Theory of Belief Functions

Chapter 2: Decision-Making with Belief Functions

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August 2019

Example of decision problem under uncertainty

Act (Purchase)	Good Economic Conditions	Poor Economic Conditions
Apartment building	50,000	30,000
Office building	100,000	-40,000
Warehouse	30,000	10,000

Formal framework

Acts, outcomes, states of nature

- A decision problem can be seen as a situation in which a decision-maker (DM) has to choose a course of action (an act) in some set $\mathcal{F} = \{f_1, \ldots, f_n\}$
- An act may have different consequences (outcomes), depending on the state of nature
- Denoting by $\Omega = \{\omega_1, \ldots, \omega_r\}$ the set of states of nature and by $\mathcal{C}$ the set of consequences (or outcomes), an act can be formalized as a mapping f from Ω to $\mathcal{C}$
- In this lecture, the three sets Ω, $\mathcal{C}$ and $\mathcal{F}$ will be assumed to be finite

Formal framework

Utilities

- The desirability of the consequences can often be modeled by a numerical utility function $u : \mathcal{C} \to \mathbb{R}$, which assigns a numerical value to each consequence
- The higher this value, the more desirable is the consequence for the DM
- In some problems, the consequences can be evaluated in terms of monetary value. The utilities can then be defined as the payoffs, or a function thereof
- If the actions are indexed by i and the states of nature by j, we will denote by u_{ij} the quantity $u[f_i(\omega_j)]$
- The $n \times r$ matrix $U = (u_{ij})$ will be called a payoff or utility matrix

Payoff matrix

Act (Purchase)	Good Economic Conditions (ω_1)	Poor Economic Conditions (ω_2)
Apartment building (f_1)	50,000	30,000
Office building (f_2)	100,000	-40,000
Warehouse (f_3)	30,000	10,000

Formal framework

Preferences

- If the true state of nature ω is known, the desirability of an act f can be deduced from that of its consequence $f(\omega)$
- Typically, the state of nature is unknown. Based on partial information, it is usually assumed that the DM can express preferences among acts, which may be represented mathematically by a preference relation $\succcurlyeq$ on $\mathcal{F}$
- This relation is interpreted as follows: given two acts f and g, $f \succcurlyeq g$ means that f is found by the DM to be at least as desirable as g
- We also define
  - The strict preference relation as $f \succ g$ iff $f \succcurlyeq g$ and not($g \succcurlyeq f$) (meaning that f is strictly more desirable than g) and
  - The indifference relation $f \sim g$ iff $f \succcurlyeq g$ and $g \succcurlyeq f$ (meaning that f and g are equally desirable)

Decision problems

- The decision problem can be formalized as building a preference relation among acts, from a utility matrix and some description of uncertainty, and finding the maximal elements of this relation
- Depending on the nature of the available information, different decision problems arise:
  1. Decision-making under ignorance
  2. Decision-making with probabilities
  3. Decision-making with belief functions

Outline

1. Classical decision theory
   - Decision-making under complete ignorance
   - Decision-making with probabilities
   - Savage's theorem

2. Decision-making with belief functions
   - Upper and lower expected utility
   - Other approaches

Outline

Problem and non-domination principle

- We assume that the DM is totally ignorant of the state of nature: all the information given to the DM is the utility matrix U
- A act f_i is said to be dominated by f_k if the outcomes of f_k are at least as desirable as those of f_i for all states, and strictly more desirable for at least one state

$$\forall j, \ u_{kj} \geq u_{ij} \text{ and } \exists j, \ u_{kj} > u_{ij}$$

- Non-domination principle: an act cannot be chosen if it is dominated by another one

Example of a dominated act

Act (Purchase)	Good Economic Conditions (ω_1)	Poor Economic Conditions (ω_2)
Apartment building (f_1)	50,000	30,000
Office building (f_2)	100,000	-40,000
~~Warehouse (f_3)~~	~~30,000~~	~~10,000~~

