)=0.1
p(1)=0.9

$\boxed{C_2 : \{0, 1\}}$

Reliable or not?

Model: structure function $\phi : C_1 \times C_2 \to S$

$m(\{0\}) = 0.05$
$m(\{1\}) = 0.75$
$m(\{0,1\}) = 0.2$

$\boxed{C_1 : \{0,1\}}$

$m(\{0\} \times \{0\}) = 0.025$
$m(\{1\} \times \{1\}) = 0.5625$
$m(\{0\} \times \{0,1\}) = 0.01$
$m(\{0,1\} \times \{0,1\}) = 0.04$
$\dots$

$\boxed{S : \{0,1\}}$

$m(\{0\}) = 0.0025$
$m(\{1\}) = 0.9575$
$m(\{0,1\}) = 0.04$

$m(\{0\}) = 0.05$
$m(\{1\}) = 0.75$
$m(\{0,1\}) = 0.2$

$\boxed{C_2 : \{0,1\}}$

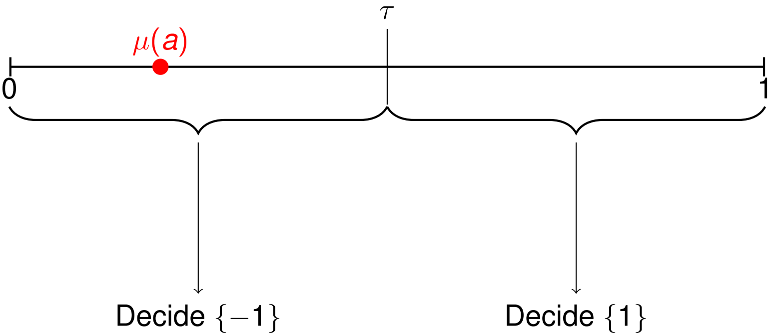
Two kinds of decision

- **Binary**: whether to take an action or not
    - risk/reliability analysis (take the risk or not)
    - logic (decide if a sentence is true)
    - binary classification
- **Non-binary**: decide among multiple choices
    - classification
    - control, planing, ...

Introducing imprecision $\simeq$ allowing for incomparability

Binary case

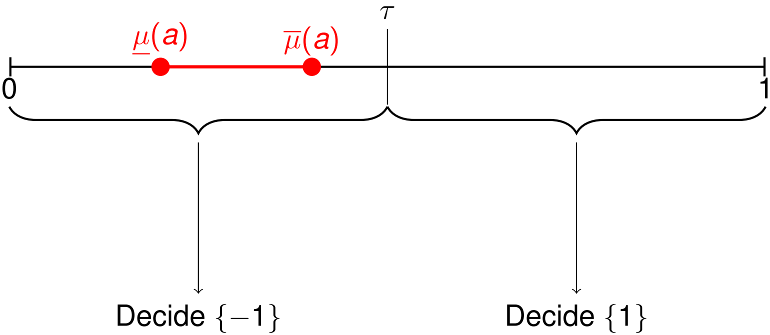
A threshold τ, two decisions $\{1, -1\}$



- Decision: -1

Binary case

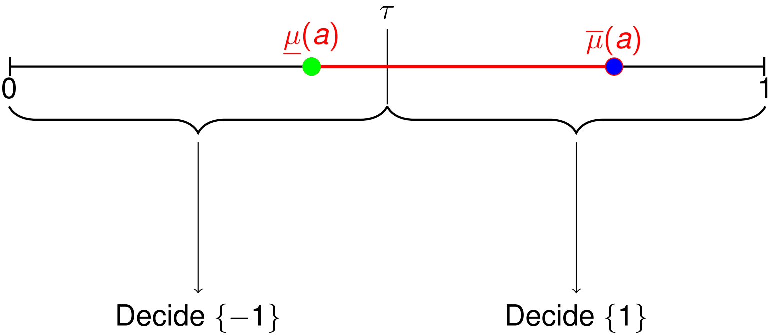
A threshold τ, two decisions $\{1, -1\}$



