Methods for building belief functions

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Building belief functions

- The basic theory tells us how to reason and compute with belief functions, but it does not tell us where belief functions come from.
- We need formalized methods for modeling expert opinions and statistical information using belief functions.
- Three general approaches:
  - Least Commitment Principle;
  - Using meta-knowledge about information sources (discounting);
  - Predictive belief functions.

Outline

Least Commitment Principle

General approach

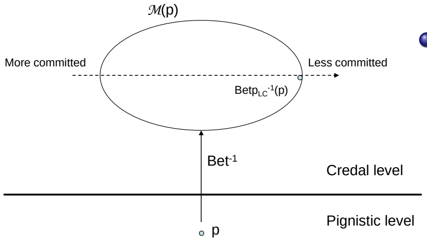
- Least commitment principle: *"When several belief functions are compatible with a set of constraints, the least informative according to some informational ordering (if it exists) should be selected"*.
- General approach:
  1. Express the available information as a set of constraints on an unknown mass function;
  2. Find the least-committed mass function (according to some ordering), compatible with the constraints.
- Three applications:
  - Inverse pignistic transformation;
  - Credal ordering constraints;
  - Deconditioning, Generalized Bayes Theorem (GBT).

Outline

Inverse pignistic transformation
Problem statement

- Assume we want to elicit a mass function m on $\Omega = \{\omega_1, \ldots, \omega_K\}$ from an expert.
- It is easier to elicit the corresponding pignistic probability:
  - For each $\omega_k \in \Omega$ ask for the fair price p_k the expert is willing to pay for a ticket that will allow him to receive 1 euro if $X = \omega_k$, and to receive nothing otherwise.
  - The pignistic probability mass function is $p(\omega_k) = p_k$, $k = 1, \ldots, K$.
- How to compute a mass function m on Ω consistent with p, i.e., such that $p = Bet(m)$?

Inverse pignistic transformation
Discrete case



- There are infinitely many mass functions m such that $Bet(m) = p$.
- The q-least committed solution is a consonant mass function defined by the following possibility distribution:

$$\pi(\omega_k) = \sum_{\ell=1}^{K} \min(p_k, p_\ell).$$

Inverse pignistic transformation

Recovering the mass function

- Let $1 = \pi_{(1)} \geq \pi_{(2)} \geq \ldots \geq \pi_{(K)}$ be the ordered possibility degrees, and $\omega_{(1)}, \ldots, \omega_{(K)}$ the elements of Ω in the corresponding order, i.e., $\pi(\omega_{(i)}) = \pi_{(i)}$, $i = 1, \ldots, K$.
- We have

$$
\begin{aligned}
m(\{\omega_{(1)}\}) &= \pi_{(1)} - \pi_{(2)} \\
&\vdots \\
m(\{\omega_{(1)}, \ldots, \omega_{(i)}\}) &= \pi_{(i)} - \pi_{(i+1)} \\
&\vdots \\
m(\Omega) &= \pi_{(K)}.
\end{aligned}
$$

Inverse pignistic transformation

Example

- Let us consider a frame $\Omega = \{\omega_1, \omega_2, \omega_3\}$ and the pignistic probability mass function

$$p(\omega_1) = 0.7, \quad p(\omega_2) = 0.2, \quad p(\omega_3) = 0.1$$

- We have

$$\begin{aligned} \pi(\omega_1) &= 0.7 + 0.2 + 0.1 = 1 \\ \pi(\omega_2) &= 0.2 + 0.2 + 0.1 = 0.5 \\ \pi(\omega_3) &= 0.1 + 0.1 + 0.1 = 0.3. \end{aligned}$$

- The corresponding mass function is

$$m(\{\omega_1\}) = 0.5, \quad m(\{\omega_1, \omega_2\}) = 0.2, \quad m(\Omega) = 0.3.$$

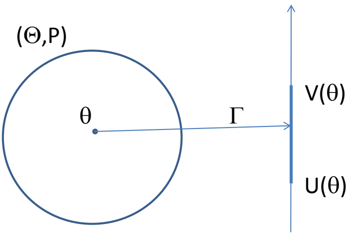
Inverse pignistic transformation

Continuous case

- Assume that the variable of interest X is a continuous variable taking values in $\mathbb{R}$.
- The expert gives us a probability distribution on $\mathbb{R}$. Can we extend the previous line of reasoning to this situation?
- We need to define belief functions on $\mathbb{R}$ and the associated notions (informational orderings, pignistic transformation, etc.).

Belief functions on $\mathbb{R}$

Random intervals



A random interval is defined by a probability space $(\Theta, \mathcal{A}, P)$ and a mapping Γ from Θ to the set $\mathcal{I}$ of closed real intervals:

$$\Gamma : \theta \to \Gamma(\theta) = [U(\theta), V(\theta)],$$

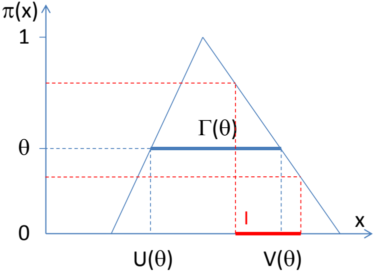
such that (U, V) is a two-dimensional random vector, with $U \leq V$.

