Belief Functions

Statistical Inference

Thierry Denœux

Summer 2023

Estimation vs. prediction

- Consider an urn with an unknown proportion θ of black balls.
- Assume that we have drawn n balls with replacement from the urn, y of which were black.
- Problems:
  1. What can we say about θ? (estimation)
  2. What can we say about the color Z of the next ball to be drawn from the urn? (prediction)
- Classical approaches
  - Frequentist: gives an answer that is correct most the time (over infinitely many replications of the random experiment)
  - Bayesian: assumes prior knowledge on θ and computes a posterior predictive probabilities $f(\theta|y)$ and $P(black|y)$

Criticism of the frequentist approach

- The frequentist approach makes a statement that is correct, say, for 95% of the samples
- The confidence level is often interpreted as a measure of confidence in the statement for a particular sample
- However, this interpretation poses some logical problems

Example

- Suppose X_1 and X_2 are iid with probability mass function

$$\mathbb{P}_\theta(X_i = \theta - 1) = \mathbb{P}_\theta(X_i = \theta + 1) = \frac{1}{2}, \quad i = 1, 2, \tag{1}$$

  where $\theta \in \mathbb{R}$ is an unknown parameter.
- Consider the following confidence set for θ,

$$C(X_1, X_2) = \begin{cases} \frac{1}{2}(X_1 + X_2) & \text{if } X_1 \neq X_2 \\ X_1 - 1 & \text{otherwise.} \end{cases} \tag{2}$$

- The corresponding confidence level is $P_\theta(\theta \in C(X_1, X_2)) = 0.75$
- Now, let (x_1, x_2) be a given realization of the random sample (X_1, X_2).
  - If $x_1 \neq x_2$, we know for sure that $\theta = (x_1 + x_2)/2$
  - If $x_1 = x_2$, we know for sure that θ is either $x_1 - 1$ or $x_1 + 1$, but we have no reason to favor any of these two hypotheses in particular.
- This problem is known as the problem of relevant subsets (there are recognizable situations in which the coverage probability is different from the stated one)

The relevant subset problem

- This phenomenon happens in the usual problem of interval estimation of the mean of a normal sample: "wide" CIs in some sense have larger coverage probability than the stated confidence level, and vice versa for "short" intervals.
- Specifically, let $X_1, \ldots, X_n$ be an iid sample from $\mathcal{N}(\mu, \sigma^2)$ with both parameters unknown, and

$$C = \left\{ (x_1, \ldots, x_n) \in \mathbb{R}^n : \frac{s}{|\overline{x}|} > k \right\}$$

  for some k
- The standard CI for μ is $\overline{x} \pm t_{n-1;1-\alpha/2} s/\sqrt{n}$
- It can be shown that, for some $\epsilon > 0$,

$$P(\mu \in CI | C) > (1-\alpha) + \epsilon$$

  for all μ and σ
- "The existence of certain relevant subsets is an embarrassment to confidence theory" (Lehmann, 1986)

Criticism of the Bayesian approach

- In the Bayesian approach, y, z and θ are seen as random variables
- Estimation: compute the posterior pdf of θ given y

$$f(\theta|y) \propto p(y|\theta)f(\theta)$$

  where $f(\theta)$ is the prior pdf on θ
- Prediction: compute the predictive posterior distribution

$$p(z|y) = \int p(z|\theta)f(\theta|y)d\theta$$

- We need the prior $f(\theta)$!
- The uniform prior is dependent on the parametrization; consequently, it is not truly noninformative (see next slide)
- Another solution: Jeffreys prior

The wine/water paradox

- Principle of Indifference (PI): in the absence of information about some quantity X, we should assign equal probability to any possible value of X
- The wine/water paradox

There is a certain quantity of liquids. All that we know about the liquid is that it is composed entirely of wine and water, and the ratio of wine to water is between 1/3 and 3.

What is the probability that the ratio of wine to water is less than or equal to 2?

The wine/water paradox (continued)

- Let X denote the ratio of wine to water. All we know is that $X \in [1/3, 3]$. According to the PI, $X \sim \mathcal{U}_{[1/3,3]}$. Consequently

$$P(X \leq 2) = (2 - 1/3)/(3 - 1/3) = 5/8$$

The wine/water paradox (continued)

- Let X denote the ratio of wine to water. All we know is that $X \in [1/3, 3]$. According to the PI, $X \sim \mathcal{U}_{[1/3,3]}$. Consequently

$$P(X \leq 2) = (2 - 1/3)/(3 - 1/3) = 5/8$$

- Now, let $Y = 1/X$ denote the ratio of water to wine. All we know is that $Y \in [1/3, 3]$. According to the PI, $Y \sim \mathcal{U}_{[1/3,3]}$. Consequently

$$P(Y \geq 1/2) = (3 - 1/2)/(3 - 1/3) = 15/16$$

The wine/water paradox (continued)

- Let X denote the ratio of wine to water. All we know is that $X \in [1/3, 3]$. According to the PI, $X \sim \mathcal{U}_{[1/3,3]}$. Consequently

$$P(X \leq 2) = (2 - 1/3)/(3 - 1/3) = 5/8$$

- Now, let $Y = 1/X$ denote the ratio of water to wine. All we know is that $Y \in [1/3, 3]$. According to the PI, $Y \sim \mathcal{U}_{[1/3,3]}$. Consequently

$$P(Y \geq 1/2) = (3 - 1/2)/(3 - 1/3) = 15/16$$

- However, $P(X \leq 2) = P(Y \geq 1/2)$!

Jeffreys prior

- The Jeffreys prior is defined objectively as being proportional to the square root of the determinant of the Fisher information

$$\pi(\theta) \propto \sqrt{\det I(\theta)},$$

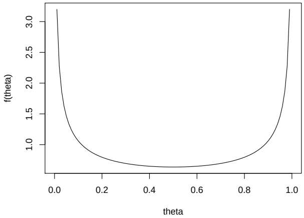
where the component (i,j) of the information matrix $I(\theta)_{ij}$ is

$$I(\theta)_{ij} = \mathbb{E}_\theta \left[\frac{\partial \log f_\theta(x)}{\partial \theta_i} \frac{\partial \log f_\theta(x)}{\partial \theta_j} \right].$$

- The motivation for this definition is that the Jeffreys prior is invariant under reparameterization: if φ is a one-to-one transformation and $\nu = \varphi(\theta)$, then the Jeffreys prior on ν is proportional to $\sqrt{\det I(\nu)}$.

Problems with Jeffrey's prior

- However, there are still some issues with this approach:
  - First, the Jeffreys prior is sometimes improper.
  - Secondly, and maybe more importantly, the Jeffreys prior can hardly be considered to be truly noninformative.
- For instance, consider an iid sample $X_1, \ldots, X_n$ from a Bernoulli distribution $\mathcal{B}(\theta)$. The Jeffreys prior on θ is the beta distribution $B(0.5, 0.5)$ whose pdf is displayed below. We can see that extreme values of θ are considered a priori more probable that central values, which does represent non vacuous knowledge about θ.



Main ideas

- None of the classical approaches to statistical inference (frequentist and Bayesian) is fully satisfactory, from a conceptual point of view
- Proposal of a new approach based on belief functions
- The new approach boils down to Bayesian inference when a probabilistic prior is available, but it does not require the user to provide such a prior
- We will apply this approach to different econometric models
- Before applying belief functions to statistical inference, we need to define belief functions on infinite spaces

Outline

Outline

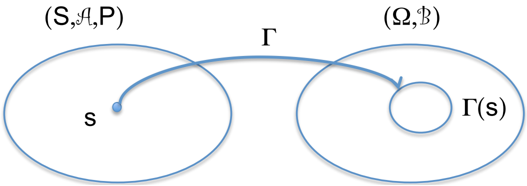
Belief function: general definition

- Let Ω be a set (finite or not) and $\mathcal{B}$ be an algebra of subsets of Ω
- A belief function (BF) on $\mathcal{B}$ is a mapping $Bel : \mathcal{B} \rightarrow [0,1]$ verifying $Bel(\emptyset) = 0$, $Bel(\Omega) = 1$ and the complete monotonicity property: for any $k \geq 2$ and any collection $B_1, \ldots, B_k$ of elements of $\mathcal{B}$,

$$Bel\left(\bigcup_{i=1}^{k} B_i\right) \geq \sum_{\emptyset \neq I \subseteq \{1,\ldots,k\}} (-1)^{|I|+1} Bel\left(\bigcap_{i \in I} B_i\right)$$

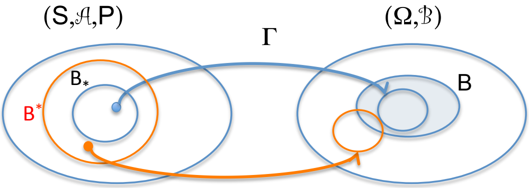
- A function $Pl : \mathcal{B} \rightarrow [0,1]$ is a plausibility function iff $Bel : B \rightarrow 1 - Pl(\overline{B})$ is a belief function

Source



- Let S be a state space, $\mathcal{A}$ an algebra of subsets of S, $\mathbb{P}$ a finitely additive probability on $(S, \mathcal{A})$
- Let Ω be a set and $\mathcal{B}$ an algebra of subsets of Ω
- Γ a multivalued mapping from S to 2^Ω
- The four-tuple $(S, \mathcal{A}, \mathbb{P}, \Gamma)$ is called a source
- Under some conditions, it induces a belief function on $(\Omega, \mathcal{B})$