- **Decision**: -1

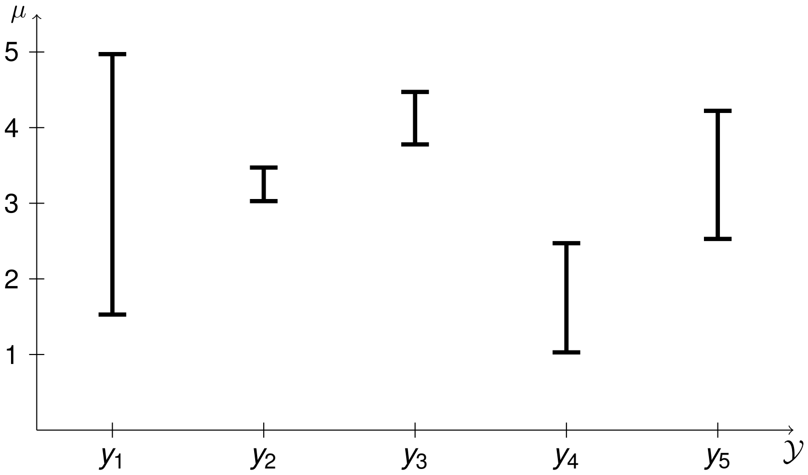
Binary case

A threshold τ, two decisions $\{1, -1\}$

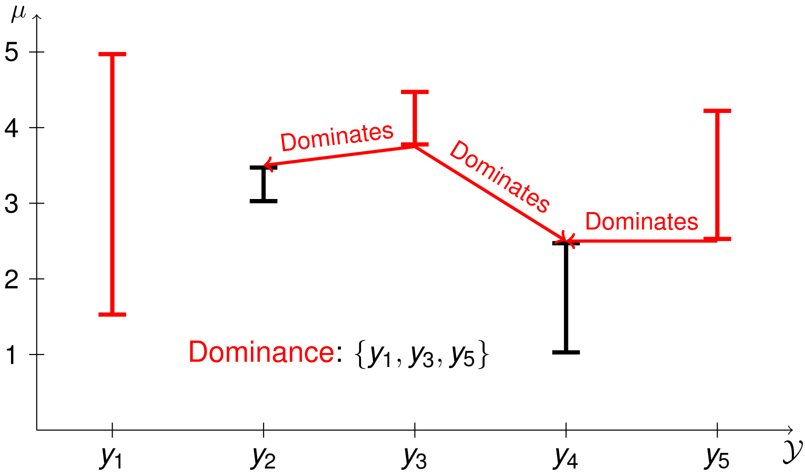


- No winner : -1,1
- Maximin : -1
- Maximax : 1

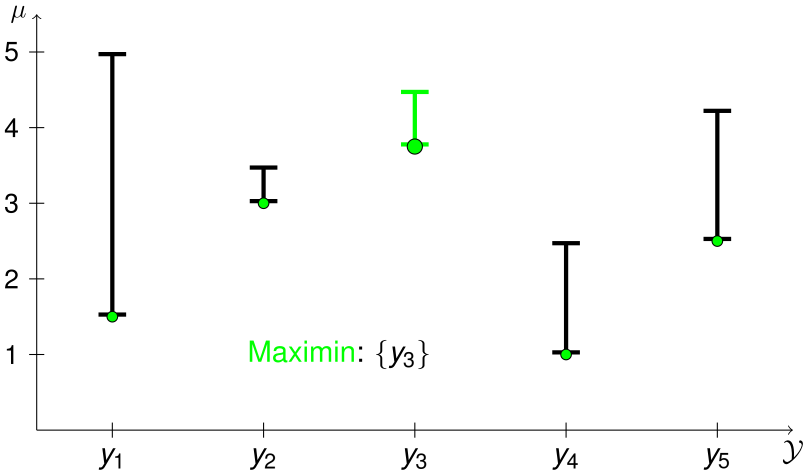
Multiple choice case



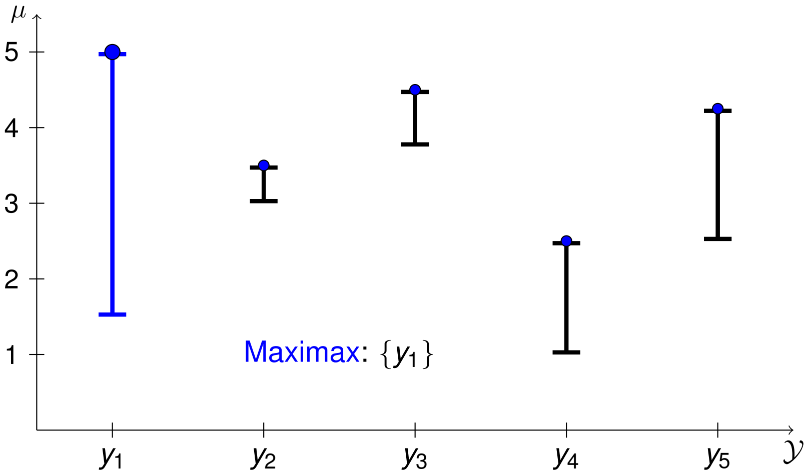
Multiple choice case



Multiple choice case



Multiple choice case



Plan

Why modelling uncertainty (outside intellectual satisfaction)?

Because . . .

- . . . you should (risk/reliability analysis)
- . . . it solves existing issues (non-monotonic reasoning)
- . . . it gives better/more robust results with acceptable computational burden

Scalability

Adding flexibility to the model $\rightarrow$ increases scalability issue

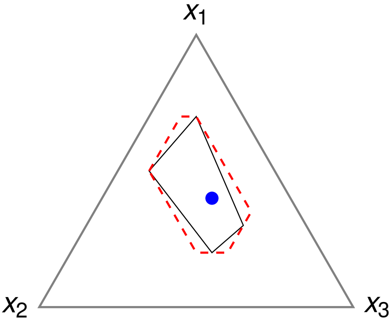
- already true for probability and intervals
- only get worse if model more complex

How to solve it? As in other domains

- approximation, model reduction, ... $\rightarrow$ make things as simple as possible (but not simpler) to answer your question
- sampling
- use flexibility only where you need it

One advantage of incompleteness

Using approximations: choice between outer/inner approximation



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