We have, for any $I \in \mathcal{I}$:

$$bel(I) = P([U, V] \subseteq I), \quad pl(I) = P([U, V] \cap I \neq \emptyset)$$

$$q(I) = P([U, V] \supseteq I)$$

Random intervals
Example: possibility distribution



- Let π be a possibility distribution on $\mathbb{R}$, $\Theta = [0,1]$, P the Lebesgues measure on $[0,1]$, and $\Gamma(\theta)$ the θ-level cut of π.
- It can be checked that

$$pl(I) = \sup_{x \in I} \pi(x) = \Pi(I)$$

$$bel(I) = 1 - \sup_{x \notin I} \pi(x) = N(I).$$

Pignistic probability density

Discrete case

- Let us assume that $\Gamma(\Theta) = \{I_1, \ldots, I_r\}$. We can define the mass function as

$$m(I_i) = P\left(\{\theta \in \Theta | \Gamma(\theta) = I_i\}\right).$$

- m is a discrete mass function with focal intervals $I_1, \ldots, I_r$.
- Assuming $0 < |I_i| < +\infty$ for all i, the pignistic probability density associated to m is:

$$p_m(x) = \sum_{i=1}^{r} m(I_i) \frac{1_{I_i}(x)}{|I_i|}, \quad \forall x \in \mathbb{R}.$$

- It is a finite mixture of continuous uniform distributions.

Pignistic probability density
Continuous case

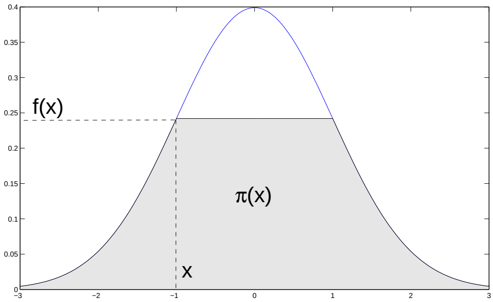
- If (U, V) is a continuous random vector with density f, we can define a "mass density"

$$m([u, v]) = f(u, v), \quad \forall (u, v) \in \mathbb{R}^2, u \leq v.$$

- The pignistic probability density is:

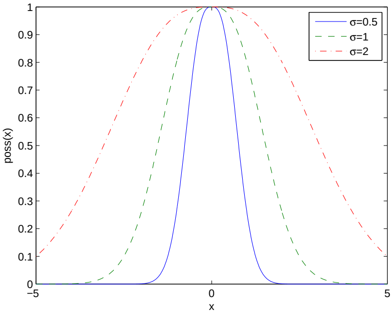
$$p_m(x) = \lim_{\epsilon \to 0} \int_{-\infty}^{x} \int_{x+\epsilon}^{+\infty} \frac{f(u, v)}{v - u} dvdu.$$

Inverse pignistic transformation
General expression



$$\pi(x) = \int_{-\infty}^{+\infty} \min(f(x), f(t))dt.$$

Inverse pignistic transformation
Example: normal distribution



$$\pi(x) = \begin{cases} \frac{2(x-\mu)}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) + 2\left(1 - \Phi\left(\frac{x-\mu}{\sigma}\right)\right) & \text{if } x \geq \mu \\ \frac{2(\mu-x)}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) + 2\Phi\left(\frac{x-\mu}{\sigma}\right) & \text{otherwise.} \end{cases}$$

Outline

Credal ordering constraint

Problem

- Consider the following problems:
  1. Let X and X' be two variables. Our beliefs on X are represented by m. Additionally, we believe that X' tends to take greater values than X. How to quantify our beliefs on X' using a mass function?
  2. We consider one variable X and two different contexts C and C'. When C holds, our beliefs on X are represented by m. When C' holds, we cannot precisely assess our beliefs on X, but we believe that X tends to take higher values than it does when C holds. How to quantify our beliefs on X in context C'?
- Approach: formalize the notion of “tending to take higher values" as a constraint on a mass function, and find the least-committed solution compatible with that constraint.

Stochastic ordering

- Given two probability distributions P and P' on $\mathbb{R}$, we say that P is stochastically less than or equal to P' ($P \preceq P'$) if

$$P((x,+\infty)) \leq P'((x,+\infty)), \quad \forall x \in \mathbb{R}$$

- Intuitively, this means that distribution P attaches less probability to larger values than P' does.
- Property: the above condition holds holds iff:

$$P \preceq P' \Leftrightarrow \mathbb{E}_P(g) \leq \mathbb{E}_{P'}(g), \quad \forall g \in \mathcal{G}$$

  where $\mathcal{G}$ is the set of measurable and non decreasing real functions.
- How to extend this notion to compare two mass functions m and m' on $\mathbb{R}$?