Criteria for rational choice

- After all dominated acts have been removed, there remains the problem of ordering them by desirability, and of finding the set of most desirable acts
- Several criteria of "rational choice" have been proposed to derive a preference relation over acts

1. Laplace criterion

$$f_i \succeq f_k \text{ iff } \frac{1}{r}\sum_j u_{ij} \geq \frac{1}{r}\sum_j u_{kj}.$$

2. Maximax criterion

$$f_i \succeq f_k \text{ iff } \max_j u_{ij} \geq \max_j u_{kj}.$$

3. Maximin (Wald) criterion

$$f_i \succeq f_k \text{ iff } \min_j u_{ij} \geq \min_j u_{kj}.$$

Example

Act	ω_1	ω_2	ave	max	min
Apartment (f_1)	50,000	30,000	**40,000**	50,000	**30,000**
Office (f_2)	100,000	-40,000	30,000	**100,00**	-40,000

Hurwicz criterion

- Hurwicz criterion: $f_i \succeq f_k$ iff

$$\alpha \min_j u_{ij} + (1-\alpha) \max_j u_{ij} \geq \alpha \min_j u_{kj} + (1-\alpha) \max_j u_{kj}$$

  where α is a parameter in $[0, 1]$, called the pessimism index
- Boils down to
  - the maximax criterion if $\alpha = 0$
  - the maximin criterion if $\alpha = 1$
- α describes the DM's attitude toward ambiguity

Minimax regret criterion criterion

- (Savage) Minimax regret criterion: an act f_i is at least as desirable as f_k if it has smaller maximal regret, where regret is defined as the utility difference with the best act, for a given state of nature
- The regret r_{ij} for act f_i and state ω_j is

$$r_{ij} = \max_{\ell} u_{\ell j} - u_{ij}$$

- The maximum regret for act f_i is $R_i = \max_j r_{ij}$
- $f_i \succeq f_k$ iff $R_i \leq R_k$

Example

- Pay-off matrix

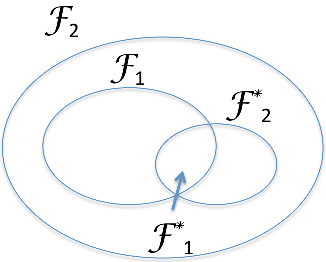
Act	ω_1	ω_2
Apartment (f_1)	50,000	30,000
Office (f_2)	100,000	-40,000

- Regret matrix

Act	ω_1	ω_2	max regret
Apartment (f_1)	50,000	0	**50,000**
Office (f_2)	0	70,000	70,000

Axioms of rational choice

- Let $\mathcal{F}^*$ denote the choice set, defined as a set of optimal acts
- Arrow and Hurwicz (1972) have proposed four axioms a choice operator $\mathcal{F} \rightarrow \mathcal{F}^*$ should verify



1. Axiom A_1: if $\mathcal{F}_1 \subset \mathcal{F}_2$ and $\mathcal{F}_2^* \cap \mathcal{F}_1 \neq \emptyset$, then $\mathcal{F}_1^* = \mathcal{F}_2^* \cap \mathcal{F}_1$
2. Axiom A_2: Relabeling actions and states does not change the optimal status of actions
3. Axiom A_3: Deletion of a duplicate state does not change the optimality status of actions (ω_j and ω_ℓ are duplicate if $u_{ij} = u_{i\ell}$ for all i)
4. Axiom A_4 (dominance): If $f \in \mathcal{F}^*$ and f' dominates f, then $f' \in \mathcal{F}^*$. If $f \notin \mathcal{F}^*$ and f dominates f', then $f' \notin \mathcal{F}^*$

Axioms of rational choice (continued)

- Under some regularity assumptions, Axioms $A_1 - A_4$ imply that the choice set depends only on the worst and the best consequences of each act
- In particular, these axioms rule out the Laplace and minimax regret criteria

