Theory of Belief Functions: Application to machine learning and statistical inference

Lecture 3: Multinomial predictive belief function

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The problem

- Research on Belief Functions (Dempster-Shafer theory) $\rightarrow$ developing new tools for manipulating belief functions:
  - Combination rules,
  - Propagation in evidential networks,
  - General Bayesian Theorem, ...
- Where do the belief functions come from?
  - Expert opinions: belief function elicitation (see paper by Ben Yaghlane *et al.* in this conference);
  - Data: the topic of this talk.

Introductory example

- Consider an urn with white (ξ_1), red (ξ_2) and black (ξ_3) balls in proportions p_1, p_2 and p_3.
- Let $X \in \mathcal{X} = \{\xi_1, \xi_2, \xi_3\}$ be the color of a ball that will be drawn from the urn: belief on X?
- Two cases:
  1. We know the proportions p_k: then $bel^{\mathcal{X}}(\{\xi_k\}) = p_k$ (Hacking's Principle);
  2. We have observed the result of n drawings from the urn with replacement, e.g. 5 white balls, 3 red balls and 2 black balls.
- How to build a belief function from data in the 2nd case ?

Formalization

- Discrete variable $X \in \mathcal{X} = \{\xi_1, \ldots, \xi_K\}$ defined as the result of a random experiment.
- X is characterized by an unknown frequency (probability) distribution $\mathbb{P}_X$.
- $\mathbb{P}_X(A)$: limit frequency of the event $A \subseteq \mathcal{X}$ in an infinite sequence of trials.
- We have observed a realization $\mathbf{x}_n$ of an iid random sample $X_n = (X_1, \ldots, X_n)$ with parent distribution $\mathbb{P}_X$.
- Problem: build a belief function $bel^{\mathcal{X}}[\mathbf{x}_n]$ with well-defined properties with respect to the unknown frequency distribution $\mathbb{P}_X \rightarrow$ predictive belief function.

Previous work

- Dempster (1966) provided a solution for the case $K = 2$:

$$m^{\mathcal{X}}(\{\xi_1\}) = \frac{N_1}{n+1}, \quad m^{\mathcal{X}}(\{\xi_2\}) = \frac{N_2}{n+1},$$

$$m^{\mathcal{X}}(\mathcal{X}) = \frac{1}{n+1},$$

  with $N_k = \#\{i | X_i = \xi_k\}$, $N_1 + N_2 = n$.
- Same result obtained by Smets (1994) in the TBM framework.
- Both approaches become intractable when $K > 2$.

Previous work (cont.)

- Same problem tackled by Walley (1996) in the imprecise probability framework $\rightarrow$ Imprecise Dirichlet Model (IDM).
- The obtained lower probability measure happens to be a belief function, with mass function:

$$
\begin{aligned}
m^{\mathcal{X}}(\{\xi_k\}|\mathbf{N}, s) &= \frac{N_k}{n+s}, \quad k = 1, \ldots, K, \\
m^{\mathcal{X}}(\mathcal{X}|\mathbf{N}, s) &= \frac{s}{s+n},
\end{aligned}
$$

  with $N_k = \#\{i|X_i = \xi_k\}$ and $s > 0$.
- Well justified in the IP framework, not in the BF framework.

New approach

- Let $bel^{\mathcal{X}}[\mathbf{x}_n]$ be the BF on X after observing a realization $\mathbf{x}_n$ of random sample $\mathbf{X}_n = (X_1, \ldots, X_n)$.
- Which properties should $bel^{\mathcal{X}}[\mathbf{x}_n]$ verify with respect to $\mathbb{P}_X$?
- Hacking's principle (1965): if $\mathbb{P}_X$ is know, then $bel^{\mathcal{X}}[\mathbf{x}_n] = \mathbb{P}_X$.
- Weak version:

$$\forall A \subseteq \mathcal{X}, \quad bel^{\mathcal{X}}[\mathbf{X}_n](A) \xrightarrow{P} \mathbb{P}_X(A), \text{ as } n \to \infty.$$

(Requirement R_1)

New approach (continued)

- Least Commitment Principle: for fixed n, $bel^{\mathcal{X}}[\mathbf{x}_n]$ should be less informative that $\mathbb{P}_X$:

$$bel^{\mathcal{X}}[\mathbf{x}_n](A) \leq \mathbb{P}_X(A), \quad \forall A \subseteq \mathcal{X}.$$

- This condition is too restrictive (it leads to the vacuous BF).
- Weaker condition (Requirement R_2):

$$\mathbb{P}(bel^{\mathcal{X}}[\mathbf{X}_n] \leq \mathbb{P}_X) \geq 1 - \alpha,$$

for some $\alpha \in (0, 1)$.

