Theory of Belief Functions
Chapter 2: Decision-Making with Belief Functions

Thierry Denœux

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Example of decision problem under uncertainty

<table>
<thead>
<tr>
<th>Act (Purchase)</th>
<th>Good Economic Conditions</th>
<th>Poor Economic Conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Apartment building</td>
<td>50,000</td>
<td>30,000</td>
</tr>
<tr>
<td>Office building</td>
<td>100,000</td>
<td>-40,000</td>
</tr>
<tr>
<td>Warehouse</td>
<td>30,000</td>
<td>10,000</td>
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</table>
A decision problem can be seen as a situation in which a decision-maker (DM) has to choose a course of action (an act) in some set \( \mathcal{F} = \{f_1, \ldots, f_n\} \).

An act may have different consequences (outcomes), depending on the state of nature.

Denoting by \( \Omega = \{\omega_1, \ldots, \omega_r\} \) the set of states of nature and by \( \mathcal{C} \) the set of consequences (or outcomes), an act can be formalized as a mapping \( f \) from \( \Omega \) to \( \mathcal{C} \).

In this lecture, the three sets \( \Omega \), \( \mathcal{C} \) and \( \mathcal{F} \) will be assumed to be finite.
Formal framework
Utilities

- The desirability of the consequences can often be modeled by a numerical utility function \( u : C \rightarrow \mathbb{R} \), which assigns a numerical value to each consequence.
- The higher this value, the more desirable is the consequence for the DM.
- In some problems, the consequences can be evaluated in terms of monetary value. The utilities can then be defined as the payoffs, or a function thereof.
- If the actions are indexed by \( i \) and the states of nature by \( j \), we will denote by \( u_{ij} \) the quantity \( u[f_i(\omega_j)] \).
- The \( n \times r \) matrix \( U = (u_{ij}) \) will be called a payoff or utility matrix.
## Payoff matrix

<table>
<thead>
<tr>
<th>Act (Purchase)</th>
<th>Good Economic Conditions ($\omega_1$)</th>
<th>Poor Economic Conditions ($\omega_2$)</th>
</tr>
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<tbody>
<tr>
<td>Apartment building ($f_1$)</td>
<td>50,000</td>
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</table>
Formal framework
Preferences

- If the true state of nature $\omega$ is known, the desirability of an act $f$ can be deduced from that of its consequence $f(\omega)$
- Typically, the state of nature is unknown. Based on partial information, it is usually assumed that the DM can express preferences among acts, which may be represented mathematically by a preference relation $\succeq$ on $\mathcal{F}$
- This relation is interpreted as follows: given two acts $f$ and $g$, $f \succeq g$ means that $f$ is found by the DM to be at least as desirable as $g$
- We also define
  - The strict preference relation as $f \succ g$ iff $f \succeq g$ and not($g \succeq f$) (meaning that $f$ is strictly more desirable than $g$) and
  - The indifference relation $f \sim g$ iff $f \succeq g$ and $g \succeq f$ (meaning that $f$ and $g$ are equally desirable)
The decision problem can be formalized as building a preference relation among acts, from a utility matrix and some description of uncertainty, and finding the maximal elements of this relation. Depending on the nature of the available information, different decision problems arise:

1. Decision-making under ignorance
2. Decision-making with probabilities
3. Decision-making with belief functions
Outline

1. Classical decision theory
   - Decision-making under complete ignorance
   - Decision-making with probabilities
   - Savage’s theorem

2. Decision-making with belief functions
   - Upper and lower expected utility
   - Other approaches
Outline

1. Classical decision theory
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Problem and non-domination principle

- We assume that the DM is **totally ignorant of the state of nature**: all the information given to the DM is the utility matrix $U$.

- A act $f_i$ is said to be **dominated** by $f_k$ if the outcomes of $f_k$ are at least as desirable as those of $f_i$ for all states, and strictly more desirable for at least one state:

$$\forall j, \quad u_{kj} \geq u_{ij} \text{ and } \exists j, \quad u_{kj} > u_{ij}$$

- **Non-domination principle**: an act cannot be chosen if it is dominated by another one.
### Example of a dominated act

<table>
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<td>10,000</td>
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Criteria for rational choice

After all dominated acts have been removed, there remains the problem of ordering them by desirability, and of finding the set of most desirable acts.

