Introduction to belief functions, Lecture 1– Exercises

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1. An urn contains 90 balls, of which 30 are white, and 60 are either black or yellow. A ball is going to be drawn from the urn. Represent the uncertainty about the outcome of this experiment using a mass function on a suitable frame. Compute the corresponding belief and plausibility functions.

Solution: The frame can be denoted as $\Omega = \{w, b, y\}$ for the three colors. The chance to get a white ball is 1/3 and the chance to get a ball that is either black or yellow is 2/3. We thus have the following mass function:

$$m(\{w\}) = 1/3, \quad m(\{b, y\}) = 2/3.$$

The corresponding belief and plausibility functions are given in the following table:

A	$\emptyset$	$\{w\}$	$\{b\}$	$\{w,b\}$	$\{y\}$	$\{w,y\}$	$\{b,y\}$	$\{w,b,y\}$
$Bel(A)$	0	1/3	0	1/3	0	1/3	2/3	1
$Pl(A)$	0	1/3	2/3	1	2/3	1	2/3	1

2. Let $\Omega = \{a, b, c\}$ and f the following function from 2^Ω to $[0, 1]$:

A	$\emptyset$	$\{a\}$	$\{b\}$	$\{a,b\}$	$\{c\}$	$\{a,c\}$	$\{b,c\}$	$\{a,b,c\}$
$f(A)$	0	0.5	0.2	0.8	0	0.5	0.5	1

Is f a belief function?

Solution: Let us assume that f is a belief function, and let us compute the corresponding mass function. We have

$$m(\{a\}) = 0.5 \quad \text{and} \quad m(\{b\}) = 0.2,$$

so

$$m(\{a, b\}) = 0.8 - 0.5 - 0.2 = 0.1.$$

Similarly, $m(\{c\}) = 0$, so $m(\{a, c\}) = 0.5 - 0.5 - 0 = 0$, and

$$m(\{b, c\}) = 0.5 - 0.2 - 0 = 0.3.$$

Finally,

$$m(\{a, b, c\}) = 1 - 0.5 - 0.2 - 0.1 - 0.3 = -0.1.$$

This mass is negative, so f is not a belief function.

We can also notice that f is not even 2-monotone. For instance,

$$f(\{a, b\} \cup \{b, c\}) = f(\{a, b, c\}) = 1$$

and

$$f(\{a, b\}) + f(\{b, c\}) - f(\{a, b\} \cap \{b, c\}) = 0.8 + 0.5 - 0.2 = 1.1,$$

so

$$f(\{a, b\} \cup \{b, c\}) < f(\{a, b\}) + f(\{b, c\}) - f(\{a, b\} \cap \{b, c\}).$$

3. An expert has given the following contour function on $\Omega = \{a, b, c, d, e, f\}$:

ω	a	b	c	d	e	f
$pl(\omega)$	0.1	0.3	0.5	1	0.7	0.3

Compute the corresponding mass function, assuming that it is consonant.

Solution: The elements of Ω are ordered by decreasing plausibility as

$$1 = pl(d) > pl(e) > pl(c) > pl(b) = pl(f) > pl(a),$$

so the focal sets are $\{d\}$, $\{d, e\}$, $\{d, e, c\}$, $\{d, e, c, b, f\}$ and Ω. We have

$$\begin{aligned} m(\{d\}) &= 1 - 0.7 = 0.3 \\ m(\{d, e\}) &= 0.7 - 0.5 = 0.2 \\ m(\{d, e, c\}) &= 0.5 - 0.3 = 0.2 \\ m(\{d, e, c, b, f\}) &= 0.3 - 0.1 = 0.2 \\ m(\Omega) &= 0.1. \end{aligned}$$

4. Let m be a consonant mass function on a frame Ω and let Bel and Pl be the corresponding belief and plausibility functions. Show that, for any subset A of Ω, $Bel(A) > 0 \Rightarrow Pl(A) = 1$.

Solution: We have $Pl(\overline{A}) = 1 - Bel(A)$, so $Bel(A) > 0$ implies $Pl(\overline{A}) < 1$. Now, since m is consonant, we have

$$Pl(\Omega) = Pl(A \cup \overline{A}) = \max(Pl(A), Pl(\overline{A})) = 1.$$

Consequently, $Pl(A) = 1$.