Credal ordering
Definitions

- Four definitions (credal orderings):
  1. $m \lesssim m'$ iff $bel((x, +\infty)) \leq pl'((x, +\infty)), \quad \forall x \in \mathbb{R}$;
  2. $m \leqslant m'$ iff $bel((x, +\infty)) \leq bel'((x, +\infty)), \quad \forall x \in \mathbb{R}$;
  3. $m \eqslantless m'$ iff $pl((x, +\infty)) \leq pl'((x, +\infty)), \quad \forall x \in \mathbb{R}$;
  4. $m \ll m'$ iff $pl((x, +\infty)) \leq bel'((x, +\infty)), \quad \forall x \in \mathbb{R}$.
- Let $\mathcal{G}_b$ denote the set of bounded, measurable and non decreasing real functions. Then we have:

$$
\begin{aligned}
m \lesssim m' &\Leftrightarrow \underline{\mathbb{E}}_m(g) \leq \overline{\mathbb{E}}_{m'}(g), \quad \forall g \in \mathcal{G}_b\\
m \leqslant m' &\Leftrightarrow \underline{\mathbb{E}}_m(g) \leq \underline{\mathbb{E}}_{m'}(g), \quad \forall g \in \mathcal{G}_b\\
m \eqslantless m' &\Leftrightarrow \overline{\mathbb{E}}_m(g) \leq \overline{\mathbb{E}}_{m'}(g), \quad \forall g \in \mathcal{G}_b\\
m \ll m' &\Leftrightarrow \overline{\mathbb{E}}_m(g) \leq \underline{\mathbb{E}}_{m'}(g), \quad \forall g \in \mathcal{G}_b.
\end{aligned}
$$

Credal ordering constraint

Example of result

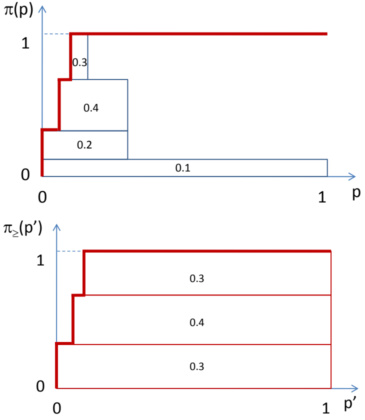
Theorem

The pl-least committed element mass function m' such that $m' \geqslant m$ exists and is unique. It is the consonant mass function $m_{\geqslant}$ with possibility distribution $\pi_{\geqslant}$ given by

$$\pi_{\geqslant}(x) = pl((-\infty, x])$$

where pl is the plausibility function associated to m.

Credal ordering constraint
Example

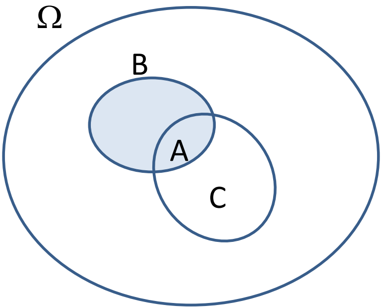


- Assume that m represents an expert's opinion regarding the failure probability p of a component in standard operating condition.
- We want to assess our beliefs regarding the failure probability p' of the same component in a more stringent environment.
- We only know that p' tends to be greater than p: $m_{p'} \geqslant m_p$.

Outline

1. Least Commitment Principle
   - Inverse pignistic transformation
   - Credal ordering constraints
   - Deconditioning
2. Using metaknowledge
   - Discounting
   - Contextual discounting
3. Predictive belief functions
   - Definition
   - Discrete case
   - Continuous case

Deconditioning

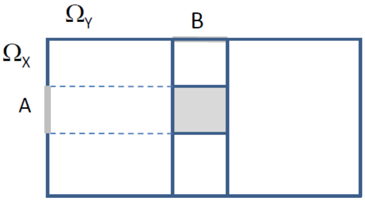


- Let m_0 be a mass function on Ω expressing our beliefs about X in a context where we know that $X \in B$.
- We want to build a mass function m verifying the constraint $m(\cdot|B) = m_0$.
- Any m built from m_0 by transferring each mass $m_0(A)$ to $A \cup C$ for some $C \subseteq \overline{B}$ satisfies the constraint.
- s-least committed solution: transfer $m_0(A)$ to the largest such set $A \cup \overline{B}$:

$$m(D) = \begin{cases} m_0(A) & \text{if } D = A \cup \overline{B} \text{ for some } A \subseteq B, \\ 0 & \text{otherwise} \end{cases}$$

Deconditioning

Ballooning extension



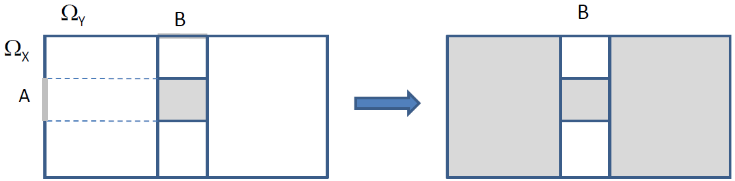
- More complex situation: two frames Ω_X and Ω_Y.
- Let $m_0^{\Omega_X}$ be a mass function on Ω_X expressing our beliefs about X in a context where we know that $Y \in B$ for some $B \subseteq \Omega_Y$.
- We want to find $m^{\Omega_{XY}}$ such that

$$\left(m^{\Omega_{XY}} \textcircled{\cap} (m_B^{\Omega_Y})^{\uparrow \Omega_{XY}}\right)^{\downarrow \Omega_X} = m_0^{\Omega_X}$$

Deconditioning

Ballooning extension (continued)

- s-least committed solution: each mass $m_0^{\Omega_X}(A)$ transferred to $(A \times B) \cup (\Omega_X \times \overline{B})$.