Strong measurability



- Lower and upper inverses: for all $B \in \mathcal{B}$,

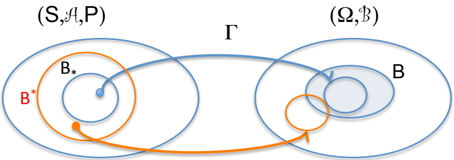
$$\Gamma_*(B) = B_* = \{s \in S | \Gamma(s) \neq \emptyset, \Gamma(s) \subseteq B\}$$

$$\Gamma^*(B) = B^* = \{s \in S | \Gamma(s) \cap B \neq \emptyset\}$$

- Γ is strongly measurable wrt $\mathcal{A}$ and $\mathcal{B}$ if, for all $B \in \mathcal{B}$, $B^* \in \mathcal{A}$
- $(\forall B \in \mathcal{B}, B^* \in \mathcal{A}) \Leftrightarrow (\forall B \in \mathcal{B}, B_* \in \mathcal{A})$
- A strongly measurable multi-valued mapping Γ is called a random set

Belief function induced by a source

Lower and upper probabilities

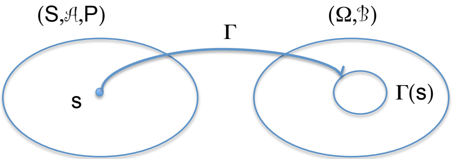


- Lower and upper probabilities:

$$\forall B \in \mathcal{B}, \quad \mathbb{P}_*(B) = \frac{\mathbb{P}(B_*)}{\mathbb{P}(\Omega^*)}, \quad \mathbb{P}^*(B) = \frac{\mathbb{P}(B^*)}{\mathbb{P}(\Omega^*)} = 1 - \mathbb{P}_*(\overline{B})$$

- $\mathbb{P}_*$ is a BF, and $\mathbb{P}^*$ is the dual plausibility function
- Conversely, for any belief function, there is a source that induces it (Shafer's thesis, 1973)

Interpretation

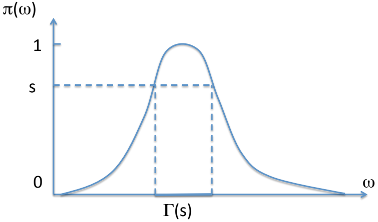


- Typically, Ω is the domain of an unknown quantity ω, and S is a set of interpretations of a given piece of evidence about ω
- If $s \in S$ holds, then the evidence tells us that $\omega \in \Gamma(s)$, and nothing more
- Then
  - $Bel(B)$ is the probability that the evidence supports B
  - $Pl(B)$ is the probability that the evidence is consistent with B

Outline

Consonant belief function

Source



- Let π be a mapping from $\Omega = \mathbb{R}^p$ to $S = [0, 1]$ s.t. $\sup \pi = 1$
- Let Γ be the multi-valued mapping from S to 2^Ω defined by

$$\forall s \in [0, 1], \quad \Gamma(s) = \{\omega \in \Omega | \pi(\omega) \geq s\}$$

- Let $\mathcal{B}([0, 1])$ be the Borel σ-field on $[0, 1]$, and P the uniform probability measure on $[0, 1]$
- We consider the source $([0, 1], \mathcal{B}([0, 1]), P, \Gamma)$

Consonant belief function

Properties

- Let Bel and Pl be the belief and plausibility functions induced by $([0,1], \mathcal{B}([0,1]), P, \Gamma)$
- The focal sets $\Gamma(s)$ are nested, i.e., for any s and s',

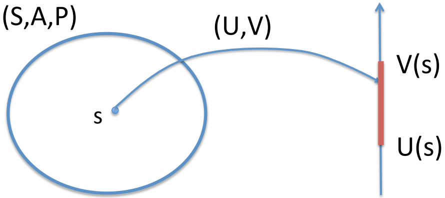
$$s \geq s' \Rightarrow \Gamma(s) \subseteq \Gamma(s')$$

  The belief function is said to be consonant.
- The corresponding contour function pl is equal to π
- The corresponding plausibility function is a possibility measure: for any $B \subseteq \Omega$,

$$Pl(B) = \sup_{\omega \in B} pl(\omega)$$

$$Bel(B) = \inf_{\omega \notin B} (1 - pl(\omega))$$

Random closed interval



- Let (U, V) be a bi-dimensional random vector from a probability space $(S, \mathcal{A}, \mathbb{P})$ to $\mathbb{R}^2$ such that $U \leq V$ a.s.
- Multi-valued mapping:

$$\Gamma : s \rightarrow \Gamma(s) = [U(s), V(s)]$$

- The source $(S, \mathcal{A}, \mathbb{P}, \Gamma)$ is a random closed interval. It defines a BF on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$

Random closed interval

Properties

- Lower/upper cdfs:

$$Bel((-\infty, x]) = \mathbb{P}([U, V] \subseteq (-\infty, x]) = \mathbb{P}(V \leq x) = F_V(x)$$
$$Pl((-\infty, x]) = \mathbb{P}([U, V] \cap (-\infty, x] \neq \emptyset) = \mathbb{P}(U \leq x) = F_U(x)$$

- Lower/upper expectation:

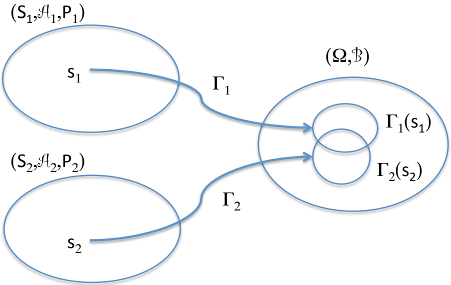
$$\mathbb{E}_*(\Gamma) = \mathbb{E}(U)$$
$$\mathbb{E}^*(\Gamma) = \mathbb{E}(V)$$

- Lower/upper quantiles

$$q_*(\alpha) = F_U^{-1}(\alpha),$$
$$q^*(\alpha) = F_V^{-1}(\alpha).$$

Outline

Dempster's rule
Definition



- Let $(S_i, \mathcal{A}_i, \mathbb{P}_i, \Gamma_i)$, $i = 1, 2$ be two sources representing independent items of evidence, inducing BF Bel_1 and Bel_2
- The combined BF $Bel = Bel_1 \oplus Bel_2$ is induced by the source $(S_1 \times S_2, \mathcal{A}_1 \otimes \mathcal{A}_2, \mathbb{P}_1 \otimes \mathbb{P}_2, \Gamma_\cap)$ with

$$\Gamma_\cap(s_1, s_2) = \Gamma_1(s_1) \cap \Gamma_2(s_2)$$

Dempster's rule

Definition

- For each $B \in \mathcal{B}$, $Bel(B)$ is the conditional probability that $\Gamma_\cap(s) \subseteq B$, given that $\Gamma_\cap(s) \neq \emptyset$:

$$Bel(B) = \frac{\mathbb{P}\left(\{(s_1, s_2) \in S_1 \times S_2 | \Gamma_\cap(s_1, s_2) \neq \emptyset, \Gamma_\cap(s_1, s_2) \subseteq B\}\right)}{\mathbb{P}(\{(s_1, s_2) \in S_1 \times S_2 | \Gamma_\cap(s_1, s_2) \neq \emptyset\})}$$

- It is well defined iff the denominator is non null
- As in the finite case, the degree of conflict between the belief functions can be defined as one minus the denominator in the above equation.

Approximate computation

Monte Carlo simulation

Require: Desired number of focal sets N

$i \leftarrow 0$

while $i < N$ **do**

Draw s_1 in S_1 from $\mathbb{P}_1$

Draw s_2 in S_2 from $\mathbb{P}_2$

$\Gamma_\cap(s_1, s_2) \leftarrow \Gamma_1(s_1) \cap \Gamma_2(s_2)$

if $\Gamma_\cap(s_1, s_2) \neq \emptyset$ **then**

$i \leftarrow i + 1$

$B_i \leftarrow \Gamma_\cap(s_1, s_2)$

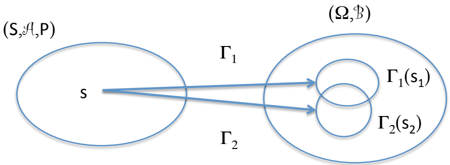
end if

end while

$\widehat{Bel}(B) \leftarrow \frac{1}{N}\#\{i \in \{1, \ldots, N\} | B_i \subseteq B\}$

$\widehat{Pl}(B) \leftarrow \frac{1}{N}\#\{i \in \{1, \ldots, N\} | B_i \cap B \neq \emptyset\}$

Combination of dependent evidence



- The case of complete dependence between two pieces of evidence can be modeled by two sources formed by different multivalued mappings Γ_1 and Γ_2 from the same probability space.
- The combined BF is induced by the source $(S, \mathcal{A}, \mathbb{P}, \Gamma_\cap)$
- This combination rule preserves consonance: the combination of two consonant BFs is still consonant.
- This is the rule used in Possibility Theory.

Outline

Parameter estimation

- Let $\boldsymbol{y} \in \mathbb{Y}$ denote the observed data and $f_{\boldsymbol{\theta}}(\boldsymbol{y})$ the probability mass or density function describing the data-generating mechanism, where $\boldsymbol{\theta} \in \Theta$ is an unknown parameter
- Having observed $\boldsymbol{y}$, how to quantify the uncertainty about Θ, without specifying a prior probability distribution?
- Different approaches have been proposed by Dempster (1968), Shafer (1976) and more recently, Martin and Liu (2016)
- Here, I will emphasize Shafer's Likelihood-based solution (Shafer, 1976; Wasserman, 1990; Denœux, 2014), which is (much) simpler to implement, and connects nicely with the "likelihoodist" approach to statistical inference.