Violation of Axiom A3 by the Laplace criterion

Act	ω_1	ω_2	ave
Apartment (f_1)	50,000	30,000	**40,000**
Office (f_2)	100,000	-40,000	30,000

Let us split the state of nature ω_1 in two states: “Good economic conditions and there is life on Mars” (ω_1') and “Good economic conditions and there is no life on Mars” (ω_1'')

Act	ω_1'	ω_1''	ω_2	ave
Apartment (f_1)	50,000	50,000	30,000	43,333
Office (f_2)	100,000	100,000	-40,000	**53,333**

Violation of Axiom A1 by minimax regret

- Pay-off matrix

Act	ω_1	ω_2
Apartment (f_1)	50,000	30,000
Office (f_2)	100,000	-40,000
f_4	130,000	-45,000

- Regret matrix

Act	ω_1	ω_2	max regret
Apartment (f_1)	80,000	0	80,000
Office (f_2)	30,000	70,000	**70,000**
f_4	0	75,000	75,000

We had $\mathcal{F}_1 = \{f_1, f_2\}$ and $\mathcal{F}_1^* = \{f_1\}$. Now, $\mathcal{F}_2 = \{f_1, f_2, f_4\}$ and $\mathcal{F}_2^* = \{f_2\}$. So, $\mathcal{F}_1^* \neq \mathcal{F}_2^* \cap \mathcal{F}_1$

Outline

1. Classical decision theory
   - Decision-making under complete ignorance
   - Decision-making with probabilities
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2. Decision-making with belief functions
   - Upper and lower expected utility
   - Other approaches

Maximum Expected Utility principle

- Let us now consider the situation where uncertainty about the state of nature is quantified by probabilities $p_1, \ldots, p_r$ on Ω
- These probabilities can be objective (decision under risk) or subjective
- We can then compute, for each act f_i, its expected utility as

$$EU(f_i) = \sum_j u_{ij} p_j$$

- Maximum Expected Utility (MEU) principle: an act f_i is more desirable than an act f_k if i it has a higher expected utility: $f_i \succeq f_k$ iff $EU(f_i) \geq EU(f_k)$

Example

Act	ω_1	ω_2
Apartment (f_1)	50,000	30,000
Office (f_2)	100,000	-40,000

Assume that there is 60% chance that the economic situation will be poor (ω_2). The expected utilities of acts f_1 and f_2 are

$$EU(f_1) = 50,000 \times 0.4 + 30,000 \times 0.6 = 38,000$$
$$EU(f_2) = 100,000 \times 0.4 - 40,000 \times 0.6 = 16,000$$

Act f_1 is thus more desirable according to the maximum expected utility criterion

Axiomatic justification of the MEU principle

- The MEU principle was first axiomatized by von Neumann and Morgenstern (1944)
- Given a probability distribution on Ω, an act $f : \Omega \rightarrow \mathcal{C}$ induces a probability measure P on the set $\mathcal{C}$ of consequences (assumed to be finite), called a lottery
- We denote by $\mathcal{L}$ the set of lotteries on $\mathcal{C}$
- If we agree that two acts providing the same lottery are equivalent, then the problem of comparing the desirability of acts becomes that of comparing the desirability of lotteries
- Let $\succeq$ be a preference relation among lotteries. Von Neumann and Morgentern argued that, to be rational, a preference relation should verify three axioms

Von Neumann and Morgenstern's axioms

1. Complete preorder: the preference relation is a complete and non trivial preorder (i.e., it is a reflexive, transitive and complete relation) on $\mathcal{L}$
2. Continuity: for any lotteries P, Q and R such that $P \succ Q \succ R$, there exists probabilities α and β in $[0, 1]$ such that

$$\alpha P + (1 - \alpha)R \succ Q \succ \beta P + (1 - \beta)R$$

   where $\alpha P + (1 - \alpha)R$ is a compound lottery, which refers to the situation where you receive P with probability α and R with probability $1 - \alpha$. This axiom implies, in particular, that there is no lottery R that is so undesirable that it cannot become desirable if mixed with some very desirable lottery P
3. Independence: for any lotteries P, Q and R and for any $\alpha \in (0, 1]$

$$P \succeq Q \Leftrightarrow \alpha P + (1 - \alpha)R \succeq \alpha Q + (1 - \alpha)R$$