Several criteria of “rational choice” have been proposed to derive a preference relation over acts:

1. **Laplace criterion**

   \[ f_i \succeq f_k \iff \frac{1}{r} \sum_j u_{ij} \geq \frac{1}{r} \sum_j u_{kj}. \]

2. **Maximax criterion**

   \[ f_i \succeq f_k \iff \max_j u_{ij} \geq \max_j u_{kj}. \]

3. **Maximin (Wald) criterion**

   \[ f_i \succeq f_k \iff \min_j u_{ij} \geq \min_j u_{kj}. \]
Example

<table>
<thead>
<tr>
<th>Act</th>
<th>$\omega_1$</th>
<th>$\omega_2$</th>
<th>ave</th>
<th>max</th>
<th>min</th>
</tr>
</thead>
<tbody>
<tr>
<td>Apartment ($f_1$)</td>
<td>50,000</td>
<td>30,000</td>
<td><strong>40,000</strong></td>
<td>50,000</td>
<td><strong>30,000</strong></td>
</tr>
<tr>
<td>Office ($f_2$)</td>
<td>100,000</td>
<td>-40,000</td>
<td>30,000</td>
<td><strong>100,000</strong></td>
<td>-40,000</td>
</tr>
</tbody>
</table>
Hurwicz criterion

- Hurwicz criterion: \( f_i \succeq f_k \) iff

\[
\alpha \min_j u_{ij} + (1 - \alpha) \max_j u_{ij} \geq \alpha \min_j u_{kj} + (1 - \alpha) \max_j u_{kj}
\]

where \( \alpha \) is a parameter in \([0, 1]\), called the pessimism index.
- Boils down to
  - the maximax criterion if \( \alpha = 0 \)
  - the maximin criterion if \( \alpha = 1 \)
- \( \alpha \) describes the DM’s attitude toward ambiguity
Minimax regret criterion criterion

(Savage) Minimax regret criterion: an act $f_i$ is at least as desirable as $f_k$ if it has smaller maximal regret, where regret is defined as the utility difference with the best act, for a given state of nature.

- The regret $r_{ij}$ for act $f_i$ and state $\omega_j$ is
  \[ r_{ij} = \max_{\ell} u_{\ell j} - u_{ij} \]

- The maximum regret for act $f_i$ is $R_i = \max_j r_{ij}$
- $f_i \succeq f_k$ iff $R_i \leq R_k$
Example

- Pay-off matrix

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- Regret matrix

<table>
<thead>
<tr>
<th>Act</th>
<th>$\omega_1$</th>
<th>$\omega_2$</th>
<th>max regret</th>
</tr>
</thead>
<tbody>
<tr>
<td>Apartment ($f_1$)</td>
<td>50,000</td>
<td>0</td>
<td>50,000</td>
</tr>
<tr>
<td>Office ($f_2$)</td>
<td>0</td>
<td>70,000</td>
<td>70,000</td>
</tr>
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Axioms of rational choice

- Let $\mathcal{F}^*$ denote the choice set, defined as a set of optimal acts.
- Arrow and Hurwicz (1972) have proposed four axioms a choice operator $\mathcal{F} \to \mathcal{F}^*$ should verify.

1. Axiom $A_1$: if $\mathcal{F}_1 \subset \mathcal{F}_2$ and $\mathcal{F}_2^* \cap \mathcal{F}_1 \neq \emptyset$, then $\mathcal{F}_1^* = \mathcal{F}_2^* \cap \mathcal{F}_1$.
2. Axiom $A_2$: Relabeling actions and states does not change the optimal status of actions.
3. Axiom $A_3$: Deletion of a duplicate state does not change the optimality status of actions ($\omega_j$ and $\omega_\ell$ are duplicate if $u_{ij} = u_{i\ell}$ for all $i$).
4. Axiom $A_4$ (dominance): If $f \in \mathcal{F}^*$ and $f'$ dominates $f$, then $f' \in \mathcal{F}^*$. If $f \notin \mathcal{F}^*$ and $f$ dominates $f'$, then $f' \notin \mathcal{F}^*$.
Axioms of rational choice (continued)