Outline

The likelihood principle

Definition (Birnbaum, 1962)

- Let E denote a statistical model representing an experimental situation. Typically, E is composed of the parameter space Θ, the sample space $\mathbb{X}$ and a probability mass or density function $f(x;\theta)$ for each $\theta \in \Theta$.
- Let us denote by $Ev(E,x)$ the evidential meaning of the specified instance (E,x) of statistical evidence.
- The likelihood Principle (L) can be stated as follows:

If E and E' are any two experiments with the same parameter space Θ, represented by probability mass or density functions $f_\theta(x)$ and $g_\theta(y)$, and if x and y are any two respective outcomes which determine likelihood functions satisfying $f_\theta(x) = cg_\theta(y)$ for some positive constant $c = c(x,y)$ and all $\theta \in \Theta$, then $Ev(E,x) = Ev(E',y)$.

Frequentist methods violate (L)

- For instance, consider an urn with a proportion θ of black balls, and the following two experiments:
  - Experiment 1: a fixed number n of balls are drawn with replacement from the urn and the number X of black balls is observed; X has a binomial distribution $\mathcal{B}(n, \theta)$.
  - Experiment 2: balls are drawn with replacement from the urn until a fixed number x of black balls have been drawn; we observe the number N of draws, which has a negative binomial distribution.
- Confidence intervals computed in these two cases are different, although the likelihood functions for these two experiments are identical.
- This is because confidence intervals (and significance tests) depend not only on the likelihood, but also on the sample space.

Justification of (L) (Birnbaum, 1962)

Birnbaum (1962) showed that (L) can be derived from the principles of sufficiency (S) and conditionality (C), which can be stated as follows:

- The principle of sufficiency (S) Let E be an experiment, with sample space $\{x\}$, and let $t(x)$ is any sufficient statistic (i.e., any statistic such that the conditional distribution of x given t does not depend on θ). Let E' be an experiment, derived from E, having the same parameter space, such that when any outcome x of E is observed the corresponding outcome $t = t(x)$ of E' is observed. Then for each x, $Ev(E, x) = Ev(E', t)$, where $t = t(x)$.
- The principle of conditionality (C) If E is mathematically equivalent to a mixture of component experiments E_h, with possible outcomes (E_h, x_h), then $Ev(E, (E_h, x_h)) = Ev(E_h, x_h)$.

Meaning of (C)

- (C) means that component experiments that might have been carried out, but in fact were not, are irrelevant once we know that E_h has been carried out.
- For instance, assume that two measuring instruments provide measurements x_1 and x_2 of an unknown quantity θ. An instrument is picked at random (experiment E). Assume we know that the first instrument ($h = 1$) is selected and we observe x_1. Then, the fact that the second instrument could have been selected is irrelevant and the over-all structure of the original experiment E can be ignored.

Outline

Likelihood-based belief function

Requirements

Let $Bel_{\boldsymbol{y}}^{\Theta}$ be a belief function representing our knowledge about θ after observing $\boldsymbol{y}$. We impose the following requirements:

1. Likelihood principle: $Bel_{\boldsymbol{y}}^{\Theta}$ should be based only on the likelihood function

$$\theta \to L_{\boldsymbol{y}}(\theta) = f_{\theta}(\boldsymbol{y})$$

2. Compatibility with Bayesian inference: when a Bayesian prior P_0 is available, combining it with $Bel_{\boldsymbol{y}}^{\Theta}$ using Dempster's rule should yield the Bayesian posterior:

$$Bel_{\boldsymbol{y}}^{\Theta} \oplus P_0 = P(\cdot|\boldsymbol{y})$$

3. Principle of minimal commitment: among all the belief functions satisfying the previous two requirements, $Bel_{\boldsymbol{y}}^{\Theta}$ should be the least committed (least informative)

Likelihood-based belief function

Solution (Denœux, 2014)

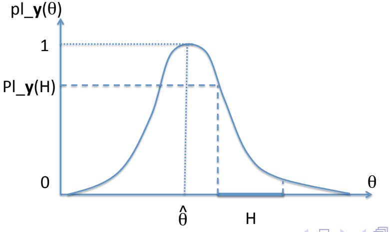
- $Bel_{\boldsymbol{y}}^{\Theta}$ is the consonant belief function induced by the relative likelihood function

$$pl_{\boldsymbol{y}}(\boldsymbol{\theta}) = \frac{L_{\boldsymbol{y}}(\boldsymbol{\theta})}{L_{\boldsymbol{y}}(\widehat{\boldsymbol{\theta}})}$$

where $\widehat{\boldsymbol{\theta}}$ is a MLE of $\boldsymbol{\theta}$, and it is assumed that $L_{\boldsymbol{y}}(\widehat{\boldsymbol{\theta}}) < +\infty$

- Corresponding plausibility function

$$Pl_{\boldsymbol{y}}^{\Theta}(H) = \sup_{\theta \in H} pl_{\boldsymbol{y}}(\theta), \quad \forall H \subseteq \Theta$$

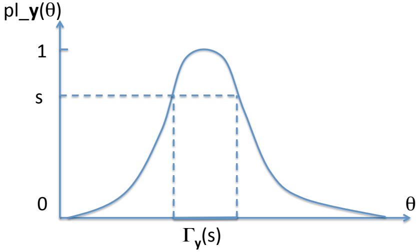


Source

- Corresponding random set:

$$\Gamma_{\mathbf{y}}(s) = \left\{ \boldsymbol{\theta} \in \Theta | \frac{L_{\mathbf{y}}(\boldsymbol{\theta})}{L_{\mathbf{y}}(\widehat{\boldsymbol{\theta}})} \geq s \right\}$$

with s uniformly distributed in $[0, 1]$



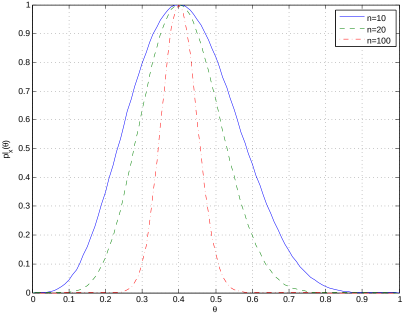
- If $\Theta \subseteq \mathbb{R}$ and if $L_{\mathbf{y}}(\theta)$ is unimodal and upper-semicontinuous, then $Bel_{\mathbf{y}}^{\Theta}$ corresponds to a random closed interval

Binomial example

In the urn model, $Y \sim \mathcal{B}(n, \theta)$ and

$$pl_y(\theta) = \frac{\theta^y (1-\theta)^{n-y}}{\widehat{\theta}^y (1-\widehat{\theta})^{n-y}} = \left(\frac{\theta}{\widehat{\theta}}\right)^{n\widehat{\theta}} \left(\frac{1-\theta}{1-\widehat{\theta}}\right)^{n(1-\widehat{\theta})}$$

for all $\theta \in \Theta = [0,1]$, where $\widehat{\theta} = y/n$ is the MLE of θ

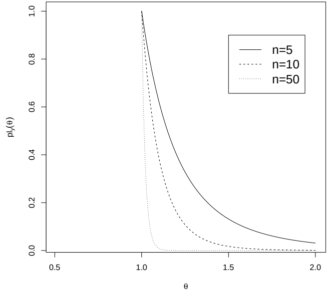


Uniform example

- Let $\boldsymbol{y} = (y_1, \ldots, y_n)$ be a realization from an iid random sample from $\mathcal{U}([0, \theta])$
- The likelihood function is

$$L_{\boldsymbol{y}}(\theta) = \theta^{-n} \mathbb{1}_{[y_{(n)}, +\infty)}(\theta)$$

- The likelihood-based BF is induced by the random closed interval $[y_{(n)}, y_{(n)} S^{-1/n}]$, with $S \sim \mathcal{U}([0, 1])$



Profile likelihood

- Assume that $\boldsymbol{\theta} = (\boldsymbol{\xi}, \boldsymbol{\nu}) \in \Omega_{\boldsymbol{\xi}} \times \Omega_{\boldsymbol{\nu}}$, where $\boldsymbol{\xi}$ is a parameter of interest and $\boldsymbol{\nu}$ is a nuisance parameter
- Then, the marginal contour function for $\boldsymbol{\xi}$ is

$$pl_{\boldsymbol{y}}(\boldsymbol{\xi}) = Pl(\{\boldsymbol{\xi}\} \times \Omega_{\boldsymbol{\nu}}) = \sup_{\boldsymbol{\nu} \in \Omega_{\boldsymbol{\nu}}} pl_{\boldsymbol{y}}(\boldsymbol{\xi}, \boldsymbol{\nu}),$$

  which is the profile relative likelihood function
- The profiling method for eliminating nuisance parameter thus has a natural justification in our approach
- When the quantities $pl_{\boldsymbol{y}}(\boldsymbol{\xi})$ cannot be derived analytically, they have to be computed numerically using an iterative optimization algorithm

Relation with likelihood-based inference

- The approach to statistical inference outlined here is very close to the "likelihoodist" approach advocated by Birnbaum (1962), Barnard (1962), and Edwards (1992), among others
- The main difference resides in the interpretation of the likelihood function as defining a belief function
- This interpretation allows us to quantify the uncertainty in statements of the form $\boldsymbol{\theta} \in H$, where H may contain multiple values. This is in contrast with the classical likelihood approach, in which only the likelihood of single hypotheses is defined
- The belief function interpretation provides an easy and natural way to combine statistical information with other information, such as expert judgements