Von Neumann and Morgenstern's theorem

The two following propositions are equivalent:

1. The preference relation $\succeq$ verifies the axioms of complete preorder, continuity, and independence
2. There exists a utility function $u : \mathcal{C} \rightarrow \mathbb{R}$ such that, for any two lotteries $P = (p_1, \ldots, p_r)$ and $Q = (q_1, \ldots, q_r)$

$$P \succeq Q \Leftrightarrow \sum_{i=1}^{r} p_i u(c_i) \geq \sum_{i=1}^{r} q_i u(c_i)$$

Function u is unique up to a strictly increasing affine transformation

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Savage's theorem

- We have reviewed some criteria for decision-making under complete ignorance, i.e., when uncertainty cannot be probabilized
- Some researchers have defended the view that a rational DM always maximizes expected utility, for some subjective probability measure and utility function
- Savage's theorem (1954): a preference relation $\succcurlyeq$ among acts verifies some rationality requirements iff there is a finitely additive probability measure P and a utility function $u : \mathcal{C} \rightarrow \mathbb{R}$ such that

$$\forall f, g \in \mathcal{F}, \quad f \succcurlyeq g \Leftrightarrow \int_\Omega u(f(\omega))dP(\omega) \geq \int_\Omega u(g(\omega))dP(\omega)$$

  Furthermore, P is unique and u is unique up to a positive affine transformation
- A strong argument for probabilism, but Savage's axioms can be questioned!

Savage's axioms

- Savage has proposed seven axioms, four of which are considered as meaningful (the other three are technical)
- Axiom 1: $\succcurlyeq$ is a total preorder (complete, reflexive and transitive)
- Axiom 2 [Sure Thing Principle]. Given $f, h \in \mathcal{F}$ and $E \subseteq \Omega$, let fEh denote the act defined by

$$(fEh)(\omega) = \begin{cases} f(\omega) & \text{if } \omega \in E \\ h(\omega) & \text{if } \omega \notin E \end{cases}$$

  Then the Sure Thing Principle states that $\forall E, \forall f, g, h, h'$

$$fEh \succcurlyeq gEh \Rightarrow fEh' \succcurlyeq gEh'$$

  The preference between two acts with a common extension outside some event E does not depend on this common extension.
- This axiom seems reasonable, but it is not verified empirically!

Ellsberg's paradox

- Suppose you have an urn containing 30 red balls and 60 balls, either black or yellow. Consider the following gambles:
  - f_1: You receive 100 euros if you draw a red ball
  - f_2: You receive 100 euros if you draw a black ball
  - f_3: You receive 100 euros if you draw a red or yellow ball
  - f_4: You receive 100 euros if you draw a black or yellow ball
- Do you prefer f_1 or f_2? f_3 or f_4?

Ellsberg's paradox

- Suppose you have an urn containing 30 red balls and 60 balls, either black or yellow. Consider the following gambles:
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  - f_2: You receive 100 euros if you draw a black ball
  - f_3: You receive 100 euros if you draw a red or yellow ball
  - f_4: You receive 100 euros if you draw a black or yellow ball
- Do you prefer f_1 or f_2? f_3 or f_4?
- Most people strictly prefer f_1 to f_2, but they strictly prefer f_4 to f_3

	R	B	Y
f_1	100	0	0
f_2	0	100	0
f_3	100	0	100
f_4	0	100	100

Now,

$$f_1 = f_1\{R, B\}0, \; f_2 = f_2\{R, B\}0$$

$$f_3 = f_1\{R, B\}100, \; f_4 = f_2\{R, B\}100$$

- The Sure Thing Principle is violated!