- Under some regularity assumptions, Axioms $A_1 - A_4$ imply that the choice set depends only on the worst and the best consequences of each act.
- In particular, these axioms rule out the Laplace and minimax regret criteria.
Violation of Axiom A3 by the Laplace criterion

<table>
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<th>$\omega_2$</th>
<th>ave</th>
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<tr>
<td>Apartment ($f_1$)</td>
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<td>30,000</td>
<td><strong>40,000</strong></td>
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<td>100,000</td>
<td>-40,000</td>
<td>30,000</td>
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Let us split the state of nature $\omega_1$ in two states: “Good economic conditions and there is life on Mars” ($\omega'_1$) and “Good economic conditions and there is no life on Mars” ($\omega''_1$)

<table>
<thead>
<tr>
<th>Act</th>
<th>$\omega'_1$</th>
<th>$\omega''_1$</th>
<th>$\omega_2$</th>
<th>ave</th>
</tr>
</thead>
<tbody>
<tr>
<td>Apartment ($f_1$)</td>
<td>50,000</td>
<td>50,000</td>
<td>30,000</td>
<td><strong>43,333</strong></td>
</tr>
<tr>
<td>Office ($f_2$)</td>
<td>100,000</td>
<td>100,000</td>
<td>-40,000</td>
<td><strong>53,333</strong></td>
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Violation of Axiom A1 by minimax regret

Pay-off matrix

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</tr>
<tr>
<td>$f_4$</td>
<td>130,000</td>
<td>-45,000</td>
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Regret matrix

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<td>Apartment ($f_1$)</td>
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<td>0</td>
<td>80,000</td>
</tr>
<tr>
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<td>30,000</td>
<td>70,000</td>
<td>70,000</td>
</tr>
<tr>
<td>$f_4$</td>
<td>0</td>
<td>75,000</td>
<td>75,000</td>
</tr>
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We had $\mathcal{F}_1 = \{f_1, f_2\}$ and $\mathcal{F}_1^* = \{f_1\}$. Now, $\mathcal{F}_2 = \{f_1, f_2, f_4\}$ and $\mathcal{F}_2^* = \{f_2\}$. So, $\mathcal{F}_1^* \neq \mathcal{F}_2^* \cap \mathcal{F}_1$
Outline

1. Classical decision theory
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2. Decision-making with belief functions
   - Upper and lower expected utility
   - Other approaches
Maximum Expected Utility principle

- Let us now consider the situation where uncertainty about the state of nature is quantified by probabilities $p_1, \ldots, p_r$ on $\Omega$.
- These probabilities can be objective (decision under risk) or subjective.
- We can then compute, for each act $f_i$, its expected utility as

$$EU(f_i) = \sum_j u_{ij}p_j$$

- Maximum Expected Utility (MEU) principle: an act $f_i$ is more desirable than an act $f_k$ if it has a higher expected utility: $f_i \succeq f_k$ iff $EU(f_i) \geq EU(f_k)$. 
Example

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Assume that there is 60% chance that the economic situation will be poor ($\omega_2$). The expected utilities of acts $f_1$ and $f_2$ are

$$EU(f_1) = 50,000 \times 0.4 + 30,000 \times 0.6 = 38,000$$

$$EU(f_2) = 100,000 \times 0.4 - 40,000 \times 0.6 = 16,000$$

Act $f_1$ is thus more desirable according to the maximum expected utility criterion.
Axiomatic justification of the MEU principle

- The MEU principle was first axiomatized by von Neumann and Morgenstern (1944).
- Given a probability distribution on $\Omega$, an act $f : \Omega \rightarrow C$ induces a probability measure $P$ on the set $C$ of consequences (assumed to be finite), called a lottery.
- We denote by $\mathcal{L}$ the set of lotteries on $C$.
- If we agree that two acts providing the same lottery are equivalent, then the problem of comparing the desirability of acts becomes that of comparing the desirability of lotteries.
- Let $\succeq$ be a preference relation among lotteries. Von Neumann and Morgenstern argued that, to be rational, a preference relation should verify three axioms.
Von Neumann and Morgenstern’s axioms