Relation with the likelihood-ratio test statistics

- We can also notice that $Pl_{\boldsymbol{y}}^{\Theta}(H)$ is identical to the likelihood ratio statistic for H
- From Wilk's theorem, we have asymptotically (under regularity conditions), when H holds,

$$-2 \ln Pl_{\boldsymbol{y}}(H) \sim \chi^2_r$$

  where r is the number of restrictions imposed by H
- Consequently,
  - rejecting hypothesis H if its plausibility is smaller than $\exp(-\chi^2_{r;1-\alpha}/2)$ is a testing procedure with significance level approximately equal to α
  - The sets $\Gamma(\exp(-\chi^2_{r;1-\alpha}/2))$ are approximate $1-\alpha$ confidence regions
- However, these properties are coincidental, as the approach outlined here is not based on frequentist inference

Combination with a Bayesian prior

- The likelihood-based method described here does not require any prior knowledge of $\boldsymbol{\theta}$.
- However, by construction, this approach boils down to Bayesian inference if a prior probability $g(\boldsymbol{\theta})$ is provided and combined with $Bel_{\boldsymbol{y}}^{\Theta}$ by Dempster's rule.
- As it will usually not be possible to compute the analytical expression of the resulting posterior distribution, it can be approximated by Monte Carlo simulation. (see next slide)
- We can see that this is just the rejection sampling algorithm with the prior $g(\theta)$ as proposal distribution.
- The rejection sampling algorithm can thus be seen, in this case, as a Monte Carlo approximation to Dempster's rule of combination.

Combination with a Bayesian prior (continued)

Monte Carlo algorithm for combining the likelihood-based belief function with a Bayesian prior by Dempster's rule

Require: Desired number of focal sets N

$i \leftarrow 0$

while $i < N$ **do**

Draw s in $[0,1]$ from the uniform probability measure λ on $[0,1]$

Draw $\boldsymbol{\theta}$ from the prior probability distribution $g(\boldsymbol{\theta})$

if $pl_{\mathbf{y}}(\boldsymbol{\theta}) \geq s$ **then**

$i \leftarrow i+1$

$\boldsymbol{\theta}_i \leftarrow \boldsymbol{\theta}$

end if

end while

Outline

Behrens-Fisher problem

- Let the observed data $\boldsymbol{y}$ be composed of two independent normal samples $\boldsymbol{y}_1 = (y_{11}, \ldots, y_{1n_1})$ and $\boldsymbol{y}_2 = (y_{21}, \ldots, y_{2n_2})$ from $\mathcal{N}(\mu_1, \sigma_1^2)$ and $\mathcal{N}(\mu_2, \sigma_2^2)$, respectively.
- We wish to compare the means μ_1 and μ_2.
- Using the frequentist approach, this is done by computing a p-value for the hypothesis $H_0 : \mu_1 = \mu_2$ of equality of means, or a confidence interval on $\mu_1 - \mu_2$. This problem, known as the Behrens-Fisher problem, only has approximate solutions
- Using our approach, the means are compared by computing the plausibility of H_0 or, more generally, of $H_\delta : \mu_1 - \mu_2 = \delta$

Belief function solution

- The marginal contour function for (μ_1, μ_2) is

$$pl_{\mathbf{y}}(\mu_1, \mu_2) = \sup_{\sigma_1, \sigma_2} pl_{\mathbf{y}}(\boldsymbol{\theta})$$
$$= \frac{\prod_{k=1}^{2} L_{\mathbf{y}_k}(\mu_k, \widehat{\sigma}_k(\mu_k))}{\prod_{k=1}^{2} L_{\mathbf{y}_k}(\overline{y}_k, s_k)},$$

where

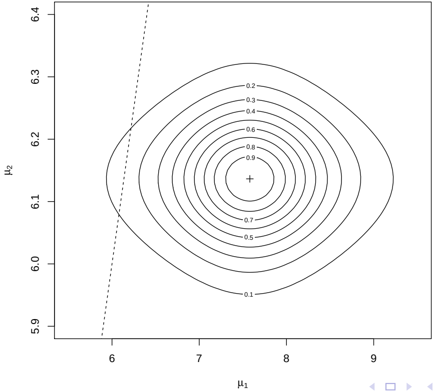
$$\widehat{\sigma}_k(\mu_k) = \frac{1}{n_k} \sum_{i=1}^{n_k} (y_{ki} - \mu_k)^2.$$

- The plausibility of $H_\delta = \{(\mu_1, \mu_2) \in \mathbb{R}^2 | \mu_1 - \mu_2 = \delta\}$ can then be computed by maximizing $pl_{\mathbf{y}}(\mu_1, \mu_2)$ under the constraint $\mu_1 - \mu_2 = \delta$, i.e.,

$$Pl_{\mathbf{y}}(H_\delta) = \max_{\mu_1} pl_{\mathbf{y}}(\mu_1, \mu_1 - \delta)$$

Example (Lehman, 1975)

We consider the following driving times from a person's house to work measured for two different routes: $\boldsymbol{y}_1 = (6.5, 6.8, 7.1, 7.3, 10.2)$ and $\boldsymbol{y}_2 = (5.8, 5.8, 5.9, 6.0, 6.0, 6.0, 6.3, 6.3, 6.4, 6.5, 6.5)$. Are the mean traveling times equal?



Linear regression
Model

We consider the following standard regression model

$$\boldsymbol{y} = X\boldsymbol{\beta} + \boldsymbol{\epsilon}$$

where

- $\boldsymbol{y} = (y_1, \ldots, y_n)'$ is the vector of n observations of the dependent variable
- X is the fixed design matrix of size $n \times (p+1)$
- $\boldsymbol{\epsilon} = (\epsilon_1, \ldots, \epsilon_n)' \sim \mathcal{N}(\mathbf{0}, I_n)$ is the vector of errors
- The vector of coefficients is $\boldsymbol{\theta} = (\boldsymbol{\beta}', \sigma)'$

Likelihood-based belief function

- The likelihood function for this model is

$$L_{\boldsymbol{y}}(\boldsymbol{\theta}) = (2\pi\sigma^2)^{-n/2} \exp\left[-\frac{1}{2\sigma^2}(\boldsymbol{y} - X\boldsymbol{\beta})'(\boldsymbol{y} - X\boldsymbol{\beta})\right]$$

- The contour function can thus be readily calculated as

$$pl_{\boldsymbol{y}}(\boldsymbol{\theta}) = \frac{L_{\boldsymbol{y}}(\boldsymbol{\theta})}{L_{\boldsymbol{y}}(\widehat{\boldsymbol{\theta}})}$$

with $\widehat{\boldsymbol{\theta}} = (\widehat{\boldsymbol{\beta}}', \widehat{\sigma})'$, where

  - $\widehat{\boldsymbol{\beta}} = (X'X)^{-1}X'\boldsymbol{y}$ is the ordinary least squares estimate of $\boldsymbol{\beta}$
  - $\widehat{\sigma}$ is the standard deviation of residuals

Plausibility of linear hypotheses

- Assertions (hypotheses) H of the form $A\boldsymbol{\beta} = \boldsymbol{q}$, where A is a $r \times (p+1)$ constant matrix and $\boldsymbol{q}$ is a constant vector of length r, for some $r \leq p+1$
- Special cases: $\{\beta_j = 0\}$, $\{\beta_j = 0, \forall j \in \{1, \ldots, p\}\}$, or $\{\beta_j = \beta_k\}$, etc.
- The plausibility of H is

$$Pl_{\boldsymbol{y}}^{\Theta}(H) = \sup_{A\boldsymbol{\beta}=\boldsymbol{q}} pl_{\boldsymbol{y}}(\boldsymbol{\theta}) = \frac{L_{\boldsymbol{y}}(\widehat{\boldsymbol{\theta}}_*)}{L_{\boldsymbol{y}}(\widehat{\boldsymbol{\theta}})}$$

where $\widehat{\boldsymbol{\theta}}_* = (\widehat{\boldsymbol{\beta}}'_*, \widehat{\sigma}_*)'$ (restricted LS estimates) with

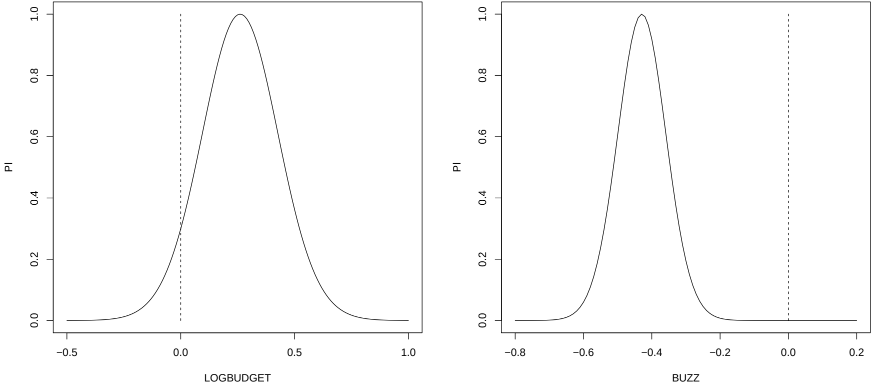
$$\widehat{\boldsymbol{\beta}}_* = \widehat{\boldsymbol{\beta}} - (X'X)^{-1}A'[A(X'X)^{-1}A']^{-1}(A\widehat{\boldsymbol{\beta}} - \boldsymbol{q})$$

$$\widehat{\sigma}_* = \sqrt{(\boldsymbol{y} - X\widehat{\boldsymbol{\beta}}_*)'(\boldsymbol{y} - X\widehat{\boldsymbol{\beta}}_*)/n}$$