Summary

- Classically, we distinguish two kinds of decision problems:
  1. Decision under ignorance: we only know, for each act, a set a possible outcomes
  2. Decision under risk: we are given, for each act, a probability distribution over the outcomes
- It has been argued that any decision problem under uncertainty should be handled as a problem of decision under risk. However, the axiomatic arguments are questionable
- In the next part: decision-making when uncertainty is described by a belief functions

Outline

How belief functions come into the picture

Belief functions become components of a decision problem in any of the following two situations (or both)

1. The decision maker's subjective beliefs concerning the state of nature are described by a belief function Bel^Ω on Ω
2. The DM is not able to precisely describe the outcomes of some acts under each state of nature

Case 1: uncertainty described by a belief function

- Let m^Ω be a mass function on Ω
- Any act $f : \Omega \to \mathcal{C}$ carries m^Ω to the set $\mathcal{C}$ of consequences, yielding a mass function $m_f^{\mathcal{C}}$, which quantifies the DM's beliefs about the outcome of act f
- Each mass $m^\Omega(A)$ is transfered to $f(A)$

$$m_f^{\mathcal{C}}(B) = \sum_{\{A \subseteq \Omega | f(A) = B\}} m^\Omega(A)$$

  for any $B \subseteq \mathcal{C}$
- $m_f^{\mathcal{C}}$ is a credibilistic lottery corresponding to act f

Case 2: partial knowledge of outcomes

- In that case, an act may formally be represented by a multi-valued mapping $f : \Omega \rightarrow 2^{\mathcal{C}}$, assigning a set of possible consequences $f(\omega) \subseteq \mathcal{C}$ to each state of nature ω
- Given a probability measure P on Ω, f then induces the following mass function $m_f^{\mathcal{C}}$ on $\mathcal{C}$,

$$m_f^{\mathcal{C}}(B) = \sum_{\{\omega \in \Omega | f(\omega) = B\}} p(\omega)$$

for all $B \subseteq \mathcal{C}$

Example

- Let $\Omega = \{\omega_1, \omega_2, \omega_3\}$ and m^Ω the following mass function

$$\begin{array}{ll} m^\Omega(\{\omega_1,\omega_2\}) = 0.3, & m^\Omega(\{\omega_2,\omega_3\}) = 0.2 \\ m^\Omega(\{\omega_3\}) = 0.4, & m^\Omega(\Omega) = 0.1 \end{array}$$

- Let $\mathcal{C} = \{c_1, c_2, c_3\}$ and f the act

$$f(\omega_1) = \{c_1\}, \quad f(\omega_2) = \{c_1, c_2\}, \quad f(\omega_3) = \{c_2, c_3\}$$

- To compute $m_f^\mathcal{C}$, we transfer the masses as follows

$$\begin{aligned} m^\Omega(\{\omega_1,\omega_2\}) &= 0.3 \rightarrow f(\omega_1) \cup f(\omega_2) = \{c_1, c_2\} \\ m^\Omega(\{\omega_2,\omega_3\}) &= 0.2 \rightarrow f(\omega_2) \cup f(\omega_3) = \{c_1, c_2, c_3\} \\ m^\Omega(\{\omega_3\}) &= 0.4 \rightarrow f(\omega_3) = \{c_2, c_3\} \\ m^\Omega(\Omega) &= 0.1 \rightarrow f(\omega_1) \cup f(\omega_2) \cup f(\omega_3) = \{c_1, c_2, c_3\} \end{aligned}$$

- Finally, we obtain the following mass function on $\mathcal{C}$

$$m^\mathcal{C}(\{c_1, c_2\}) = 0.3, \quad m^\mathcal{C}(\{c_2, c_3\}) = 0.4, \quad m^\mathcal{C}(\mathcal{C}) = 0.3$$

Decision problem

- In the two situations considered above, we can assign to each act f a credibilistic lottery, defined as a mass function on $\mathcal{C}$
- Given a utility function u on $\mathcal{C}$, we then need to extend the MEU model
- Several such extensions will now be reviewed