- Notation $m^{\Omega_{XY}} = (m_0^{\Omega_X})^{\Uparrow \Omega_{XY}}$ (ballooning extension).

Application: Generalized Bayes Theorem

Problem statement

- Two variables $X \in \Omega$ et $\theta \in \Theta = \{\theta_1, \ldots, \theta_K\}$.
- Typically:
  - X is observed (sensor measurement),
  - θ is not observed (class, unknown parameter).
- Partial knowledge of X given $\theta = \theta_k$ for each k: $m^\Omega(\cdot|\theta_k)$.
- Prior knowledge about θ: m_0^Θ (may be vacuous).
- We observe $X \in A$.
- Belief function on Θ?

Generalized Bayes Theorem

Solution

- Solution:

$$m^{\Theta}(\cdot|A) = \left(\textcircled{\cap}_{k=1}^{K} m^{\Omega}(\cdot|\theta_k)^{\Uparrow \Omega\times\Theta} \textcircled{\cap} m_A^{\Omega \uparrow \Omega\times\Theta} \textcircled{\cap} m_0^{\Theta \uparrow \Omega\times\Theta} \right)^{\downarrow \Theta}$$

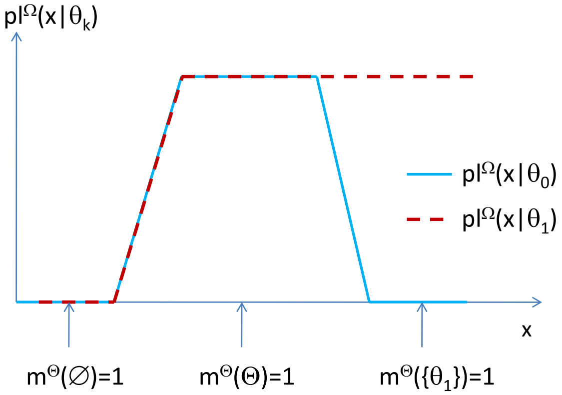
- Expression:

$$m^{\Theta}(\cdot|A) = \textcircled{\cap}_{k=1}^{K} m_k^{\Theta} \textcircled{\cap} m_0^{\Theta},$$

where

$$m_k^{\Theta}(\overline{\{\theta_k\}}) = 1 - pl^{\Omega}(A|\theta_k)$$
$$m_k^{\Theta}(\Theta) = pl^{\Omega}(A|\theta_k)$$

Generalized Bayes Theorem
Example



Generalized Bayes Theorem

Properties

- Property 1: Bayes' theorem is recovered as a special case when the conditional mass functions $m^{\Omega}(\cdot|\theta_k)$ and m_0^{Θ} are Bayesian mass functions.
- Property 2: If X and Y are cognitively independent conditionally on θ, i.e.:

$$pl^{\Omega_X \times \Omega_Y}(A \times B|\theta_k) = pl^{\Omega_X}(A|\theta_k) \cdot pl^{\Omega_Y}(B|\theta_k),$$

for all k, $A \subseteq \Omega_X$ and $B \subseteq \Omega_Y$, then

$$m^{\Theta}(\cdot|X \in A, Y \in B) = m^{\Theta}(\cdot|X \in A) \cap\!\!\!\!\bigcirc m^{\Theta}(\cdot|Y \in B).$$

Outline

Discounting
Problem statement

- A source of information provides:
  - a value;
  - a set of values;
  - a probability distribution, etc..
- The information is:
  - not fully reliable or
  - not fully relevant.
- Examples:
  - Possibly faulty sensor;
  - Measurement performed in unfavorable experimental conditions;
  - Information is related to a situation or an object that only has some similarity with the situation or the object considered (case-based reasoning).

Discounting
Formalization

- A source S provides a mass function m_S^Ω.
- S may be reliable or not. Let $\mathcal{R} = \{R, NR\}$.
- Assumptions:
  - If S is reliable, we accept m_S^Ω as a representation of our beliefs:
  $$m^\Omega(\cdot|R) = m_S^\Omega$$
  - If S is not reliable, we know nothing:
  $$m^\Omega(\cdot|NR) = m_\Omega^\Omega$$
  - The source has a probability $1 - \alpha$ of being reliable:
  $$m^\mathcal{R}(\{NR\}) = \alpha, \quad m^\mathcal{R}(\{R\}) = 1 - \alpha$$

Discounting
Solution

- Solution:

$$^{\alpha}m^{\Omega} = \left(m^{\mathcal{R}\uparrow\Omega\times\mathcal{R}} \textcircled{\cap} m^{\Omega}(\cdot|R)^{\Uparrow\Omega\times\mathcal{R}} \right)^{\downarrow\Omega}.$$

- Simple expressions:

$$\begin{aligned} ^{\alpha}m^{\Omega} &= (1-\alpha)m^{\Omega}_{S} + \alpha m^{\Omega}_{\Omega} \\ &= m^{\Omega}_{S} \textcircled{\cup} m^{\Omega}_{0} \end{aligned}$$

with $m^{\Omega}_{0}(\Omega) = \alpha$ and $m^{\Omega}_{0}(\emptyset) = 1-\alpha$.