1. **Complete preorder**: the preference relation is a complete and non trivial preorder (i.e., it is a reflexive, transitive and complete relation) on $\mathcal{L}$

2. **Continuity**: for any lotteries $P$, $Q$ and $R$ such that $P \succ Q \succ R$, there exists probabilities $\alpha$ and $\beta$ in $[0, 1]$ such that

$$\alpha P + (1 - \alpha) R \succ Q \succ \beta P + (1 - \beta) R$$

where $\alpha P + (1 - \alpha) R$ is a compound lottery, which refers to the situation where you receive $P$ with probability $\alpha$ and $R$ with probability $1 - \alpha$. This axiom implies, in particular, that there is no lottery $R$ that is so undesirable that it cannot become desirable if mixed with some very desirable lottery $P$

3. **Independence**: for any lotteries $P$, $Q$ and $R$ and for any $\alpha \in (0, 1]$

$$P \succeq Q \iff \alpha P + (1 - \alpha) R \succeq \alpha Q + (1 - \alpha) R$$
Von Neumann and Morgenstern’s theorem

The two following propositions are equivalent:

1. The preference relation $\succeq$ verifies the axioms of complete preorder, continuity, and independence

2. There exists a utility function $u : C \rightarrow \mathbb{R}$ such that, for any two lotteries $P = (p_1, \ldots, p_r)$ and $Q = (q_1, \ldots, q_r)$

$$P \succeq Q \iff \sum_{i=1}^{r} p_i u(c_i) \geq \sum_{i=1}^{r} q_i u(c_i)$$

Function $u$ is unique up to a strictly increasing affine transformation
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Savage’s theorem

- We have reviewed some criteria for decision-making under complete ignorance, i.e., when uncertainty cannot be probabilized.
- Some researchers have defended the view that a rational DM always maximizes expected utility, for some subjective probability measure and utility function.
- **Savage’s theorem (1954):** A preference relation \( \succsim \) among acts verifies some rationality requirements iff there is a finitely additive probability measure \( P \) and a utility function \( u : C \to \mathbb{R} \) such that

\[
\forall f, g \in F, \quad f \succsim g \iff \int_{\Omega} u(f(\omega))dP(\omega) \geq \int_{\Omega} u(g(\omega))dP(\omega)
\]

Furthermore, \( P \) is unique and \( u \) is unique up to a positive affine transformation.
- A strong argument for probabilism, but Savage’s axioms can be questioned!
Savage’s axioms

- Savage has proposed seven axioms, four of which are considered as meaningful (the other three are technical).
- Axiom 1: \( \succsim \) is a total preorder (complete, reflexive and transitive).
- Axiom 2 [Sure Thing Principle]. Given \( f, h \in \mathcal{F} \) and \( E \subseteq \Omega \), let \( fEh \) denote the act defined by

\[
(fEh)(\omega) = \begin{cases} 
  f(\omega) & \text{if } \omega \in E \\
  h(\omega) & \text{if } \omega \notin E
\end{cases}
\]

Then the Sure Thing Principle states that \( \forall E, \forall f, g, h, h' \)

\[
fEh \succ gEh \Rightarrow fEh' \succ gEh'
\]

*The preference between two acts with a common extension outside some event \( E \) does not depend on this common extension.*

- This axiom seems reasonable, but it is not verified empirically!
Ellsberg’s paradox

- Suppose you have an urn containing 30 red balls and 60 balls, either black or yellow. Consider the following gambles:
  - $f_1$: You receive 100 euros if you draw a red ball
  - $f_2$: You receive 100 euros if you draw a black ball
  - $f_3$: You receive 100 euros if you draw a red or yellow ball
  - $f_4$: You receive 100 euros if you draw a black or yellow ball

- Do you prefer $f_1$ or $f_2$? $f_3$ or $f_4$?
Ellsberg’s paradox

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Do you prefer $f_1$ or $f_2$? $f_3$ or $f_4$?

