Exercises on statistical inference using belief functions

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Exercise 1

Question 1

For a Gaussian contour function with parameters m and h, the set $\Gamma(s)$ is the closed whose bounds are the solutions of the quadratic equation

$$-\frac{h}{2}(x-m)^2 = \log(s).$$

We thus have $\Gamma(s) = m \pm \sqrt{-2\frac{\log(s)}{h}}$. Now, the intersection of two closed intervals $[a,b]$ and $[c,d]$ is

$$[a,b]\cap[c,d] = \begin{cases} \emptyset & \text{if } \max(a,c) > \min(b,d) \\ [\max(a,c),\min(b,d)] & \text{otherwise.} \end{cases}$$

Finally, the degree of conflict between two random sets Γ_1 and Γ_2 is

$$\kappa = P((s_1,s_2)\in S^2 : \Gamma_1(s_1)\cap\Gamma_2(s_2)=\emptyset).$$

We can now write the following function, which returns N focal sets of the combined belief function:

```
comb_Gaussian <- function(m1,h1,m2,h2,N=10^4){
  X <- matrix(0,N,2)
  n <- 0
  nc <- 0
  while(n<N){
    s1 <- runif(1)
    s2 <- runif(1)
    X1 <- c(m1-sqrt(-2*log(s1)/h1),m1+sqrt(-2*log(s1)/h1)) # Gamma(s1)
    X2 <- c(m2-sqrt(-2*log(s2)/h2),m2+sqrt(-2*log(s2)/h2)) # Gamma(s2)
    if(max(X1[1],X2[1])<= min(X1[2],X2[2])){  # The intersection is nonempty
      n <- n+1
      X[n,] <- c(max(X1[1],X2[1]), min(X1[2],X2[2])) # The inetrsection is a focal set
    } else nc <- nc+1
  }
  return(list(X=X,conf=nc/(n+nc)))
}
```

Question 2

Let us compute 10^4 focal sets:

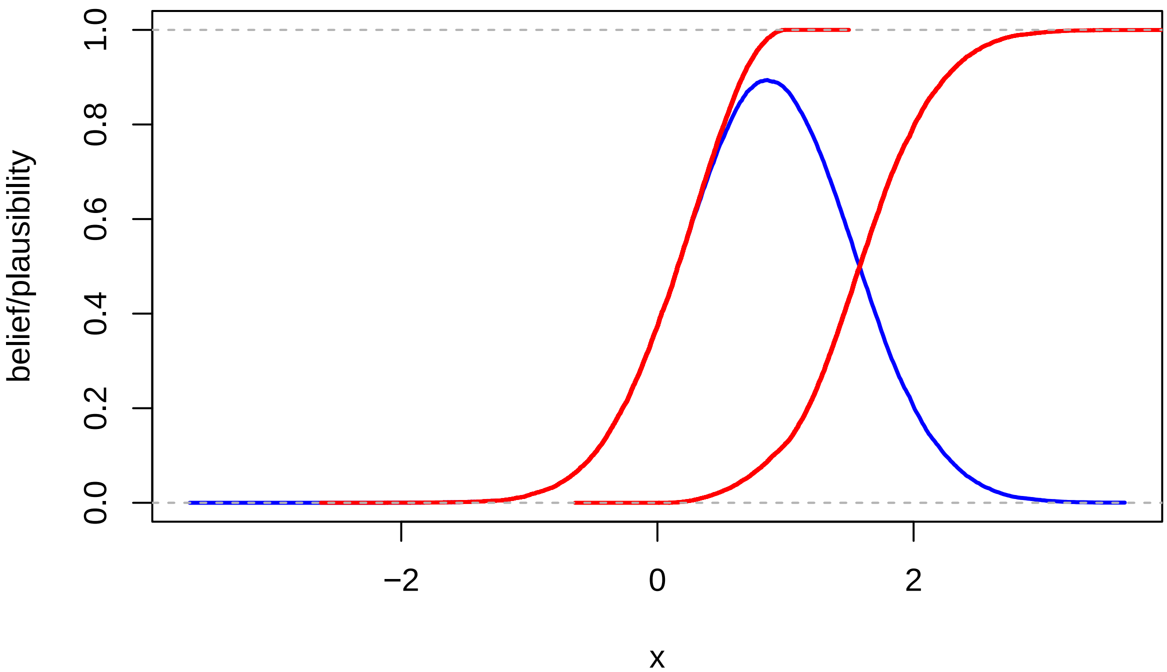
```r
m1<-0; h1<-0.3
m2<-1; h2<-2
comb<-comb_Gaussian(m1,h1,m2,h2,N=10^4)
X<-comb$X
conf<-comb$conf
```

