Theory of belief functions:
Application to machine learning and statistical inference
Decision Analysis – Exercises

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1. An oil company must decide whether or not to drill for oil. They are uncertain whether the hole will be dry (D), have a trickle of oil (T), or be a gusher (G). Drilling a hole costs \$70,000. The payoffs for hitting a gusher, a trickle or a dry hole are \$270,000, \$120,000, and \$0, respectively.

   (a) Which act do we select using the Laplace, maximax, maximin criteria?

   Solution: We have the following payoff matrix (in 1000\$):

	D	T	G
drill (f_1)	-70	50	200
not drill (f_2)	0	0	0

   Using the Laplace criterion, the average utility of drilling is

   $$\frac{1}{3}(-70 + 50 + 200) = 60$$

   and the utility of not drilling is 0, so $f_1 \succcurlyeq f_2$.
   Using the maximax criterion, the maximum utilities of drilling and not drilling are, respectively, 200 and 0, so again $f_1 \succcurlyeq f_2$.
   Using the maximin criterion, the minimum utilities of drilling and not drilling are, respectively, -70 and 0, so $f_2 \succcurlyeq f_1$.

   (b) Discuss the decision based on the Hurwicz criterion, for different values of the pessimism index.

   Solution: Let α denote the degree of pessimism. Act f_1 is preferred to f_2 iff

   $$-70\alpha + 200(1 - \alpha) \geq 0 \Leftrightarrow \alpha \leq \frac{20}{27} \approx 0.74.$$

   (c) Based on seismic soundings, we have obtained the following mass function on $\Omega = \{D, T, G\}$:

   $$m(\{D\}) = 0.1, \quad m(\{T, D\}) = 0.4, \quad m(\{G, T\}) = 0.2, \quad m(\Omega) = 0.3$$

   Compute the lower and upper expected utilities for each of the two acts, as well as the pignistic expected utilities.

   Solution: If f_2 is chosen, the loss is surely zero, so the three expectations are equal to zero.
   For f_1, the induced mass function on $\mathcal{C} = \{-70, 0, 50, 200\}$ is

   $$m_1(\{-70\}) = 0.1, \quad m_1(\{-70, 50\}) = 0.4, \quad m_1(\{50, 200\}) = 0.2,$$

   $$m_1(\{-70, 50, 200\}) = 0.3.$$

The lower and upper expected utilities are, respectively,

$$\underline{\mathbb{E}}_{m_1}(u) = -70 \times 0.1 - 70 \times 0.4 + 50 \times 0.2 - 70 \times 0.3 = -46$$

and

$$\overline{\mathbb{E}}_{m_1}(u) = -70 \times 0.1 + 50 \times 0.4 + 200 \times 0.2 + 200 \times 0.3 = 113.$$

The pignistic probability distribution corresponding to m_1 is

$$p_1(-70) = 0.1 + 0.4/2 + 0.3/3 = 0.4$$

$$p_1(50) = 0.4/2 + 0.2/2 + 0.3/3 = 0.4$$

$$p_1(200) = 0.2/2 + 0.3/3 = 0.2.$$

The pignistic expected utility for act f_1 is, thus,

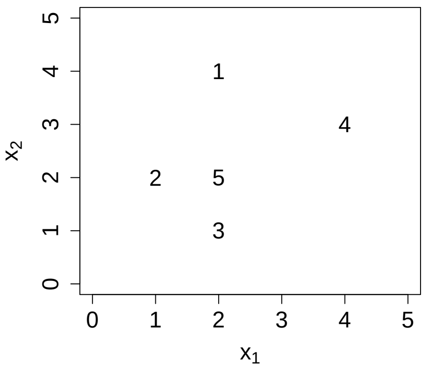
$$\mathbb{E}_{p_1}(u) = -70 \times 0.4 + 50 \times 0.4 + 200 \times 0.2 = 32.$$

(d) Discuss the decisions made using the generalized Hurwicz criterion, as a function of the pessimism index.