Most people strictly prefer $f_1$ to $f_2$, but they strictly prefer $f_4$ to $f_3$

<table>
<thead>
<tr>
<th></th>
<th>$R$</th>
<th>$B$</th>
<th>$Y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f_1$</td>
<td>100</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$f_2$</td>
<td>0</td>
<td>100</td>
<td>0</td>
</tr>
<tr>
<td>$f_3$</td>
<td>100</td>
<td>0</td>
<td>100</td>
</tr>
<tr>
<td>$f_4$</td>
<td>0</td>
<td>100</td>
<td>100</td>
</tr>
</tbody>
</table>

Now,

$f_1 = f_1\{R, B\}0$, $f_2 = f_2\{R, B\}0$

$f_3 = f_1\{R, B\}100$, $f_4 = f_2\{R, B\}100$

The Sure Thing Principle is violated!
Classically, we distinguish two kinds of decision problems:

1. **Decision under ignorance**: we only know, for each act, a set of possible outcomes.
2. **Decision under risk**: we are given, for each act, a probability distribution over the outcomes.

It has been argued that any decision problem under uncertainty should be handled as a problem of decision under risk. However, the axiomatic arguments are questionable.

In the next part: decision-making when uncertainty is described by a belief functions.
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Belief functions become components of a decision problem in any of the following two situations (or both)

1. The decision maker’s subjective beliefs concerning the state of nature are described by a belief function $Bel^\Omega$ on $\Omega$

2. The DM is not able to precisely describe the outcomes of some acts under each state of nature
Case 1: uncertainty described by a belief function

- Let $m^\Omega$ be a mass function on $\Omega$.
- Any act $f : \Omega \rightarrow C$ carries $m^\Omega$ to the set $C$ of consequences, yielding a mass function $m_f^C$, which quantifies the DM’s beliefs about the outcome of act $f$.
- Each mass $m^\Omega(A)$ is transferred to $f(A)$

$$m_f^C(B) = \sum_{\{A \subseteq \Omega | f(A) = B\}} m^\Omega(A)$$

for any $B \subseteq C$.

- $m_f^C$ is a credibilistic lottery corresponding to act $f$. 
Case 2: partial knowledge of outcomes

- In that case, an act may formally be represented by a multi-valued mapping \( f : \Omega \rightarrow 2^C \), assigning a set of possible consequences \( f(\omega) \subseteq C \) to each state of nature \( \omega \).
- Given a probability measure \( P \) on \( \Omega \), \( f \) then induces the following mass function \( m_f^C \) on \( C \),

\[
m_f^C(B) = \sum_{\{\omega \in \Omega | f(\omega) = B\}} p(\omega)
\]

for all \( B \subseteq C \).
Example

- Let $\Omega = \{\omega_1, \omega_2, \omega_3\}$ and $m^\Omega$ the following mass function:
  \[ m^\Omega(\{\omega_1, \omega_2\}) = 0.3, \quad m^\Omega(\{\omega_2, \omega_3\}) = 0.2 \]
  \[ m^\Omega(\{\omega_3\}) = 0.4, \quad m^\Omega(\Omega) = 0.1 \]

- Let $C = \{c_1, c_2, c_3\}$ and $f$ the act:
  \[ f(\omega_1) = \{c_1\}, \quad f(\omega_2) = \{c_1, c_2\}, \quad f(\omega_3) = \{c_2, c_3\} \]

- To compute $m^C_f$, we transfer the masses as follows:
  \[ m^\Omega(\{\omega_1, \omega_2\}) = 0.3 \rightarrow f(\omega_1) \cup f(\omega_2) = \{c_1, c_2\} \]
  \[ m^\Omega(\{\omega_2, \omega_3\}) = 0.2 \rightarrow f(\omega_2) \cup f(\omega_3) = \{c_1, c_2, c_3\} \]
  \[ m^\Omega(\{\omega_3\}) = 0.4 \rightarrow f(\omega_3) = \{c_2, c_3\} \]
  \[ m^\Omega(\Omega) = 0.1 \rightarrow f(\omega_1) \cup f(\omega_2) \cup f(\omega_3) = \{c_1, c_2, c_3\} \]

- Finally, we obtain the following mass function on $C$:
  \[ m^C(\{c_1, c_2\}) = 0.3, \quad m^C(\{c_2, c_3\}) = 0.4, \quad m^C(C) = 0.3 \]
Decision problem

- In the two situations considered above, we can assign to each act $f$ a credibilistic lottery, defined as a mass function on $C$.
- Given a utility function $u$ on $C$, we then need to extend the MEU model.
- Several such extensions will now be reviewed.
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Let $m$ be a mass function on $C$, and $u$ a utility function $C \rightarrow \mathbb{R}$.