We compute the estimated contour function at equally spaced values in a vector x:

```r
x<-seq(min(m1-2/sqrt(h1),m2-2/sqrt(h2)),max(m1+2/sqrt(h1),m2+2/sqrt(h2)),0.01)
nx<-length(x)
pl<-rep(0,nx)
for(i in 1:nx){
  pl[i]<-mean((X[,1]<=x[i])&(x[i]<=X[,2]))
}
```

The estimated lower and upper cfs can be computed as the empirical cdfs of, respectively, the upper and lower bounds of the N focal sets:

```r
plot(x,pl,col="blue",type="l",lwd=2,xlab="x",ylab="belief/plausibility",ylim=c(0,1))
lines(ecdf(X[,1]),do.points=FALSE, verticals=TRUE,col="red",lwd=2)
lines(ecdf(X[,2]),do.points=FALSE, verticals=TRUE,col="red",lwd=2)
```



Question 3

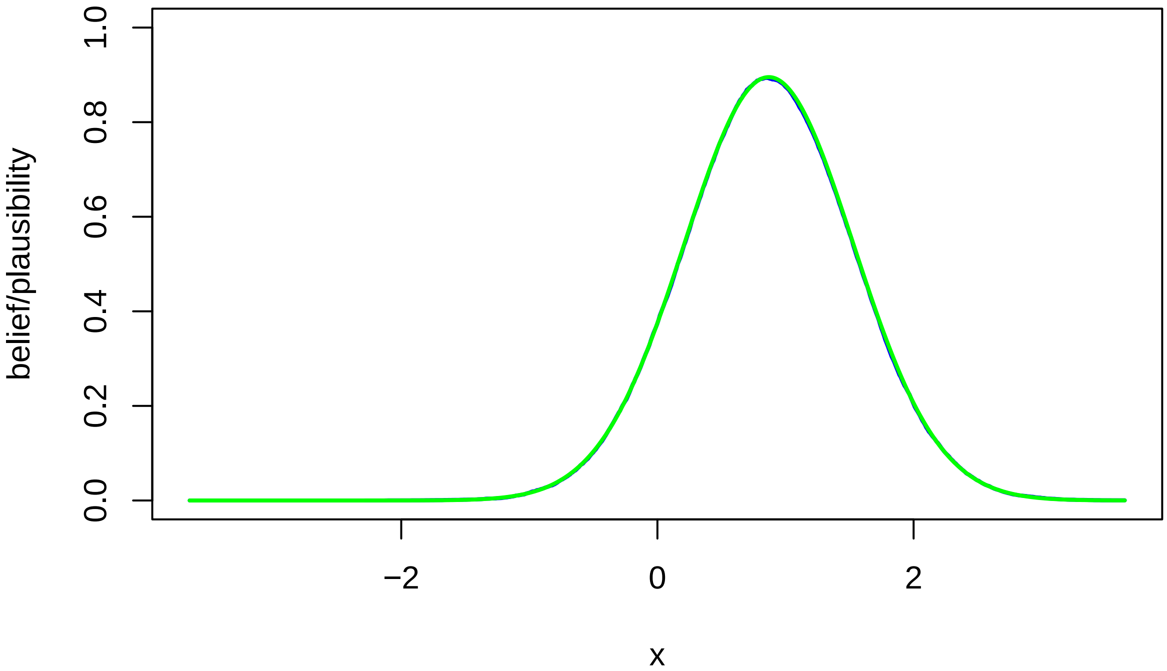
We now that the combined contour function is equal to the product of the contour functions, devided by one minus the degree of conflict:

$$pl(x) = \frac{pl_1(x)pl_2(x)}{1-\kappa}$$

Let us verify this equality:

```r
pl1<-exp(-0.5*h1*(x-m1)^2)
pl2<-exp(-0.5*h2*(x-m2)^2)
```

```
plot(x,pl,col="blue",type="l",lwd=2,xlab="x",ylab="belief/plausibility",ylim=c(0,1))
lines(x,pl1*pl2/(1-conf),lwd=2,col="green")
```



