# Introduction to belief functions, Lecture 1– Exercises

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1. An urn contains 90 balls, of which 30 are white, and 60 are either black or yellow. A ball is going to be drawn from the urn. Represent the uncertainty about the outcome of this experiment using a mass function on a suitable frame. Compute the corresponding belief and plausibility functions.

**Solution:** The frame can be denoted as $\Omega = \{w, b, y\}$ for the three colors. The chance to get a white ball is 1/3 and the chance to get a ball that is either black or yellow is 2/3. We thus have the following mass function:

$$m(\{w\}) = 1/3, \quad m(\{b, y\}) = 2/3.$$

The corresponding belief and plausibility functions are given in the following table:

| $A$ | $\emptyset$ | $\{w\}$ | $\{b\}$ | $\{w,b\}$ | $\{y\}$ | $\{w,y\}$ | $\{b,y\}$ | $\{w,b,y\}$ |
|---|---|---|---|---|---|---|---|---|
| $Bel(A)$ | 0 | 1/3 | 0 | 1/3 | 0 | 1/3 | 2/3 | 1 |
| $Pl(A)$ | 0 | 1/3 | 2/3 | 1 | 2/3 | 1 | 2/3 | 1 |

2. Let $\Omega = \{a, b, c\}$ and $f$ the following function from $2^\Omega$ to $[0, 1]$:

| $A$ | $\emptyset$ | $\{a\}$ | $\{b\}$ | $\{a,b\}$ | $\{c\}$ | $\{a,c\}$ | $\{b,c\}$ | $\{a,b,c\}$ |
|---|---|---|---|---|---|---|---|---|
| $f(A)$ | 0 | 0.5 | 0.2 | 0.8 | 0 | 0.5 | 0.5 | 1 |

Is $f$ a belief function?

**Solution:** Let us assume that $f$ is a belief function, and let us compute the corresponding mass function. We have

$$m(\{a\}) = 0.5 \quad \text{and} \quad m(\{b\}) = 0.2,$$

so

$$m(\{a, b\}) = 0.8 - 0.5 - 0.2 = 0.1.$$

Similarly, $m(\{c\}) = 0$, so $m(\{a, c\}) = 0.5 - 0.5 - 0 = 0$, and

$$m(\{b, c\}) = 0.5 - 0.2 - 0 = 0.3.$$

Finally,

$$m(\{a, b, c\}) = 1 - 0.5 - 0.2 - 0.1 - 0.3 = -0.1.$$

This mass is negative, so $f$ is not a belief function.

We can also notice that $f$ is not even 2-monotone. For instance,

$$f(\{a, b\} \cup \{b, c\}) = f(\{a, b, c\}) = 1$$

and

$$f(\{a, b\}) + f(\{b, c\}) - f(\{a, b\} \cap \{b, c\}) = 0.8 + 0.5 - 0.2 = 1.1,$$

so

$$f(\{a, b\} \cup \{b, c\}) < f(\{a, b\}) + f(\{b, c\}) - f(\{a, b\} \cap \{b, c\}).$$

3. An expert has given the following contour function on $\Omega = \{a, b, c, d, e, f\}$:

| $\omega$ | $a$ | $b$ | $c$ | $d$ | $e$ | $f$ |
|---|---|---|---|---|---|---|
| $pl(\omega)$ | 0.1 | 0.3 | 0.5 | 1 | 0.7 | 0.3 |

Compute the corresponding mass function, assuming that it is consonant.

**Solution:** The elements of $\Omega$ are ordered by decreasing plausibility as

$$1 = pl(d) > pl(e) > pl(c) > pl(b) = pl(f) > pl(a),$$

so the focal sets are $\{d\}$, $\{d, e\}$, $\{d, e, c\}$, $\{d, e, c, b, f\}$ and $\Omega$. We have

$$\begin{aligned} m(\{d\}) &= 1 - 0.7 = 0.3 \\ m(\{d, e\}) &= 0.7 - 0.5 = 0.2 \\ m(\{d, e, c\}) &= 0.5 - 0.3 = 0.2 \\ m(\{d, e, c, b, f\}) &= 0.3 - 0.1 = 0.2 \\ m(\Omega) &= 0.1. \end{aligned}$$

4. Let $m$ be a consonant mass function on a frame $\Omega$ and let $Bel$ and $Pl$ be the corresponding belief and plausibility functions. Show that, for any subset $A$ of $\Omega$, $Bel(A) > 0 \Rightarrow Pl(A) = 1$.

**Solution:** We have $Pl(\overline{A}) = 1 - Bel(A)$, so $Bel(A) > 0$ implies $Pl(\overline{A}) < 1$. Now, since $m$ is consonant, we have

$$Pl(\Omega) = Pl(A \cup \overline{A}) = \max(Pl(A), Pl(\overline{A})) = 1.$$

Consequently, $Pl(A) = 1$.