The **lower and upper expectations** of $u$ are defined, respectively, as the averages of the minima and the maxima of $u$ within each focal set of $m$.

$$
\underline{E}_m(u) = \sum_{A \subseteq C} m(A) \min_{c \in A} u(c)
$$

$$
\overline{E}_m(u) = \sum_{A \subseteq C} m(A) \max_{c \in A} u(c)
$$

It is clear that $\underline{E}_m(u) \leq \overline{E}_m(u)$, with the inequality becoming an equality when $m$ is Bayesian, in which case the lower and upper expectations collapse to the usual expectation.

If $m = m_A$ is logical with focal set $A$, then $\underline{E}_m(u)$ and $\overline{E}_m(u)$ are, respectively, the minimum and the maximum of $u$ in $A$. 

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**Upper and lower expected utility**

Decision-making with belief functions

Thierry Denœux

BF - Decision-Making

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Imprecise probability interpretation

- The lower and upper expectations are lower and upper bounds of expectations with respect to probability measures compatible with $m$

$$\underline{E}_m(u) = \min_{P \in \mathcal{P}(m)} E_P(u)$$

$$\overline{E}_m(u) = \max_{P \in \mathcal{P}(m)} E_P(u)$$

- The mean of minima (res., maxima) is also the minimum (resp., maximum) of means with respect to all compatible probability measures
Having defined the notions of lower and upper expectations, we can define two preference relations among credibilistic lotteries as

\[ m_1 \succeq m_2 \text{ iff } \mathbb{E}_{m_1}(u) \geq \mathbb{E}_{m_2}(u) \]

and

\[ m_1 \succeq m_2 \text{ iff } \mathbb{E}_{m_1}(u) \leq \mathbb{E}_{m_2}(u) \]

Relation \( \succeq \) corresponds to a pessimistic (or conservative) attitude of the DM. When \( m \) is logical, it corresponds to the maximin criterion.

Symmetrically, \( \succeq \) corresponds to an optimistic attitude and extends the maximax criterion.

Both criteria boil down to the MEU criterion when \( m \) is Bayesian.
Back to Ellsberg’s paradox

- Here, $\Omega = \{R, B, Y\}$ and $m^\Omega(\{R\}) = 1/3$, $m^\Omega(\{B, Y\}) = 2/3$
- The mass functions on $C = \{0, 100\}$ induced by the four acts are

$$
m_1(\{100\}) = 1/3, \quad m_1(\{0\}) = 2/3
$$
$$
m_2(\{0\}) = 1/3, \quad m_2(\{0, 100\}) = 2/3
$$
$$
m_3(\{100\}) = 1/3, \quad m_3(\{0, 100\}) = 2/3
$$
$$
m_4(\{0\}) = 1/3, \quad m_4(\{100\}) = 2/3
$$

- Corresponding lower and upper expectations

<table>
<thead>
<tr>
<th>$R$</th>
<th>$B$</th>
<th>$Y$</th>
<th>$\underline{E}_m(u)$</th>
<th>$\overline{E}_m(u)$</th>
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<tbody>
<tr>
<td>$f_1$</td>
<td>100</td>
<td>0</td>
<td>0</td>
<td>$u(100)/3$</td>
</tr>
<tr>
<td>$f_2$</td>
<td>0</td>
<td>100</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$f_3$</td>
<td>100</td>
<td>0</td>
<td>100</td>
<td>$u(100)/3$</td>
</tr>
<tr>
<td>$f_4$</td>
<td>0</td>
<td>100</td>
<td>100</td>
<td>$2u(100)/3$</td>
</tr>
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</table>
Interval dominance