Outline

Upper and lower expectations

- Let m be a mass function on $\mathcal{C}$, and u a utility function $\mathcal{C} \to \mathbb{R}$
- The lower and upper expectations of u are defined, respectively, as the averages of the minima and the maxima of u within each focal set of m

$$\underline{\mathbb{E}}_m(u) = \sum_{A \subseteq \mathcal{C}} m(A) \min_{c \in A} u(c)$$

$$\overline{\mathbb{E}}_m(u) = \sum_{A \subseteq \mathcal{C}} m(A) \max_{c \in A} u(c)$$

- It is clear that $\underline{\mathbb{E}}_m(u) \leq \overline{\mathbb{E}}_m(u)$, with the inequality becoming an equality when m is Bayesian, in which case the lower and upper expectations collapse to the usual expectation
- If $m = m_A$ is logical with focal set A, then $\underline{\mathbb{E}}_m(u)$ and $\overline{\mathbb{E}}_m(u)$ are, respectively, the minimum and the maximum of u in A

Imprecise probability interpretation

- The lower and upper expectations are lower and upper bounds of expectations with respect to probability measures compatible with m

$$\underline{\mathbb{E}}_m(u) = \min_{P \in \mathcal{P}(m)} \mathbb{E}_P(u)$$
$$\overline{\mathbb{E}}_m(u) = \max_{P \in \mathcal{P}(m)} \mathbb{E}_P(u)$$

- The mean of minima (res., maxima) is also the minimum (resp., maximum) of means with respect to all compatible probability measures

Corresponding decision criteria

- Having defined the notions of lower and upper expectations, we can define two preference relations among credibilistic lotteries as

$$m_1 \underline{\succcurlyeq} m_2 \text{ iff } \underline{\mathbb{E}}_{m_1}(u) \geq \underline{\mathbb{E}}_{m_2}(u)$$

  and

$$m_1 \overline{\succcurlyeq} m_2 \text{ iff } \overline{\mathbb{E}}_{m_1}(u) \geq \overline{\mathbb{E}}_{m_2}(u)$$

- Relation $\underline{\succcurlyeq}$ corresponds to a pessimistic (or conservative) attitude of the DM. When m is logical, it corresponds to the maximin criterion
- Symmetrically, $\overline{\succcurlyeq}$ corresponds to an optimistic attitude and extends the maximax criterion
- Both criteria boil down to the MEU criterion when m is Bayesian

Back to Ellsberg's paradox

- Here, $\Omega = \{R, B, Y\}$ and $m^\Omega(\{R\}) = 1/3$, $m^\Omega(\{B, Y\}) = 2/3$
- The mass functions on $\mathcal{C} = \{0, 100\}$ induced by the four acts are

$$\begin{aligned} m_1(\{100\}) &= 1/3, & m_1(\{0\}) &= 2/3 \\ m_2(\{0\}) &= 1/3, & m_2(\{0, 100\}) &= 2/3 \\ m_3(\{100\}) &= 1/3, & m_3(\{0, 100\}) &= 2/3 \\ m_4(\{0\}) &= 1/3, & m_4(\{100\}) &= 2/3 \end{aligned}$$

- Corresponding lower and upper expectations

	R	B	Y	$\underline{\mathbb{E}}_m(u)$	$\overline{\mathbb{E}}_m(u)$
f_1	100	0	0	$\mathbf{u(100)/3}$	$u(100)/3$
f_2	0	100	0	0	$\mathbf{2u(100)/3}$
f_3	100	0	100	$u(100)/3$	$\mathbf{u(100)}$
f_4	0	100	100	$\mathbf{2u(100)/3}$	$2u(100)/3$

Interval dominance

- If we drop the requirement that the preference relation among acts be complete, then we can consider the interval dominance relation,

$$m_1 \succcurlyeq_{ID} m_2 \text{ iff } \underline{\mathbb{E}}_{m_1}(u) \geq \overline{\mathbb{E}}_{m_2}(u)$$