Example: movie Box office data

- Dataset about 62 movies released in 2009 (from Greene, 2012)
- Dependent variable: logarithm of Box Office receipts
- 11 covariates:
  - 3 dummy variables (G, PG, PG13) to encode the MPAA (Motion Picture Association of America) rating, logarithm of budget (LOGBUDGET), star power (STARPOWR),
  - a dummy variable to indicate if the movie is a sequel (SEQUEL),
  - four dummy variables to describe the genre (ACTION, COMEDY, ANIMATED, HORROR)
  - one variable to represent internet buzz (BUZZ)

Some marginal contour functions



Regression coefficients

	Estimate	Std. Error	t-value	p-value	$Pl(\beta_j = 0)$
(Intercept)	15.400	0.643	23.960	$<$ 2e-16	1.0e-34
G	0.384	0.553	0.695	0.49	0.74
PG	0.534	0.300	1.780	0.081	0.15
PG13	0.215	0.219	0.983	0.33	0.55
LOGBUDGET	0.261	0.185	1.408	0.17	0.30
STARPOWR	4.32e-3	0.0128	0.337	0.74	0.93
SEQUEL	0.275	0.273	1.007	0.32	0.54
ACTION	-0.869	0.293	-2.964	4.7e-3	6.6e-3
COMEDY	-0.0162	0.256	-0.063	0.95	0.99
ANIMATED	-0.833	0.430	-1.937	0.058	0.11
HORROR	0.375	0.371	1.009	0.32	0.54
BUZZ	0.429	0.0784	5.473	1.4e-06	4.8e-07

Outline

1. Belief functions on infinite spaces
   - Definition
   - Practical models
   - Combination and propagation
2. Estimation
   - Justification
   - Likelihood-based belief function
   - Examples
   - Consistency
3. Prediction
   - Predictive belief function
   - Examples

Consistency of the likelihood-based belief function

- Assume that the observed data $\boldsymbol{y} = (y_1, \ldots, y_n)$ is a realization of an iid sample $\boldsymbol{Y} = (Y_1, \ldots, Y_n)$ from $Y \sim f_{\boldsymbol{\theta}}(y)$
- From Fraser (1968):

Theorem

If $\mathbb{E}_{\boldsymbol{\theta}_0}[\log f_{\boldsymbol{\theta}}(Y)]$ exists, is finite for all $\boldsymbol{\theta}$, and has a unique maximum at $\boldsymbol{\theta}_0$, then, for any $\boldsymbol{\theta} \neq \boldsymbol{\theta}_0$, $pl_n(\boldsymbol{\theta}) \rightarrow 0$ almost surely under the law determined by $\boldsymbol{\theta}_0$

Consistency of the likelihood-based belief function (continued)

- The property $pl_n(\boldsymbol{\theta}_0) \to 1$ a.s. does not hold in general (under regularity assumptions, $-2\log pl_n(\boldsymbol{\theta}_0)$ converges in distribution to χ^2_p)
- But we have the following theorem:

Theorem

Under some assumptions (Fraser, 1968), for any neighborhood N of $\boldsymbol{\theta}_0$, $Bel_n^{\Theta}(N) \to 1$ and $Pl_n^{\Theta}(N) \to 1$ almost surely under the law determined by $\boldsymbol{\theta}_0$

Outline

Outline

Prediction problem

- Observed (past) data: $\boldsymbol{y}$ from $\boldsymbol{Y} \sim f_{\boldsymbol{\theta}}(\boldsymbol{y})$
- Future data: $Z|\boldsymbol{y} \sim F_{\boldsymbol{\theta},\boldsymbol{y}}(z)$ (real random variable)
- Problem: quantify the uncertainty of Z using a predictive belief function

Outline of the approach (1/2)

- Let us come back to the urn example
- Let $Z \sim \mathcal{B}(\theta)$ be defined as

$$Z = \begin{cases} 1 & \text{if next ball is black} \\ 0 & \text{otherwise} \end{cases}$$

- We can write Z as a function of θ and a pivotal variable $W \sim \mathcal{U}([0,1])$,

$$\begin{aligned} Z &= \begin{cases} 1 & \text{if } W \leq \theta \\ 0 & \text{otherwise} \end{cases} \\ &= \varphi(\theta, W) \end{aligned}$$

Outline of the approach (2/2)

- The equality

$$Z = \varphi(\theta, W)$$

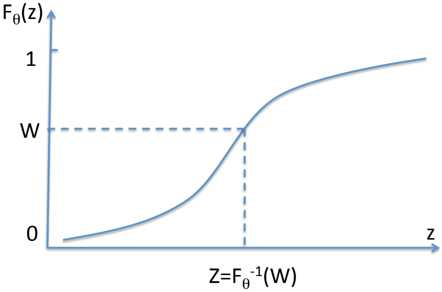
allows us to separate the two sources of uncertainty on Z

1. uncertainty on W (random/aleatory uncertainty)
2. uncertainty on θ (estimation/epistemic uncertainty)

- Two-step method:

1. Represent uncertainty on θ using a likelihood-based belief function Bel_y^{Θ} constructed from the observed data y (estimation problem)
2. Combine Bel_y^{Θ} with the probability distribution of W to obtain a predictive belief function $Bel_y^{\mathbb{Z}}$

φ-equation



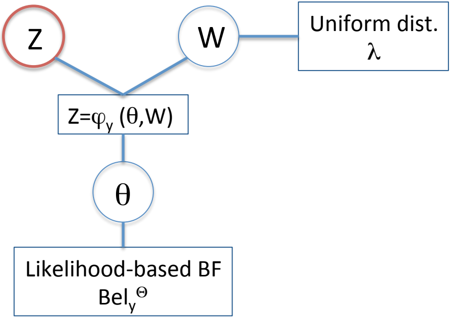
We can always write Z as a function of $\boldsymbol{\theta}$ and W as

$$Z = F^{-1}_{\boldsymbol{\theta},\mathbf{y}}(W) = \varphi_{\mathbf{y}}(\boldsymbol{\theta}, W)$$

where $W \sim \mathcal{U}([0,1])$ and $F^{-1}_{\boldsymbol{\theta},\mathbf{y}}$ is the generalized inverse of $F_{\boldsymbol{\theta},\mathbf{y}}$,

$$F^{-1}_{\boldsymbol{\theta},\mathbf{y}}(W) = \inf\{z | F_{\boldsymbol{\theta},\mathbf{y}}(z) \geq W\}$$

Main result

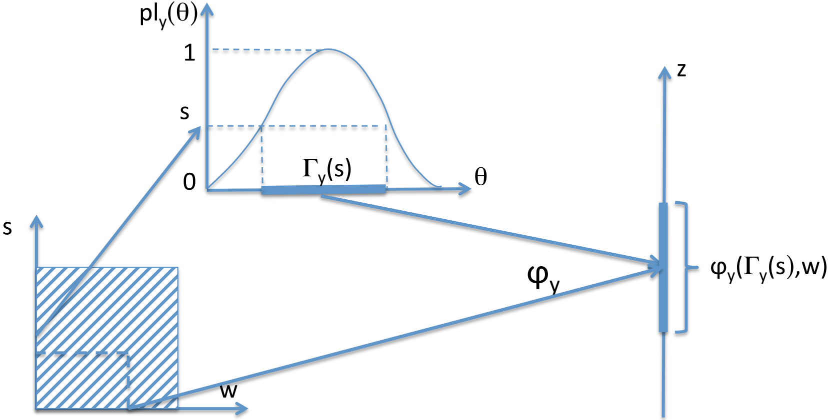


After combination by Dempster's rule and marginalization on $\mathbb{Z}$, we obtain the predictive BF on Z induced by the multi-valued mapping

$$(s, w) \to \varphi_{\boldsymbol{y}}(\Gamma_{\boldsymbol{y}}(s), w).$$

with (s, w) uniformly distributed in $[0, 1]^2$

Graphical representation



Practical computation

- Analytical expression when possible (simple cases), or
- Monte Carlo simulation:
  1. Draw N pairs (s_i, w_i) independently from a uniform distribution
  2. compute (or approximate) the focal sets $\varphi_{\boldsymbol{y}}(\Gamma_{\boldsymbol{y}}(s_i), w_i)$
- The predictive belief and plausibility of any subset $A \subseteq \mathbb{Z}$ are then estimated by

$$\widehat{Bel}_{\boldsymbol{y}}^{\mathbb{Z}}(A) = \frac{1}{N}\#\{i \in \{1, \ldots, N\} | \varphi_{\boldsymbol{y}}(\Gamma_{\boldsymbol{y}}(s_i), w_i) \subseteq A\}$$

$$\widehat{Pl}_{\boldsymbol{y}}^{\mathbb{Z}}(A) = \frac{1}{N}\#\{i \in \{1, \ldots, N\} | \varphi_{\boldsymbol{y}}(\Gamma_{\boldsymbol{y}}(s_i), w_i) \cap A \neq \emptyset\}$$