- $^{\alpha}m^{\Omega}$ is a s-less committed than (a generalization of) m^{Ω}_{S}:

$$^{\alpha}m^{\Omega} \sqsupseteq_{s} m^{\Omega}_{S}.$$

Outline

Generalization: Contextual Discounting
Formalization

- A more general model allowing us to take into account richer meta-information about the source.
- Let $\Theta = \{\theta_1, \ldots, \theta_L\}$ be a partition of Ω, representing different contexts.
- Let $m^{\mathcal{R}}(\cdot|\theta_k)$ denote the mass function on $\mathcal{R}$ quantifying our belief in the reliability of source S, when we know that the actual value of X is in θ_k.
- We assume that:

$$m^{\mathcal{R}}(\{R\}|\theta_k) = 1 - \alpha_k, \quad m^{\mathcal{R}}(\{NR\}|\theta_k) = \alpha_k.$$

  for each $k \in \{1, \ldots, L\}$.
- Let $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_L)$.

Contextual Discounting
Example

- Let us consider a simplified aerial target recognition problem, in which we have three classes: airplane ($\omega_1 \equiv a$), helicopter ($\omega_2 \equiv h$) and rocket ($\omega_3 \equiv r$).
- Let $\Omega = \{a, h, r\}$.
- The sensor provides the following mass function: $m_S^\Omega(\{a\}) = 0.5$, $m_S^\Omega(\{r\}) = 0.5$.
- We assume that
  - The probability that the source is reliable when the target is an airplane is equal to $1 - \alpha_1 = 0.4$;
  - The probability that the source is reliable when the target is either a helicopter, or a rocket is equal to $1 - \alpha_2 = 0.9$.
- We have $\Theta = \{\theta_1, \theta_2\}$, with $\theta_1 = \{a\}$, $\theta_2 = \{h, r\}$, and $\boldsymbol{\alpha} = (0.6, 0.1)$.

Contextual Discounting
Solution

- Solution:

$$ {}^{\alpha}m^{\Omega} = \left(\textcircled{\cap}_{k=1}^{L} m^{\mathcal{R}}(\cdot|\theta_k)^{\Uparrow \Omega \times \mathcal{R}} \textcircled{\cap} m^{\Omega}(\cdot|R)^{\Uparrow \Omega \times \mathcal{R}} \right)^{\downarrow \Omega}. $$

- Result:

$$ {}^{\alpha}m^{\Omega} = m_S^{\Omega} \textcircled{\cup} m_1^{\Omega} \textcircled{\cup} \ldots \textcircled{\cup} m_L^{\Omega} $$

with $m_k^{\Omega}(\theta_k) = \alpha_k$ and $m_k^{\Omega}(\emptyset) = 1 - \alpha_k$.

- Standard discounting is recovered as a special case when $\Theta = \{\Omega\}$.

Contextual Discounting

Example (continued)

- The discounted mass function can be obtained by combining disjunctively 3 mass functions:
  - $m_S^\Omega(\{a\}) = 0.5$, $m_S^\Omega(\{r\}) = 0.5$;
  - $m_1^\Omega(\{a\}) = 0.6$, $m_1^\Omega(\emptyset) = 0.4$;
  - $m_1^\Omega(\{h, r\}) = 0.1$, $m_1^\Omega(\emptyset) = 0.9$.
- Result:

A	h	a	r	h, a	h, r	a, r	Ω
$m_S^\Omega(A)$	0	0.5	0.5	0	0	0	0
${}^\alpha m^\Omega(A)$	0	0.45	0.18	0	0.02	0.27	0.08

Outline

Predictive belief functions
Motivation

- Let X be random variable (defined from a repeatable random experiment), with unknown probability $\mathbb{P}_X$.
- We have observed n independent replicates of X:

$$\mathbf{X} = (X_1, \ldots, X_n).$$

- Problem: quantify our beliefs regarding a future realization of X using a belief function $bel(\cdot; \mathbf{X})$: predictive belief function.

Predictive belief functions
Examples

1. Example 1:
   - We have drawn r black balls in n drawings from an urn with replacement:
   - What is our belief that the next ball to be drawn from the urn will be black?
2. Example 2:
   - The lifetimes of 20 bearings have been observed:

     2398, 2812, 3113, 3212, 3523, 5236, 6215,
     6278, 7725, 8604, 9003, 9350, 9460, 11584,
     11825, 12628, 12888, 13431, 14266, 17809.
   - Let X be the lifetime of a bearing taken at random from the same population. Belief function on X?