5. Let m_1 and m_2 be two mass functions on $\Omega = \{a, b, c, d\}$ defined as follows

$$m_1(\{a\}) = 0.3 \quad m_1(\{a, c\}) = 0.5 \quad m_1(\{b, c, d\}) = 0.2$$

and

$$m_2(\{b, c\}) = 0.4 \quad m_2(\{a, c, d\}) = 0.5 \quad m_2(\{d\}) = 0.1.$$

Compute the combined mass function by Dempster's rule. What is the degree of conflict between m_1 and m_2?

Solution:

		m_2		
		$\{b, c\}, 0.4$	$\{a, c, d\}, 0.5$	$\{d\}, 0.1$
	$\{a\}, 0.3$	$\emptyset, 0.12$	$\{a\}, 0.15$	$\emptyset, 0.03$
m_1	$\{a, c\}, 0.5$	$\{c\}, 0.2$	$\{a, c\}, 0.25$	$\emptyset, 0.05$
	$\{b, c, d\}, 0.2$	$\{b, c\}, 0.08$	$\{c, d\}, 0.1$	$\{d\}, 0.02$

The degree of conflict is

$$\kappa = 0.12 + 0.03 + 0.05 = 0.2$$

and the combined mass function is

$$
\begin{aligned}
(m_1 \oplus m_2)(\{a\}) &= 0.15/0.8 = 0.1875 \\
(m_1 \oplus m_2)(\{c\}) &= 0.2/0.8 = 0.25 \\
(m_1 \oplus m_2)(\{a, c\}) &= 0.25/0.8 = 0.3125 \\
(m_1 \oplus m_2)(\{b, c\}) &= 0.08/0.8 = 0.1 \\
(m_1 \oplus m_2)(\{c, d\}) &= 0.1/0.8 = 0.125 \\
(m_1 \oplus m_2)(\{d\}) &= 0.02/0.8 = 0.025.
\end{aligned}
$$

6. Let $\Omega = \{a, b\}$, and let m and m' be the following mass functions on Ω,

$$m = \{a\}^\alpha \oplus \{b\}^\beta, \quad m' = \{a\}^{\alpha'} \oplus \{b\}^{\beta'},$$

where A^w denotes the mass function m such that $m(A) = 1 - w$ and $m(\Omega) = w$.

(a) Compute m and m'.

Solution: To compute m we can make the following table:

		$\{b\}^\beta$	
		$\{b\}, 1-\beta$	Ω, β
$\{a\}^\alpha$	$\{a\}, 1-\alpha$	$\emptyset, (1-\alpha)(1-\beta)$	$\{a\}, \beta(1-\alpha)$
	Ω, α	$\{b\}, \alpha(1-\beta)$	$\Omega, \alpha\beta$

The degree of conflict is $\kappa = (1-\alpha)(1-\beta) = 1 - \alpha - \beta + \alpha\beta$, so

$$m(\{a\}) = \frac{\beta(1-\alpha)}{\alpha + \beta - \alpha\beta}, \quad m(\{b\}) = \frac{\alpha(1-\beta)}{\alpha + \beta - \alpha\beta},$$

$$m(\{a, b\}) = \frac{\alpha\beta}{\alpha + \beta - \alpha\beta}.$$

A similar expression is obtained for m' by replacing α and β by α' and β'.

(b) Compute $m \oplus m'$.

Solution: We observe that

$$A^w \oplus A^{w'} = A^{ww'}.$$

Consequently, using the commutativity and associativity of $\oplus$,

$$m \oplus m' = \{a\}^\alpha \oplus \{b\}^\beta \oplus \{a\}^{\alpha'} \oplus \{b\}^{\beta'} = \{a\}^{\alpha\alpha'} \oplus \{b\}^{\beta\beta'}.$$

Consequently,

$$(m \oplus m')(\{a\}) = \frac{\beta\beta'(1-\alpha\alpha')}{\alpha\alpha' + \beta\beta' - \alpha\beta\alpha'\beta'}, \quad (m \oplus m')(\{b\}) = \frac{\alpha\alpha'(1-\beta\beta')}{\alpha\alpha' + \beta\beta' - \alpha\beta\alpha'\beta'},$$

$$(m \oplus m')(\{a, b\}) = \frac{\alpha\alpha'\beta\beta'}{\alpha\alpha' + \beta\beta' - \alpha\beta\alpha'\beta'}.$$