- Given a collection of credibilistic lotteries, we can then compute the set of maximal (i.e., non dominated) elements of $\succcurlyeq_{ID}$
- Imprecise probability view

$$m_1 \succcurlyeq_{ID} m_2 \Leftrightarrow \forall P_1 \in \mathcal{P}(m_1), \forall P_2 \in \mathcal{P}(m_2), \mathbb{E}_{P_1}(u) \geq \mathbb{E}_{P_2}(u)$$

- The justification for this preference relation is not so clear from the point of view of belief function theory (i.e., if one does not interpret a belief function as a lower probability)

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Generalized Hurwicz criterion

- The Hurwicz criterion can be generalized as

$$\mathbb{E}_{m,\alpha}(u) = \sum_{A \subseteq \mathcal{C}} m(A) \left(\alpha \min_{c \in A} u(c) + (1-\alpha) \max_{c \in A} u(c) \right)$$
$$= \alpha \underline{\mathbb{E}}_m(u) + (1-\alpha)\overline{\mathbb{E}}(u)$$

  where $\alpha \in [0, 1]$ is a pessimism index
- This criterion was introduced and justified axiomatically by Jaffray (1988)
- Strat (1990) who proposed to interpret α as the DM's subjective probability that the ambiguity will be resolved unfavorably

Transferable belief model

- A completely different approach to decision-making with belief function was advocated by Smets, as part of the Transferable Belief Model
- Smets defended a two-level mental model
  1. a credal level, where an agent's belief are represented by belief functions, and
  2. a pignistic level, where decisions are made by maximizing the EU with respect to a probability measure derived from a belief function
- The rationale for introducing probabilities at the decision level is the avoidance of Dutch books
- Smets argued that the belief-probability transformation T should be linear, i.e., it should verify

$$T(\alpha m_1 + (1-\alpha)m_2) = \alpha T(m_1) + (1-\alpha)T(m_2),$$

for any mass functions m_1 and m_2 and for any $\alpha \in [0, 1]$

Pignistic transformation

- The only linear belief-probability transformation T is the pignistic transformation, with $p_m = T(m)$ given by

$$p_m(c) = \sum_{\{A \subseteq \mathcal{C} | c \in A\}} \frac{m(A)}{|A|}, \quad \forall c \in \mathcal{C}$$

- The expected utility w.r.t. the pignistic probability is

$$\mathbb{E}_p(u) = \sum_{c \in \mathcal{C}} p_m(c) u(c) = \sum_{A \subseteq \mathcal{C}} m(A) \left(\frac{1}{|A|} \sum_{c \in A} u(c) \right)$$

- The maximum pignistic expected utility criterion thus extends the Laplace criterion

Generalized minimax regret

- Yager (2004) also extended the minimax regret criterion to belief functions
- We need to consider n acts $f_1, \ldots, f_n$, and we write $u_{ij} = u[f_i(\omega_j)]$
- The regret if act f_i is selected, and state ω_j occurs, is $r_{ij} = \max_k u_{kj} - u_{ij}$
- For a non-empty subset A of Ω, the maximum regret of act f_i is

$$R_i(A) = \max_{\omega_j \in A} r_{ij}$$

- The expected maximal regret for act f_i is

$$\overline{R}_i = \sum_{\emptyset \neq A \subseteq \Omega} m^\Omega(A) R_i(A)$$

- Act f_i is preferred over act f_k if $\overline{R}_i \leq \overline{R}_k$
- The minimax regret criterion is recovered when m^Ω is logical
- The MEU model is recovered when m^Ω is Bayesian

Summary

non-probabilized		belief functions	probabilized
maximin	$\longleftrightarrow$	lower expectation	
maximax	$\longleftrightarrow$	upper expectation	
Laplace	$\longleftrightarrow$	pignistic expectation	expected utility
Hurwicz	$\longleftrightarrow$	generalized Hurwicz	
minimax regret	$\longleftrightarrow$	generalized minimax regret	