Predictive belief functions
Requirements

- Requirement 1 (Hacking's frequency principle):
  - If $\mathbb{P}_X$ were known, we would equate our beliefs with probabilities: $bel(\cdot;\mathbb{P}_X) = \mathbb{P}_X$.
  - Weaker version when $\mathbb{P}_X$ is unknown:

$$\forall A \subset \Omega, \quad bel(A;\mathbf{X}) \xrightarrow{P} \mathbb{P}_X(A), \text{ as } n \to \infty,$$

- Requirement 2 (LCP):
  - As n is finite, $bel(\cdot;\mathbf{X})$ should be less committed than $\mathbb{P}_X$. However, the condition $bel(\cdot;\mathbf{X}) \leq \mathbb{P}_X$ is too strong.
  - Weaker requirement:

$$\mathbb{P}\left(bel(A;\mathbf{X}) \leq \mathbb{P}_X(A), \forall A \subset \Omega\right) \geq 1-\alpha.$$

"$bel(\cdot;\mathbf{X})$ is less committed than $\mathbb{P}_X$ most of the time"

Predictive belief functions

Meaning of Requirement 2

$$\mathbf{x} = (x_1, \ldots, x_n) \rightarrow bel(\cdot, \mathbf{x})$$
$$\mathbf{x}' = (x'_1, \ldots, x'_n) \rightarrow bel(\cdot; \mathbf{x}')$$
$$\mathbf{x}'' = (x''_1, \ldots, x''_n) \rightarrow bel(\cdot; \mathbf{x}'')$$
$$\vdots$$

- As the number of realizations of the random sample tends to ∞, the proportion of belief functions less committed than $\mathbb{P}_X$ should tend to a limit at least equal to $1 - \alpha$.

Outline

Using simultaneous confidence intervals

IJAR 42(3):228-252, 2006

- If X is discrete, $\Omega = \{\omega_1, \ldots, \omega_K\}$: a solution can be obtained using a simultaneous confidence intervals on probabilities $p_k = \mathbb{P}(X = \omega_k)$.
- Random intervals $[P_k^-, P_k^+]$, $k = 1, \ldots, K$ are simultaneous confidence intervals at level $1 - \alpha$ if

$$\mathbb{P}\left(P_k^- \leq p_k \leq P_k^+, k = 1, \ldots, K\right) \geq 1 - \alpha$$

- They are asymptotic simultaneous confidence intervals if the above inequality holds in the limit as $n \to \infty$.

Goodman's simultaneous confidence intervals

Asymptotic simultaneous confidence intervals were proposed by Goodman (1965):

$$P_k^- = \frac{b + 2N_k - \sqrt{\Delta_k}}{2(n+b)},$$

$$P_k^+ = \frac{b + 2N_k + \sqrt{\Delta_k}}{2(n+b)},$$

with $N_k = \#\{i | X_i = \omega_k\}$, $b = \chi^2_{1;1-\alpha/K}$ and $\Delta_k = b\left(b + \frac{4N_k(n-N_k)}{n}\right)$.

Goodman's simultaneous confidence intervals
Example

- 220 psychiatric patients categorized as either neurotic, depressed, schizophrenic or having a personality disorder.
- Observed counts: $\mathbf{n} = (91, 49, 37, 43)$.
- Goodman' confidence intervals at confidence level $1 - \alpha = 0.95$:

Diagnosis	N_k/n	P_k^-	P_k^+
Neurotic	0.41	0.33	0.50
Depressed	0.22	0.16	0.30
Schizophrenic	0.17	0.11	0.24
Personality disorder	0.20	0.14	0.27

From confidence intervals to lower probabilities

- To each $\mathbf{p} = (p_1, \ldots, p_K)$ corresponds a probability measure $\mathbb{P}_X$ s.t. $\mathbb{P}_X(\{\omega_k\}) = p_k$ for each k.
- Consequently, simultaneous confidence intervals define a family of probability measures described by the following lower probability measure:

$$P^-(A) = \max\left(\sum_{\omega_k \in A} P_k^-, 1 - \sum_{\omega_k \notin A} P_k^+\right)$$

- P^- satisfies requirements R_1 and R_2:
  - $P^-(A) \xrightarrow{P} \mathbb{P}_X(A)$ as $n \to \infty$, for all $A \subseteq \Omega$,
  - $\mathbb{P}(P^- \leq \mathbb{P}_X) \geq 1 - \alpha$.
- Is it a belief function?

From lower probabilities to belief functions
$K = 2$ or $K = 3$

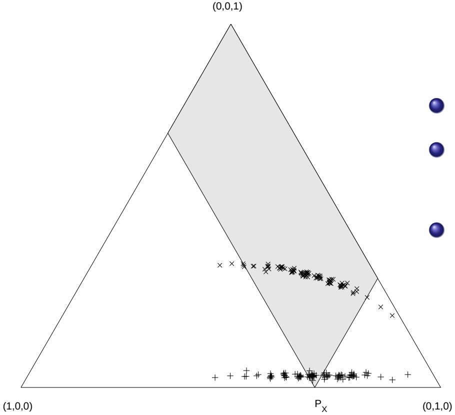
- If $K = 2$ or $K = 3$, P^- is a belief function.
- Case $K = 2$:

$$
\begin{aligned}
m(\{\omega_1\}) &= P_1^- \approx \widehat{p} - u_{1-\alpha/2}\sqrt{\frac{\widehat{p}(1-\widehat{p})}{n}} \\
m(\{\omega_2\}) &= P_2^- \approx 1 - \widehat{p} - u_{1-\alpha/2}\sqrt{\frac{\widehat{p}(1-\widehat{p})}{n}} \\
m(\Omega) &= 1 - P_1^- - P_2^- \approx 2u_{1-\alpha/2}\sqrt{\frac{\widehat{p}(1-\widehat{p})}{n}},
\end{aligned}
$$

with $\widehat{p} = N_1/n$.