- If we drop the requirement that the preference relation among acts be complete, then we can consider the interval dominance relation,

\[ m_1 \succ_{ID} m_2 \iff \underline{E}_{m_1}(u) \geq \underline{E}_{m_2}(u) \]

- Given a collection of credibilistic lotteries, we can then compute the set of maximal (i.e., non-dominated) elements of \( \succ_{ID} \)

- Imprecise probability view

\[ m_1 \succ_{ID} m_2 \iff \forall P_1 \in \mathcal{P}(m_1), \forall P_2 \in \mathcal{P}(m_2), \underline{E}_{P_1}(u) \geq \underline{E}_{P_2}(u) \]

- The justification for this preference relation is not so clear from the point of view of belief function theory (i.e., if one does not interpret a belief function as a lower probability)
Outline

1. Classical decision theory
   - Decision-making under complete ignorance
   - Decision-making with probabilities
   - Savage’s theorem

2. Decision-making with belief functions
   - Upper and lower expected utility
   - Other approaches
The Hurwicz criterion can be generalized as

$$E_{m,\alpha}(u) = \sum_{A \subseteq C} m(A) \left( \alpha \min_{c \in A} u(c) + (1 - \alpha) \max_{c \in A} u(c) \right)$$

$$= \alpha E_m(u) + (1 - \alpha) E(u)$$

where $\alpha \in [0, 1]$ is a pessimism index.

This criterion was introduced and justified axiomatically by Jaffray (1988).

Strat (1990) who proposed to interpret $\alpha$ as the DM’s subjective probability that the ambiguity will be resolved unfavorably.
Transferable belief model

- A completely different approach to decision-making with belief function was advocated by Smets, as part of the Transferable Belief Model.
- Smets defended a two-level mental model:
  1. A credal level, where an agent’s belief are represented by belief functions, and
  2. A pignistic level, where decisions are made by maximizing the EU with respect to a probability measure derived from a belief function.
- The rationale for introducing probabilities at the decision level is the avoidance of Dutch books.
- Smets argued that the belief-probability transformation $T$ should be linear, i.e., it should verify

$$T(\alpha m_1 + (1 - \alpha)m_2) = \alpha T(m_1) + (1 - \alpha) T(m_2),$$

for any mass functions $m_1$ and $m_2$ and for any $\alpha \in [0, 1]$. 
Pignistic transformation

- The only linear belief-probability transformation \( T \) is the pignistic transformation, with \( p_m = T(m) \) given by

\[
p_m(c) = \sum_{\{A \subseteq C \mid c \in A\}} \frac{m(A)}{|A|}, \quad \forall c \in C
\]

- The expected utility w.r.t. the pignistic probability is

\[
E_p(u) = \sum_{c \in C} p_m(c) u(c) = \sum_{A \subseteq C} m(A) \left( \frac{1}{|A|} \sum_{c \in A} u(c) \right)
\]

- The maximum pignistic expected utility criterion thus extends the Laplace criterion
Generalized minimax regret

- Yager (2004) also extended the minimax regret criterion to belief functions
- We need to consider $n$ acts $f_1, \ldots, f_n$, and we write $u_{ij} = u[f_i(\omega_j)]$
- The regret if act $f_i$ is selected, and state $\omega_j$ occurs, is $r_{ij} = \max_k u_{kj} - u_{ij}$
- For a non-empty subset $A$ of $\Omega$, the maximum regret of act $f_i$ is

$$R_i(A) = \max_{\omega_j \in A} r_{ij}$$

- The expected maximal regret for act $f_i$ is

$$\bar{R}_i = \sum_{\emptyset \neq A \subseteq \Omega} m^\Omega(A) R_i(A)$$

- Act $f_i$ is preferred over act $f_k$ if $\bar{R}_i \leq \bar{R}_k$
- The minimax regret criterion is recovered when $m^\Omega$ is logical
- The MEU model is recovered when $m^\Omega$ is Bayesian
### Summary

<table>
<thead>
<tr>
<th>non-probabilized</th>
<th>belief functions</th>
<th>probabilized</th>
</tr>
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<tbody>
<tr>
<td>maximin</td>
<td>lower expectation</td>
<td></td>
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<tr>
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<td>minimax regret</td>
<td>generalized minimax regret</td>
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