Example: the urn model

- Here, $Y \sim \mathcal{B}(n, \theta)$. The likelihood-based BF is induced by a random interval

$$\Gamma(s) = \{\theta : pl_y(\theta) \geq s\} = [\underline{\theta}(s), \overline{\theta}(s)]$$

- We have

$$Z = \varphi(\theta, W) = \begin{cases} 1 & \text{if } W \leq \theta \\ 0 & \text{otherwise} \end{cases}$$

- Consequently,

$$\varphi\left(\Gamma(s), W\right) = \varphi\left(\left[\underline{\theta}(s), \overline{\theta}(s)\right], W\right) = \begin{cases} \{1\} & \text{if } W \leq \underline{\theta}(s) \\ \{0\} & \text{if } \overline{\theta}(s) < W \\ \{0, 1\} & \text{otherwise} \end{cases}$$

Example: the urn model

Properties

We have

$$Bel_y(\{1\}) = \mathbb{E}(\underline{\theta}(S)) = \widehat{\theta} - \int_0^{\widehat{\theta}} pl_y(\theta)d\theta$$

$$Pl_y(\{1\}) = \mathbb{E}(\overline{\theta}(S)) = \widehat{\theta} + \int_{\widehat{\theta}}^1 pl_y(\theta)d\theta$$

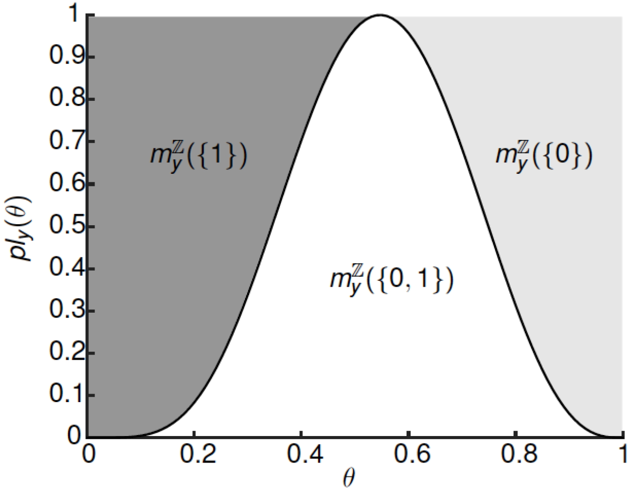
So

$$m(\{0,1\}) = \int_0^1 pl_y(\theta)d\theta$$

As $n \to \infty$, $m(\{1\}) \to 1$ and $m(\{0,1\}) \to 0$ in probability.

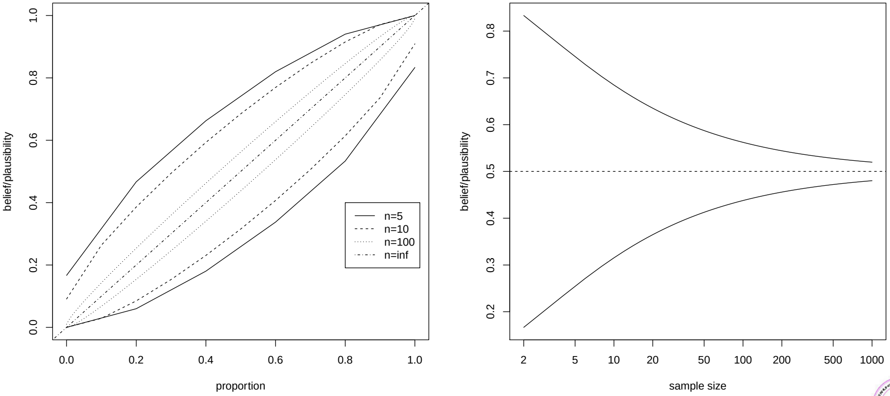
Example: the urn model

Geometric representation



Example: the urn model

Belief/plausibility intervals



Uniform example

- Assume that $Y_1, \ldots, Y_n, Z$ is iid from $\mathcal{U}([0, \theta])$
- Then $F_\theta(z) = z/\theta$ for all $0 \leq z \leq \theta$ and we can write $Z = \theta W$ with $W \sim \mathcal{U}([0, 1])$
- We have seen that the belief function $Bel_{\boldsymbol{y}}^{\Theta}$ after observing $\boldsymbol{Y} = \boldsymbol{y}$ is induced by the random interval $[y_{(n)}, y_{(n)} S^{-1/n}]$
- Each focal set of $Bel_{\boldsymbol{y}}^{\mathbb{Z}}$ is an interval

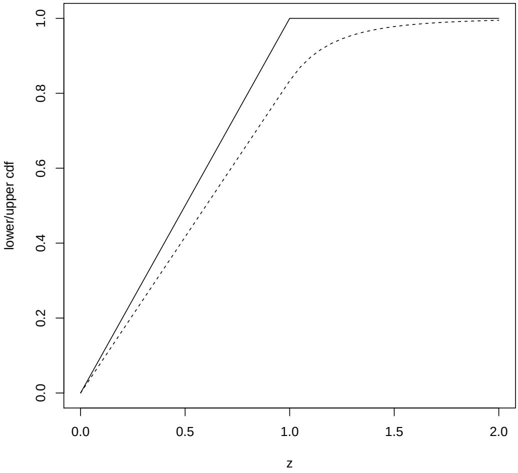
$$\varphi(\Gamma_{\boldsymbol{y}}(s), w) = [y_{(n)} w, y_{(n)} s^{-1/n} w]$$

- The predictive belief function $Bel_{\boldsymbol{y}}^{\mathbb{Z}}$ is induced by the random interval

$$[\widehat{Z}_{\boldsymbol{y}*}, \widehat{Z}_{\boldsymbol{y}}^{*}] = [y_{(n)} W, y_{(n)} S^{-1/n} W]$$

Uniform example

Lower and upper cdfs



Uniform example

Consistency

- From the consistency of the MLE, $Y_{(n)}$ converges in probability to θ_0, so

$$\widehat{Z}_{\mathbf{Y}*} = Y_{(n)}W \xrightarrow{d} \theta_0 W = Z$$

- We have $\mathbb{E}(S^{-1/n}) = n/(n-1)$, and

$$\mathrm{Var}(S^{-1/n}) = \frac{n}{(n-2)(n-1)^2}$$

- Consequently, $\mathbb{E}(S^{-1/n}) \to 1$ and $\mathrm{Var}(S^{-1/n}) \to 0$, so $S^{-1/n} \xrightarrow{P} 1$
- Hence,

$$\widehat{Z}^*_{\mathbf{Y}} = Y_{(n)}S^{-1/n}W \xrightarrow{d} \theta_0 W = Z$$

Consistency (general case)

- Assume that
  - The observed data $\boldsymbol{y} = (y_1, \ldots, y_n)$ is a realization of an iid sample $\boldsymbol{Y} = (Y_1, \ldots, Y_n)$
  - The likelihood function $L_n(\boldsymbol{\theta})$ is unimodal and upper-semicontinuous, so that its level sets $\Gamma_n(s)$ are closed and connected, and that function $\varphi(\boldsymbol{\theta}, w)$ is continuous
- Under these conditions, the random set $\varphi(\Gamma_n(S), W)$ is a closed random interval $[\widehat{Z}_{*n}, \widehat{Z}_n^*]$
- Then:

Theorem

Assume that the conditions of the previous theorem hold, and that the predictive belief function $Bel_n^{\mathbb{Z}}$ is induced by a random closed interval $[\widehat{Z}_{*n}, \widehat{Z}_n^*]$. Then $\widehat{Z}_{*n}$ and $\widehat{Z}_n^*$ both converge in distribution to Z when n tends to infinity.

Outline

Linear model: prediction

- Let z be a not-yet observed value of the dependent variable for a vector $\boldsymbol{x}_0$ of covariates:

$$z = \boldsymbol{x}_0'\boldsymbol{\beta} + \epsilon_0,$$

with $\epsilon_0 \sim \mathcal{N}(0, \sigma^2)$

- We can write, equivalently,

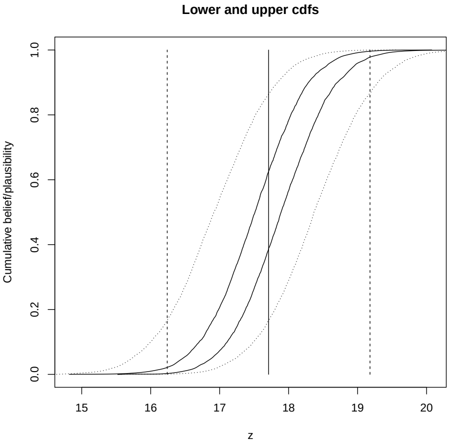
$$z = \boldsymbol{x}_0'\boldsymbol{\beta} + \sigma\Phi^{-1}(w) = \varphi_{\boldsymbol{x}_0,\boldsymbol{y}}(\boldsymbol{\theta}, w),$$

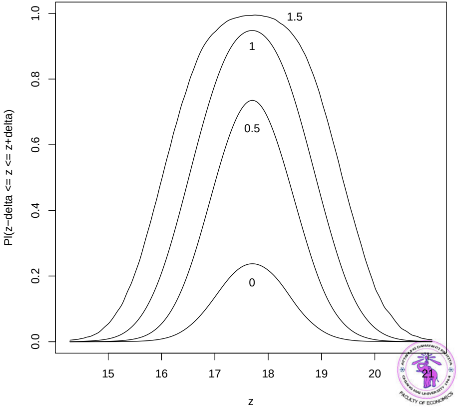
where w has a standard uniform distribution

- The predictive belief function on z can then be approximated using Monte Carlo simulation

Movie example

BO success of an action sequel film rated PG13 by MPAA, with LOGBUDGET=5.30, STARPOWER=23.62 and BUZZ= 2.81?