Exercise 2

Question 1

To use the probability integral transform, we need to compute the inverse of the cdf:

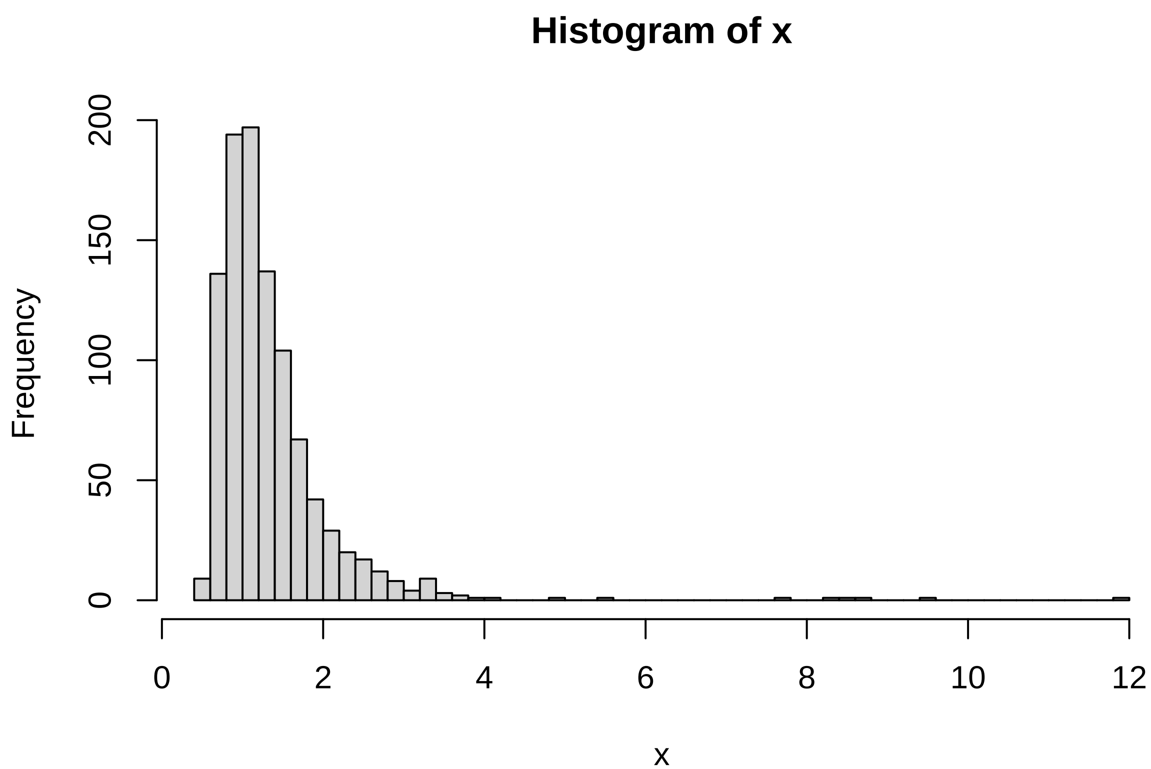
$$\exp(-X^{-\alpha}) = U \Leftrightarrow X = (-\ln U)^{-1/\alpha}.$$

We can then write the following function:

```
rfrechet1 <- function(n,alpha) (-log(runif(n)))^(-1/alpha)
```

Let us generate a sample of size $n = 1000$ and draw the histogram as well as the empirical cdf together with the theoretical cdf:

```
alpha<-3
x<-rfrechet1(1000,alpha)
hist(x,breaks=50)
```




```
plot(ecdf(x))
u<-seq(0,max(x),0.1)
lines(u,exp(-u^(-alpha)),col="red")
```