The case $K = 2$

Example



- $K = 2$, $p_1 = \mathbb{P}_X(\{\omega_1\}) = 0.3$.
- 100 realizations of a random sample of size $n = 30$.
- 100 predictive belief functions at level $1 - \alpha = 0.95$.

From lower probabilities to belief functions

$K = 2$ or $K = 3$

- If $K > 3$, P^- is not a belief function in general. We can find the most committed belief function satisfying $bel \leq P^-$ by solving the following linear optimization problem:

$$\max_m J(m) = \sum_{A \subseteq \Omega} bel(A) = \sum_{A \subseteq \Omega} \sum_{B \subseteq A} m(B)$$

under the constraints:

$$\sum_{B \subseteq A} m(B) \leq P^-(A), \quad \forall A \subset \Omega,$$

$$\sum_{A \subseteq \Omega} m(A) = 1, \quad m(A) \geq 0, \quad \forall A \subseteq \Omega \, .$$

- The solution satisfies requirements R_1 and R_2: it is a predictive belief function (at confidence level $1 - \alpha$).

The case $K > 3$

Psychiatric Data

A	$P^-(A)$	$bel(A)$	$m(A)$
$\{\omega_1\}$	0.33	0.33	0.33
$\{\omega_2\}$	0.16	0.14	0.14
$\{\omega_1, \omega_2\}$	0.50	0.50	0.021
$\{\omega_3\}$	0.11	0.097	0.097
$\{\omega_1, \omega_3\}$	0.45	0.45	0.020
$\{\omega_2, \omega_3\}$	0.28	0.28	0.036
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\{\omega_1, \omega_3, \omega_4\}$	0.70	0.66	0.038
$\{\omega_2, \omega_3, \omega_4\}$	0.50	0.48	0.019
Ω	1	1	0

Case of ordered data

- Assume Ω is ordered: $\omega_1 < \ldots < \omega_K$.
- The focal sets of *bel* can be constrained to be intervals $A_{k,r} = \{\omega_k, \ldots, \omega_r\}$.
- Under this additional constraint, an analytical solution to the previous optimization problem can be found:

$$m(A_{k,k}) = P_k^-,$$

$$m(A_{k,k+1}) = P^-(A_{k,k+1}) - P^-(A_{k+1,k+1}) - P^-(A_{k,k}),$$

$$m(A_{k,r}) = P^-(A_{k,r}) - P^-(A_{k+1,r}) - P^-(A_{k,r-1}) + P^-(A_{k+1,r-1})$$

for $r > k+1$, and $m(B) = 0$, for all $B \notin \mathcal{I}$.

Example: rain data

- January precipitation in Arizona (in inches), recorded during the period 1895-2004.

class ω_k	n_k	n_k/n	p_k^-	p_k^+
< 0.75	48	0.44	0.32	0.56
$[0.75, 1.25)$	17	0.15	0.085	0.27
$[1.25, 1.75)$	19	0.17	0.098	0.29
$[1.75, 2.25)$	11	0.10	0.047	0.20
$[2.25, 2.75)$	6	0.055	0.020	0.14
≥ 2.75	9	0.082	0.035	0.18

- Degree of belief that the precipitation in Arizona next January will exceed, say, 2.25 inches?

Rain data
Result

$m(A_{k,r})$	1	2	3	4	5	6
1	0.32	0	0	0.13	0.11	0
2	-	0.085	0	0	0.012	0.14
3	-	-	0.098	0	0	0
4	-	-	-	0.047	0	0
5	-	-	-	-	0.020	0
6	-	-	-	-	-	0.035

- We get $bel(X \geq 2.25) = bel(\{\omega_5, \omega_6\}) = 0.055$ and $pl(X \geq 2.25) = 0.317$.
- In 95 % of cases, the interval $[bel(A), pl(A)]$ computed using this method contains $\mathbb{P}_X(A)$.

Outline

Predictive belief functions

Proceedings ISIPTA '07, 11-20, 2007

- If X is absolutely continuous, $\Omega = \mathbb{R}$: a solution can be obtained using a confidence band on the cumulative distribution function F_X of X.
- Let $\mathbf{X} = (X_1, \ldots, X_n)$ be an iid sample from X with cdf F_X.
- A pair of functions $(\underline{F}(\cdot; \mathbf{X}), \overline{F}(\cdot; \mathbf{X}))$ computed from $\mathbf{X}$ and such that $\underline{F}(\cdot; \mathbf{X}) \leq \overline{F}(\cdot; \mathbf{X})$ is a confidence band at level $\alpha \in (0, 1)$ if

$$P\left\{\underline{F}(x; \mathbf{X}) \leq F_X(x) \leq \overline{F}(x; \mathbf{X}),\ \forall x \in \mathbb{R}\right\} = 1 - \alpha,$$

Predictive belief functions
Kolmogorov Confidence band

- A non parametric confidence band can be computed using the Kolmogorov statistic:

$$D_n = \sup_x |S_n(x; \mathbf{X}) - F_X(x)|,$$

where $S_n(\cdot; \mathbf{X})$ is the sample cdf.