Ex ante forecasting

Problem and classical approach

- Consider the situation where some explanatory variables are unknown at the time of the forecast and have to be estimated or predicted
- Classical approach: assume that $\boldsymbol{x}_0$ has been estimated with some variance, which has to be taken into account in the calculation of the forecast variance
- According to Green (Econometric Analysis, 7th edition, 2012)
  - "*This vastly complicates the computation. Many authors view it as simply intractable*"
  - "*analytical results for the correct forecast variance remain to be derived except for simple special cases*"

Ex ante forecasting

Belief function approach

- In contrast, this problem can be handled very naturally in our approach by modeling partial knowledge of $\boldsymbol{x}_0$ by a belief function $Bel^{\mathbb{X}}$ in the sample space $\mathbb{X}$ of $\boldsymbol{x}_0$
- We then have
$$Bel_{\boldsymbol{y}}^{\mathbb{Z}} = \left(Bel_{\boldsymbol{y}}^{\Theta} \oplus Bel_{\boldsymbol{y}}^{\mathbb{Z}\times\Theta} \oplus Bel^{\mathbb{X}}\right)^{\downarrow\mathbb{Z}}$$
- Assume that the belief function $Bel^{\mathbb{X}}$ is induced by a source $(\Omega, \mathcal{A}, \mathbb{P}^{\Omega}, \Lambda)$, where Λ is a multi-valued mapping from Ω to $2^{\mathbb{X}}$
- The predictive belief function $Bel_{\boldsymbol{y}}^{\mathbb{Z}}$ is then induced by the multi-valued mapping
$$(\omega, s, w) \to \varphi_y(\Lambda(\omega), \Gamma_{\boldsymbol{y}}(s), w)$$
- $Bel_{\boldsymbol{y}}^{\mathbb{Z}}$ can be approximated by Monte Carlo simulation

Monte Carlo algorithm

Require: Desired number of focal sets N

for $i = 1$ **to** N **do**

Draw (s_i, w_i) uniformly in $[0,1]^2$

Draw ω from $\mathbb{P}^\Omega$

Search for $z_{*i} = \min_{\boldsymbol{\theta}} \varphi_{\mathbf{y}}(\boldsymbol{x}_0, \boldsymbol{\theta}, w_i)$ such that $pl_{\mathbf{y}}(\boldsymbol{\theta}) \geq s_i$ and $\boldsymbol{x}_0 \in \Lambda(\omega)$

Search for $z_i^* = \max_{\boldsymbol{\theta}} \varphi_{\mathbf{y}}(\boldsymbol{x}_0, \boldsymbol{\theta}, w_i)$ such that $pl_{\mathbf{y}}(\boldsymbol{\theta}) \geq s_i$ and $\boldsymbol{x}_0 \in \Lambda(\omega)$

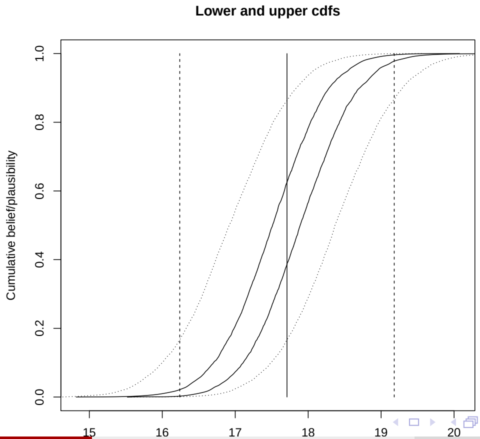
$B_i \leftarrow [z_{*i}, z_i^*]$

end for

Movie example

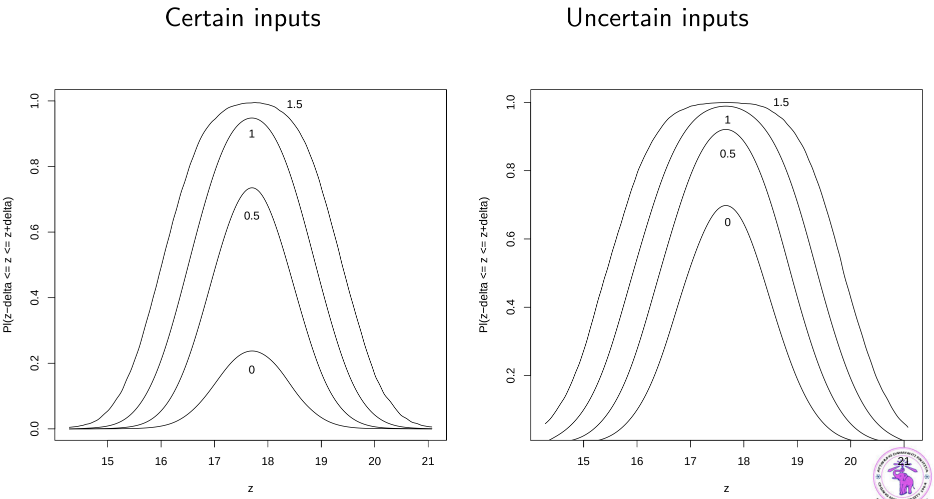
Lower and upper cdfs

BO success of an action sequel film rated PG13 by MPAA, with LOGBUDGET=5.30, STARPOWER=23.62 and BUZZ= (0,2.81,5) (triangular possibility distribution)?



Movie example

Pl-plots



Innovation diffusion

- Forecasting the diffusion of an innovation has been a topic of considerable interest in marketing research
- Typically, when a new product is launched, sale forecasts have to be based on little data and uncertainty has to be quantified to avoid making wrong business decisions based on unreliable forecasts
- Our approach uses the Bass model (Bass, 1969) for innovation diffusion together with past sales data to quantify the uncertainty on future sales using the formalism of belief functions

Bass model

- Fundamental assumption (Bass, 1969): for eventual adopters, the probability $f(t)$ of purchase at time t, given that no purchase has yet been made, is an affine function of the number of previous buyers

$$\frac{f(t)}{1-F(t)} = p + qF(t)$$

where p is a coefficient of innovation, q is a coefficient of imitation and $F(t) = \int_0^t f(u)du$.

- Solving this differential equation, the probability that an individual taken at random from the population will buy the product before time t is

$$\Phi_\theta(t) = cF(t) = \frac{c(1-\exp[-(p+q)t])}{1+(p/q)\exp[-(p+q)t]}$$

where c is the probability of eventually adopting the product and $\theta = (p, q, c)$

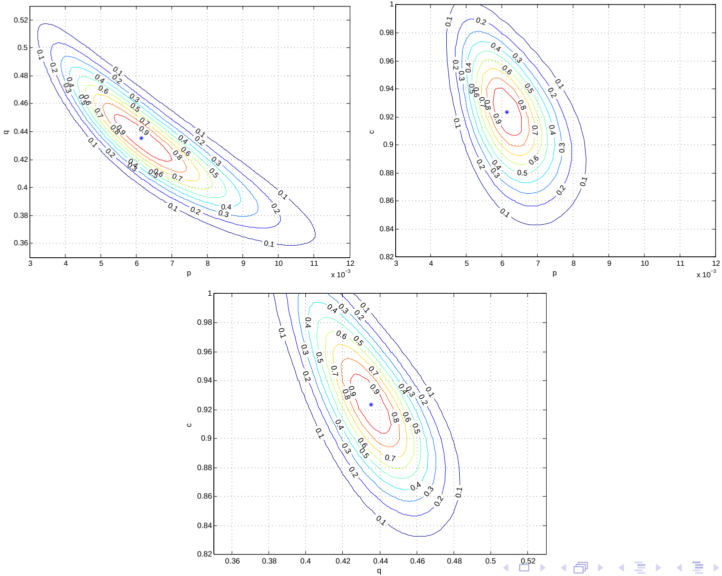
Parameter estimation

- Data: $y_1, \ldots, y_{T-1}$, where y_i = observed number of adopters in time interval $[t_{i-1}, t_i)$
- The number of individuals in the sample of size M who did not adopt the product at time t_{T-1} is $y_T = M - \sum_{i=1}^{T-1} y_i$
- The probability of adopting the innovation between times t_{i-1} and t_i is $p_i = \Phi_\theta(t_i) - \Phi_\theta(t_{i-1})$ for $1 \leq i \leq T-1$, and the probability of not adopting the innovation before t_{T-1} is $p_T = 1 - \Phi_\theta(t_{T-1})$
- Consequently, $\boldsymbol{y} = (y_1, \ldots, y_T)$ is a realization of $\boldsymbol{Y} \sim \mathcal{M}(M, p_1, \ldots, p_T)$ and the likelihood function is

$$L_{\boldsymbol{y}}(\theta) \propto \prod_{i=1}^{T} p_i^{y_i} = \left(\prod_{i=1}^{T-1} [\Phi_\theta(t_i) - \Phi_\theta(t_{i-1})]^{y_i} \right) [1 - \Phi_\theta(t_{T-1})]^{y_T}$$

- The belief function on θ is defined by $pl_{\boldsymbol{y}}(\theta) = L_{\boldsymbol{y}}(\theta)/L_{\boldsymbol{y}}(\widehat{\theta})$

Results



Sales forecasting

- Let us assume we are at time t_{T-1} and we wish to forecast the number Z of sales between times τ_1 and τ_2, with $t_{T-1} \leq \tau_1 < \tau_2$
- Z has a binomial distribution $\mathcal{B}(Q, \pi_\theta)$, where
  - Q is the number of potential adopters at time $T-1$
  - π_θ is the probability of purchase for an individual in $[\tau_1, \tau_2]$, given that no purchase has been made before t_{T-1}

$$\pi_\theta = \frac{\Phi_\theta(\tau_2) - \Phi_\theta(\tau_1)}{1 - \Phi_\theta(t_{T-1})}$$

- Z can be written as $Z = \varphi(\theta, \boldsymbol{W}) = \sum_{i=1}^{Q} \mathbb{1}_{[0,\pi_\theta]}(W_i)$ where

$$\mathbb{1}_{[0,\pi_\theta]}(W_i) = \begin{cases} 1 & \text{if } W_i \leq \pi_\theta \\ 0 & \text{otherwise} \end{cases}$$

and $\boldsymbol{W} = (W_1, \ldots, W_Q)$ has a uniform distribution in $[0, 1]^Q$.