5. Let $m_1$ and $m_2$ be two mass functions on $\Omega = \{a, b, c, d\}$ defined as follows

$$m_1(\{a\}) = 0.3 \quad m_1(\{a, c\}) = 0.5 \quad m_1(\{b, c, d\}) = 0.2$$

and

$$m_2(\{b, c\}) = 0.4 \quad m_2(\{a, c, d\}) = 0.5 \quad m_2(\{d\}) = 0.1.$$

Compute the combined mass function by Dempster's rule. What is the degree of conflict between $m_1$ and $m_2$?

**Solution:**

| | | $m_2$ | | |
|---|---|---|---|---|
| | | $\{b, c\}, 0.4$ | $\{a, c, d\}, 0.5$ | $\{d\}, 0.1$ |
| $m_1$ | $\{a\}, 0.3$ | $\emptyset, 0.12$ | $\{a\}, 0.15$ | $\emptyset, 0.03$ |
| | $\{a, c\}, 0.5$ | $\{c\}, 0.2$ | $\{a, c\}, 0.25$ | $\emptyset, 0.05$ |
| | $\{b, c, d\}, 0.2$ | $\{b, c\}, 0.08$ | $\{c, d\}, 0.1$ | $\{d\}, 0.02$ |

The degree of conflict is

$$\kappa = 0.12 + 0.03 + 0.05 = 0.2$$

and the combined mass function is

$$\begin{aligned}
(m_1 \oplus m_2)(\{a\}) &= 0.15/0.8 = 0.1875\\
(m_1 \oplus m_2)(\{c\}) &= 0.2/0.8 = 0.25\\
(m_1 \oplus m_2)(\{a,c\}) &= 0.25/0.8 = 0.3125\\
(m_1 \oplus m_2)(\{b,c\}) &= 0.08/0.8 = 0.1\\
(m_1 \oplus m_2)(\{c,d\}) &= 0.1/0.8 = 0.125\\
(m_1 \oplus m_2)(\{d\}) &= 0.02/0.8 = 0.025.
\end{aligned}$$

6. Let $\Omega = \{a, b\}$, and let $m$ and $m'$ be the following mass functions on $\Omega$,

$$m = \{a\}^\alpha \oplus \{b\}^\beta, \quad m' = \{a\}^{\alpha'} \oplus \{b\}^{\beta'},$$

where $A^w$ denotes the mass function $m$ such that $m(A) = 1 - w$ and $m(\Omega) = w$.

(a) Compute $m$ and $m'$.

**Solution:** To compute $m$ we can make the following table:

| | | $\{b\}^\beta$ | |
|---|---|---|---|
| | | $\{b\}, 1-\beta$ | $\Omega, \beta$ |
| $\{a\}^\alpha$ | $\{a\}, 1-\alpha$ | $\emptyset, (1-\alpha)(1-\beta)$ | $\{a\}, \beta(1-\alpha)$ |
| | $\Omega, \alpha$ | $\{b\}, \alpha(1-\beta)$ | $\Omega, \alpha\beta$ |

The degree of conflict is $\kappa = (1-\alpha)(1-\beta) = 1 - \alpha - \beta + \alpha\beta$, so

$$m(\{a\}) = \frac{\beta(1-\alpha)}{\alpha+\beta-\alpha\beta}, \quad m(\{b\}) = \frac{\alpha(1-\beta)}{\alpha+\beta-\alpha\beta},$$

$$m(\{a,b\}) = \frac{\alpha\beta}{\alpha+\beta-\alpha\beta}.$$

A similar expression is obtained for $m'$ by replacing $\alpha$ and $\beta$ by $\alpha'$ and $\beta'$.

(b) Compute $m \oplus m'$.

**Solution:** We observe that

$$A^w \oplus A^{w'} = A^{ww'}.$$

Consequently, using the commutativity and associativity of $\oplus$,

$$m \oplus m' = \{a\}^\alpha \oplus \{b\}^\beta \oplus \{a\}^{\alpha'} \oplus \{b\}^{\beta'} = \{a\}^{\alpha\alpha'} \oplus \{b\}^{\beta\beta'}.$$

Consequently,

$$(m \oplus m')(\{a\}) = \frac{\beta\beta'(1-\alpha\alpha')}{\alpha\alpha'+\beta\beta'-\alpha\beta\alpha'\beta'}, \quad (m \oplus m')(\{b\}) = \frac{\alpha\alpha'(1-\beta\beta')}{\alpha\alpha'+\beta\beta'-\alpha\beta\alpha'\beta'},$$

$$(m \oplus m')(\{a,b\}) = \frac{\alpha\alpha'\beta\beta'}{\alpha\alpha'+\beta\beta'-\alpha\beta\alpha'\beta'}.$$