Meaning of Requirement R_2

$$\mathbf{x}_n = (x_1, \ldots, x_n) \rightarrow bel^{\mathcal{X}}[\mathbf{x}_n]$$
$$\mathbf{x}'_n = (x'_1, \ldots, x'_n) \rightarrow bel^{\mathcal{X}}[\mathbf{x}'_n]$$
$$\mathbf{x}''_n = (x''_1, \ldots, x''_n) \rightarrow bel^{\mathcal{X}}[\mathbf{x}''_n]$$
$$\vdots$$

- As the number of realizations of the random sample tends to ∞, the proportion of belief functions less committed than $\mathbb{P}_X$ should tend to $1 - \alpha$.
- To achieve this property: use of a multinomial confidence region.

Multinomial Confidence Region

- Let $N_k = \#\{i | X_i = \xi_k\}$. Vector $\mathbf{N} = (N_1, \ldots, N_K)$ has a multinomial distribution $\mathcal{M}(n, p_1, \ldots, p_K)$, with $p_k = \mathbb{P}_X(\{\xi_k\})$.
- Let $\mathcal{S}(\mathbf{N}) \subseteq [0,1]^K$ a random region of $[0,1]^K$. It is a confidence region for $\mathbf{p}$ at level $1-\alpha$ if
$$\mathbb{P}(\mathcal{S}(\mathbf{N}) \ni \mathbf{p}) \geq 1 - \alpha.$$
- $\mathcal{S}(\mathbf{N})$ is an asymptotic confidence region if the above inequality holds in the limit as $n \to \infty$.
- Simultaneous confidence intervals: $\mathcal{S}(\mathbf{N}) = [P_1^-, P_1^+] \times \ldots \times [P_K^-, P_K^+]$

Multinomial Confidence Region (cont.)

- Goodman's simultaneous confidence intervals:

$$P_k^- = \frac{b + 2N_k - \sqrt{\Delta_k}}{2(n+b)},$$

$$P_k^+ = \frac{b + 2N_k + \sqrt{\Delta_k}}{2(n+b)},$$

with $b = \chi^2_{1;1-\alpha/K}$ and $\Delta_k = b\left(b + \frac{4N_k(n-N_k)}{n}\right)$.

Example

- 220 psychiatric patients categorized as either neurotic, depressed, schizophrenic or having a personality disorder.
- Observed counts: $\mathbf{n} = (91, 49, 37, 43)$.
- Goodman' confidence intervals at confidence level $1 - \alpha = 0.95$:

Diagnosis	N_k/n	P_k^-	P_k^+
Neurotic	0.41	0.33	0.50
Depressed	0.22	0.16	0.30
Schizophrenic	0.17	0.11	0.24
Personality disorder	0.20	0.14	0.27

From ConfidenceRegions to Lower Probabilities

- To each $\mathbf{p} = (p_1, \ldots, p_K)$ corresponds a probability measure $\mathbb{P}_X$.
- Consequently, $\mathcal{S}(\mathbf{N})$ may be seen as defining a family of probability measures, uniquely defined by the following lower probability measure:

$$P^-(A) = \max\left(\sum_{\xi_k \in A} P_k^-, 1 - \sum_{\xi_k \notin A} P_k^+\right)$$

- P^- satisfies requirements R_1 and R_2:
  - $P^-(A) \xrightarrow{P} \mathbb{P}_X(A)$ as $n \to \infty$, for all $A \subseteq \mathcal{X}$,
  - $\mathbb{P}(P^- \leq \mathbb{P}_X) \geq 1 - \alpha$.

From lower probabilities to belief functions

Case $K \leq 3$

- Is P^- a belief function ?
- If $K = 2$ or $K = 3$, P^- is a belief function.
- Case $K = 2$:

$$m^{\mathcal{X}}(\{\xi_1\}) = P_1^-, \quad m^{\mathcal{X}}(\{\xi_2\}) = P_2^-, \quad m^{\mathcal{X}}(\mathcal{X}) = 1 - P_1^- - P_2^-.$$

- With Goodman intervals:

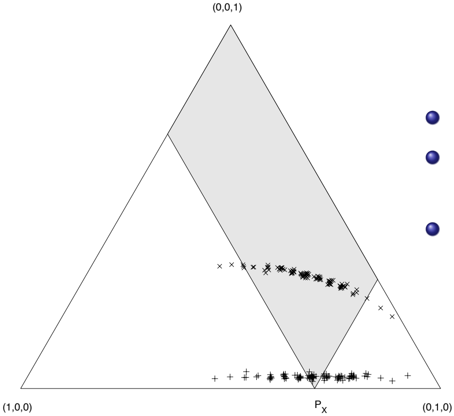
$$m(\{\xi_1\}) \approx \widehat{p} - u_{1-\alpha/2}\sqrt{\frac{\widehat{p}(1-\widehat{p})}{n}}$$

$$m(\{\xi_2\}) \approx 1 - \widehat{p} - u_{1-\alpha/2}\sqrt{\frac{\widehat{p}(1-\widehat{p})}{n}}$$

$$m(\mathcal{X}) \approx 2u_{1-\alpha/2}\sqrt{\frac{\widehat{p}(1-\widehat{p})}{n}},$$

where $\widehat{p} = N_1/n$.