Predictive belief function

Multi-valued mapping

- The predictive belief function on Z is induced by the multi-valued mapping $(s, \boldsymbol{w}) \to \varphi(\Gamma_{\boldsymbol{y}}(s), \boldsymbol{w})$ with

$$\Gamma_{\boldsymbol{y}}(s) = \{\theta \in \Theta : pl_{\boldsymbol{y}}(\theta) \geq s\}$$

- When θ varies in $\Gamma_{\boldsymbol{y}}(s)$, the range of π_θ is $[\underline{\pi}_\theta(s), \overline{\pi}_\theta(s)]$, with

$$\underline{\pi}_\theta(s) = \min_{\{\theta | pl_{\boldsymbol{y}}(\theta) \geq s\}} \pi_\theta, \quad \overline{\pi}_\theta(s) = \max_{\{\theta | pl_{\boldsymbol{y}}(\theta) \geq s\}} \pi_\theta$$

- We have

$$\varphi(\Gamma_{\boldsymbol{y}}(s), \boldsymbol{w}) = [\underline{Z}(s, \boldsymbol{w}), \overline{Z}(s, \boldsymbol{w})],$$

  where $\underline{Z}(s, \boldsymbol{w})$ and $\overline{Z}(s, \boldsymbol{w})$ are, respectively, the number of w_i's that are less than $\underline{\pi}_\theta(s)$ and $\overline{\pi}_\theta(s)$

- For fixed s, $\underline{Z}(s, \boldsymbol{W}) \sim \mathcal{B}(Q, \underline{\pi}_\theta(s))$ and $\overline{Z}(s, \boldsymbol{W}) \sim \mathcal{B}(Q, \overline{\pi}_\theta(s))$

Predictive belief function

Calculation

- The belief and plausibilities that Z will be less than z are

$$Bel_{\mathbf{y}}^{\mathbb{Z}}([0,z]) = \int_0^1 F_{Q,\underline{\pi}_\theta(s)}(z)ds$$

$$Pl_{\mathbf{y}}^{\mathbb{Z}}([0,z]) = \int_0^1 F_{Q,\overline{\pi}_\theta(s)}(z)ds$$

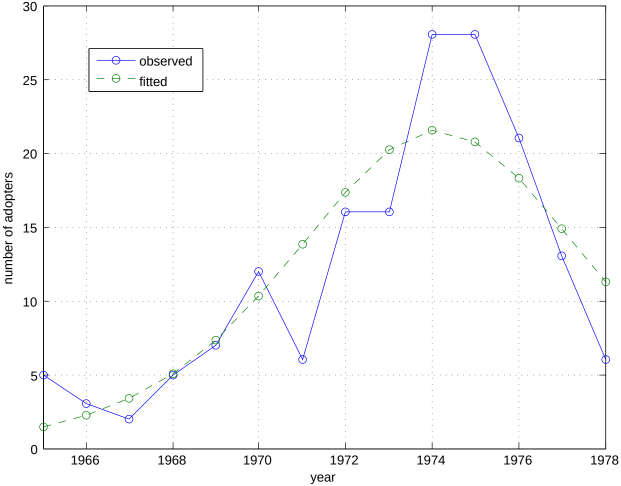
  where $F_{Q,p}$ denotes the cdf of the binomial distribution $\mathcal{B}(Q,p)$
- The contour function of Z is

$$pl_{\mathbf{y}}(z) = \int_0^1 \left(F_{Q,\underline{\pi}_\theta(s)}(z) - F_{Q,\overline{\pi}_\theta(s)}(z-1)\right) ds$$

- Theses integrals can be approximated by Monte-Carlo simulation

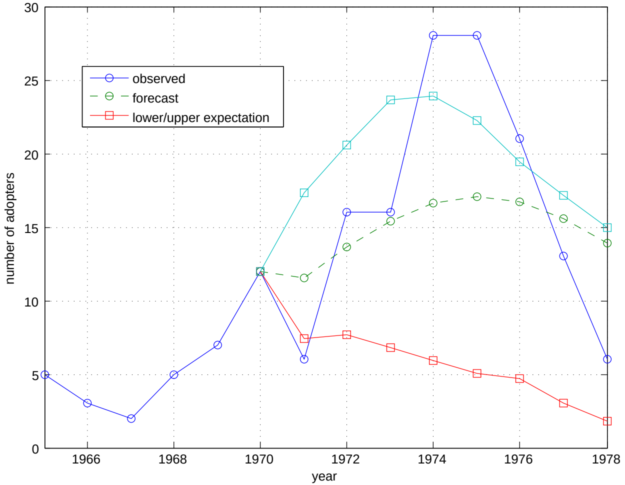
Ultrasound data

Data collected from 209 hospitals through the U.S.A. (Schmittlein and Mahajan, 1982) about adoption of an ultrasound equipment



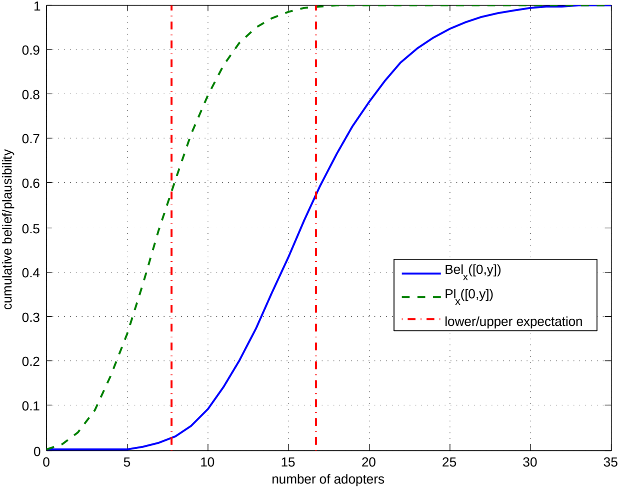
Forecasting

Predictions made in 1970 for the number of adopters in the period 1971-1978, with their lower and upper expectations



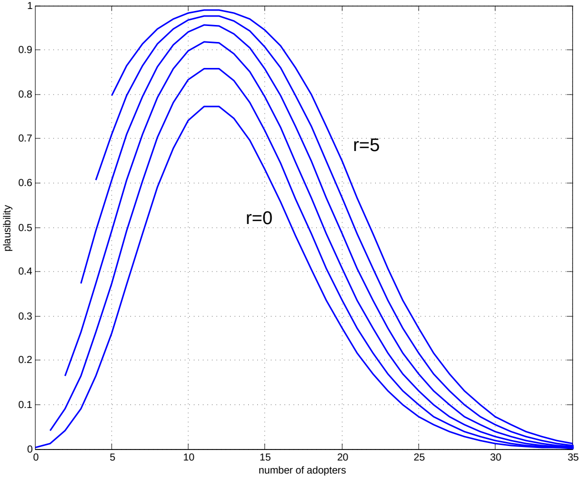
Cumulative belief and plausibility functions

Lower and upper cumulative distribution functions for the number of adopters in 1971, forecasted in 1970



Pl-plot

Plausibilities $Pl_{\boldsymbol{y}}^{\mathbb{Y}}([z-r, z+r])$ as functions of z, from $r=0$ (lower curve) to $r=5$ (upper curve), for the number of adopters in 1971, forecasted in 1970:



Conclusions

- Uncertainty quantification is an important component of any forecasting methodology. The approach introduced in this lecture allows us to represent forecast uncertainty in the belief function framework, based on past data and a statistical model
- The proposed method is conceptually simple and computationally tractable
- The belief function formalism makes it possible to combine information from several sources (such as expert opinions and statistical data)
- The Bayesian predictive probability distribution is recovered when a prior on θ is available
- The consistency of the method has been established under some conditions

References

cf. https://www.hds.utc.fr/~tdenoeux

T. Denoeux.
Likelihood-based belief function: justification and some extensions to low-quality data
International Journal of Approximate Reasoning, 55(7):1535–1547, 2014

O. Kanjanatarakul, S. Sriboonchitta and T. Denœux
Forecasting using belief functions. An application to marketing econometrics.
International Journal of Approximate Reasoning, 55(5):1113–1128, 2014

O. Kanjanatarakul, T. Denœux and S. Sriboonchitta
Prediction of future observations using belief functions: a likelihood-based approach
International Journal of Approximate Reasoning, Vol. 72, pages 71-94, 2016.