Solution: Let α denote the degree of pessimism. Act f_1 is preferred to f_2 iff

$$-46\alpha + 113(1-\alpha) \geq 0 \Leftrightarrow \alpha \leq \frac{113}{157} \approx 0.72.$$

2. We consider a classification problem with three classes $\Omega = \{\omega_1, \omega_2, \omega_3\}$ and two attributes. The following figure shows the feature vectors for five objects.



We have partial information about the class labels of objects 1 to 4, and we want to classify object 5 using the evidential K-nearest neighbor rule with $K = 3$ and function φ defined as follows:

$$\varphi(d) = \frac{1}{1+d}.$$

Denoting by y_i the class of object i, we have the following *partial class labels*:

$$y_1 \in \{\omega_1, \omega_2\}, \quad y_2 = \omega_2, \quad y_3 \in \{\omega_2, \omega_3\}, \quad y_4 \in \{\omega_1, \omega_3\}.$$

This means that, for instance, we only know that object 1 belongs either to class ω_1 or ω_2; we know that object 2 belongs to ω_2 for sure, etc.

(a) We wish to classify object 5. Compute the corresponding mass function.

Solution: The 3 nearest neighbors of object 5 are vectors 1, 2 and 3. We have

$$m_1(\{\omega_1,\omega_2\}) = 1/(1+2) = 1/3, \quad m_1(\Omega) = 2/3$$

$$m_2(\{\omega_2\}) = 1/(1+1) = 0.5, \quad m_2(\Omega) = 0.5$$

$$m_3(\{\omega_2,\omega_3\}) = 0.5, \quad m_3(\Omega) = 0.5$$

Combining m_1 and m_2 yields

$$m_{12}(\{\omega_2\}) = 0.5, \quad m_{12}(\{\omega_1,\omega_2\}) = 1/6, \quad m_{12}(\Omega) = 1/3$$

After combination with m_3 we get

$$m_{123}(\{\omega_2\}) = 7/12, \quad m_{123}(\{\omega_1,\omega_2\}) = 1/12, \quad m_{123}(\{\omega_2,\omega_3\}) = 1/6, \quad m_{123}(\Omega) = 1/6$$

(b) We consider four acts: f_0, f_1 , f_2 and f_3, where f_0 means rejection, and f_k assignment of object 5 to class ω_k. We have the following loss matrix:

	ω_1	ω_2	ω_3
f_0	0.6	0.6	0.6
f_1	0	1	1
f_2	1	0	2
f_3	1	0.5	0

Compute the lower and upper risks for each of the four acts. Which decision do we make for object 5, using the pessimistic and optimistic decision rules?

Solution:

- f_0: lower=upper=0.6
- f_1:
  - lower: $7/12 + 0 + 1/6 + 0 = 9/12 = 3/4$
  - upper: $7/12 + 1/12 + 1/6 + 1/6 = 1$
- f_2:
  - lower: $0 + 0 + 0 + 0 = 0$
  - upper: $0 + 1/12 + 2/6 + 2/6 = 9/12 = 3/4$
- f_3:
  - lower: $7/12 \times 0.5 + 1/12 \times 0.5 + 0 + 0 = 4/12 = 1/3$
  - upper: $7/12 \times 0.5 + 1/12 \times 1 + 0.5/6 + 1/6 = 15/24 = 5/8$

	f_0	f_1	f_2	f_3
lower risk	0.6	3/4	**0**	1/3
upper risk	**0.6**	1	3/4	$5/8 \approx 0.625$

The pessimistic rule (minimization of upper risk) selects f_0. The optimistic rule (minimization of lower risk) selects f_2.

(c) Which decision do we make for object 5, using the pignistic decision rule?

Solution: Pignistic probabilities:

$p(\omega_1) = 1/24 + 1/18 = 7/72 \approx 0.097$, $\quad p(\omega_2) = 7/12 + 1/24 + 1/12 + 1/18 = 55/72 \approx 0.764$

$$p(\omega_3) = 1/12 + 1/18 = 10/72 \approx 0.139$$

Risk:

- f_0: 0.6
- f_1: 0.764+0.139=0.903 (=65/72)
- f_2: 0.097+2*0.139= **0.375** (=27/72)
- f_3: 0.097+0.5*0.764=0.479 (=69/144)

We select f_2.