Simulation example



- $K = 2$, $p_1 = \mathbb{P}_X(\{\omega_1\}) = 0.3$.
- 100 realizations of a random sample of size $n = 30$.
- 100 predictive belief functions at level $1 - \alpha = 0.95$.

From lower probabilities to belief functions

Case $K > 3$

- If $K > 3$, P^- is not a belief function in general. We can find the most committed belief function satisfying $bel \leq P^-$ by solving the following linear optimization problem:

$$\max_m J(m) = \sum_{A \subseteq \Omega} bel(A) = \sum_{A \subseteq \Omega} \sum_{B \subseteq A} m(B)$$

under the constraints:

$$\sum_{B \subseteq A} m(B) \leq P^-(A), \quad \forall A \subset \Omega,$$

$$\sum_{A \subseteq \Omega} m(A) = 1, \quad m(A) \geq 0, \quad \forall A \subseteq \Omega\,.$$

- The solution satisfies requirements R_1 and R_2: it is a predictive belief function (at confidence level $1 - \alpha$).

Example: Psychiatric Data

A	$P^-(A)$	$bel^{\mathcal{X}*}(A)$	$m^{\mathcal{X}*}(A)$
$\{\xi_1\}$	0.33	0.33	0.33
$\{\xi_2\}$	0.16	0.14	0.14
$\{\xi_1, \xi_2\}$	0.50	0.50	0.021
$\{\xi_3\}$	0.11	0.097	0.097
$\{\xi_1, \xi_3\}$	0.45	0.45	0.020
$\{\xi_2, \xi_3\}$	0.28	0.28	0.036
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\{\xi_1, \xi_3, \xi_4\}$	0.70	0.66	0.038
$\{\xi_2, \xi_3, \xi_4\}$	0.50	0.48	0.019
$\mathcal{X}$	1	1	0

Case of ordered data

- Assume $\mathcal{X}$ is ordered: $\xi_1 < \ldots < \xi_K$.
- The focal sets of $bel^{\mathcal{X}}[\mathbf{x}_n]$ can be constrained to be intervals $A_{k,r} = \{\xi_k, \ldots, \xi_r\}$.
- Under this additional constraint, an analytical solution to the previous optimization problem can be found:

$$m^{\mathcal{X}*}(A_{k,k}) = P_k^-,$$

$$m^{\mathcal{X}*}(A_{k,k+1}) = P^-(A_{k,k+1}) - P^-(A_{k+1,k+1}) - P^-(A_{k,k}),$$

$$m^{\mathcal{X}*}(A_{k,r}) = P^-(A_{k,r}) - P^-(A_{k+1,r}) - P^-(A_{k,r-1}) + P^-(A_{k+1,r-1})$$

for $r > k+1$, and $m^{\mathcal{X}*}(B) = 0$, for all $B \notin \mathcal{I}$.

Example: rain data

- January precipitation in Arizona (in inches), recorded during the period 1895-2004.

class ξ_k	n_k	n_k/n	p_k^-	p_k^+
< 0.75	48	0.44	0.32	0.56
$[0.75, 1.25)$	17	0.15	0.085	0.27
$[1.25, 1.75)$	19	0.17	0.098	0.29
$[1.75, 2.25)$	11	0.10	0.047	0.20
$[2.25, 2.75)$	6	0.055	0.020	0.14
≥ 2.75	9	0.082	0.035	0.18

- Degree of belief that the precipitation in Arizona next January will exceed, say, 2.25 inches?

Rain data: Result

$m(A_{k,r})$	1	2	3	4	5	6
1	0.32	0	0	0.13	0.11	0
2	-	0.085	0	0	0.012	0.14
3	-	-	0.098	0	0	0
4	-	-	-	0.047	0	0
5	-	-	-	-	0.020	0
6	-	-	-	-	-	0.035

- We get $bel^{\mathcal{X}}(X \geq 2.25) = bel^{\mathcal{X}*}(\{\xi_5, \xi_6\}) = 0.055$ and $pl(X \geq 2.25) = 0.317$.
- In 95 % of cases, the interval $[bel^{\mathcal{X}}(A), pl^{\mathcal{X}}(A)]$ computed using this method contains $\mathbb{P}_X(A)$.

Conclusions

- A "frequentist" approach, based on multinomial confidence regions, for building a belief function quantifying the uncertainty about a discrete random variable X with unknown probability distribution, based on observed data.
- Two "reasonable" properties of the solution with respect to the true frequency distribution $\mathbb{P}_X$:
  - it is less committed than $\mathbb{P}_X$ with some user-defined probability, and
  - it converges towards $\mathbb{P}_X$ in probability as the size of the sample tends to infinity.
- Another approach based on the likelihood function will be described later.