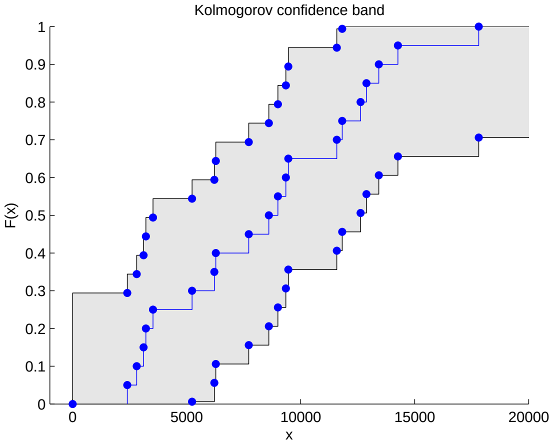
- The probability distribution of D_n can be computed exactly. Let $d_{n,\alpha}$ by the α-critical value of D_n, i.e., $\mathbb{P}(D_n \geq d_{n,\alpha}) = \alpha$.
- The two step functions

$$\begin{aligned} \underline{F}(x; \mathbf{X}) &= \max(0, S_n(x; \mathbf{X}) - d_{n,\alpha}), \\ \overline{F}(x; \mathbf{X}) &= \min(1, S_n(x; \mathbf{X}) + d_{n,\alpha}) \end{aligned}$$

form a confidence band at level $1 - \alpha$.

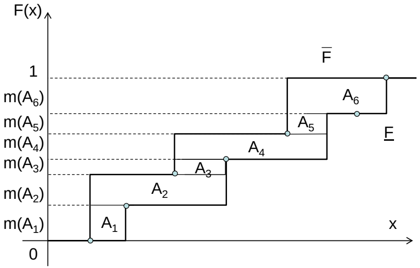
Kolmogorov Confidence band

Bearing data ($1 - \alpha = 0.95$)



Predictive belief functions

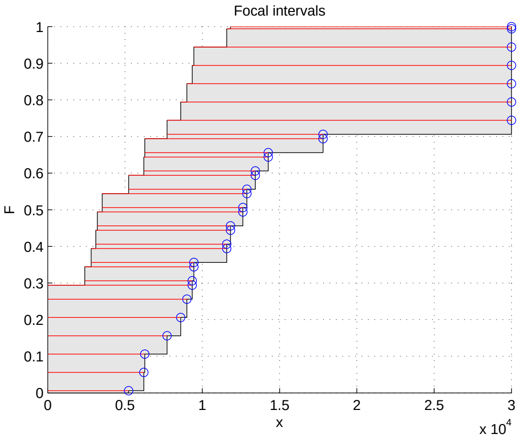
p-boxes and belief functions



- A Kolmogorov confidence band defines a p-box (a set of probability measures with cdf constrained by 2 step functions).
- A p-box defines a discrete random interval.
- The belief function constructed from a Kolmogorov confidence band at level $1-\alpha$ is a predictive belief function at level $1-\alpha$.

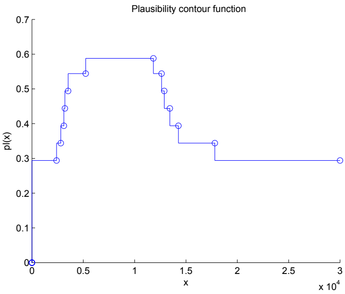
Construction of a mass function from a p-box

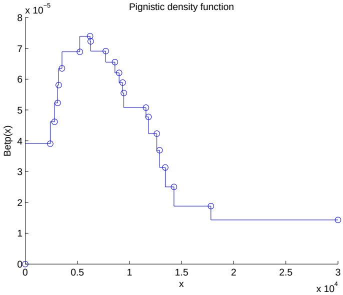
Bearing data



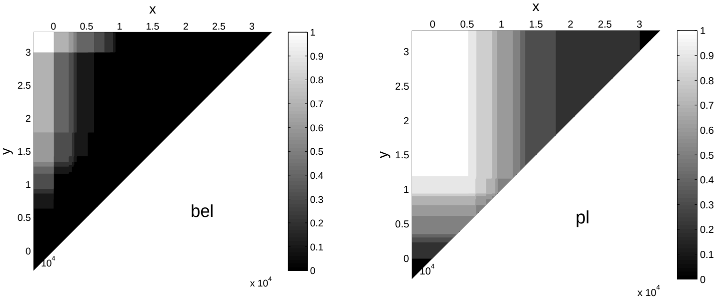
Contour and pignistic density functions

Bearing data





Belief and plausibility functions
Bearing data



Summary

- Developing engineering applications using the Dempster-Shafer framework requires modeling expert knowledge and statistical information using belief functions.
- Systematic and principled methods now exist:
  - Least-commitment principle
  - Discounting
  - GBT
  - Predictive belief functions
  - etc.
- Specific methods will be studied in following lectures (parametric statistical inference, classification, etc.).
- More research on expert knowledge elicitation and statistical inference is needed.

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cf. `http://www.hds.utc.fr/˜tdenoeux`

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