Allowing imprecision in belief representation using fuzzy-valued belief structures*

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Abstract

This paper extends the theory of Evidence, by allowing subjective degrees of belief in crisp or fuzzy propositions to be represented in the form of intervals or fuzzy numbers. The usual concepts of credibility and plausibility, as well as combination rules and normalization procedures, are generalized in this framework.

1 Introduction

Uncertainty representation is a critical issue in many areas of science and engineering. In the last thirty years, several alternatives to Probability theory have been advocated as suitable frameworks for encoding and manipulating uncertain knowledge. In particular, the Dempster-Shafer theory of evidence [7] has attracted considerable interest, essentially for its ability to represent states of partial or total ignorance. A particularly coherent interpretation of this theory was proposed by Smets [9] as the Transferable Belief Model (TBM), a normative, non probabilistic approach to the numerical representation of partial beliefs. In the TBM, it is assumed that the state of belief of a rational agent may be represented by a function assigning *precise* degrees of belief to individual propositions. Although this assumption is well justified axiomatically [8], it raises the question of the practical elicitation of beliefs in real applications. As argued by Walley [10], imprecision cannot always be avoided in models of uncertainty, because of many factors such as lack of introspection, assessment strategy (precise degrees of belief may be too difficult or too costly to elicit with great precision), belief instability, ambiguity, etc. In this paper, an attempt is made to extend the TBM so as to allow imprecision in belief representation, by providing the possibility to assign intervals, or fuzzy numbers, to crisp or fuzzy propositions. Whereas the possibility of such an extension had already been evoked by a few authors [15, 6], no sound and computationally tractable framework was available. Such a framework is proposed in this paper, based on the concepts of interval-valued and fuzzy-valued belief structures, defined, respectively, as crisp and fuzzy sets of generalized belief structures.

*In Proc. of IPMU'98, vol. 1, p. 48–55, Paris, July 1998. EDK.

2 Background

Let Ω denote a finite set of possible answers to a certain question, and Y a variable describing the correct (but unknown) answer. According to the TBM, it is assumed that the beliefs held by a rational agent (denoted by "You") regarding Y, given a certain body of evidence, may be represented by a *belief structure* (BS), defined as a function m from 2^Ω to $[0,1]$, verifying:

$$\sum_{A\subseteq\Omega} m(A) = 1. \qquad (1)$$

The subsets A of Ω such that $m(A) > 0$ are the *focal elements* of m, and the quantities $m(A)$ are called *belief numbers*. A BS m such that $m(\emptyset) = 0$ is said to be normal. The normality condition corresponds to the certainty that Y lies in Ω. If Ω is assumed to be exhaustive (closed-world assumption), then such a condition should be imposed (this is the situation initially considered by Shafer [7]). In the most general case, however, the allocation of a positive belief number to the empty set is interpreted in the TBM as quantifying You belief that $Y \notin \Omega$ [9].

Assuming Your state of belief to be described by m, Your *total belief* in the proposition $Y \in A$ is represented by a number, called the *credibility* of A, and defined as:

$$\mathrm{bel}_m(A) \triangleq \sum_{\emptyset \neq B \subseteq A} m(B). \quad (2)$$

Such a function $\mathrm{bel}_m : 2^\Omega \mapsto [0,1]$, called a *belief function*, may be shown to have the property of complete monotonicity [7]. Smets recently proposed a set of axioms justifying the use of belief functions for representing degrees of belief [8]. Closely related to the notion of belief function is that of *plausibility function*, defined for each $A \subseteq \Omega$ as:

$$\mathrm{pl}_m(A) \triangleq \mathrm{bel}_m(\Omega) - \mathrm{bel}_m(\bar{A}) \quad (3)$$

where $\bar{A}$ denotes the complement of A. The quantity $\mathrm{pl}_m(A)$ receives a natural interpretation as the amount of potential support that could be given to A, if further evidence became available.

Let us now assume that you collect two distinct pieces of evidence coming from two different sources. Let m_1 and m_2 denote the BS's induced by each of these pieces of evidence considered individually. Then, m_1 and m_2 may be combined in several different ways, depending on Your knowledge concerning the reliability of the two sources. If you know that they both are reliable, then You may combine m_1 and m_2 conjunctively by defining a new BS $m_1 \cap m_2$ as:

$$(m_1 \cap m_2)(A) \triangleq \sum_{B \cap C = A} m_1(B) m_2(C), \quad (4)$$

for all $A \subseteq \Omega$. On the other hand, if You only know that *at least* one of the two sources is reliable, then the corresponding BS's should rather be combined in a disjunctive fashion, replacing $\cap$ by $\cup$ in the above equation.

Note that the conjunctive sum as described by (4) may produce a subnormal BS (i.e., it is possible to have $(m_1 \cap m_2)(\emptyset) > 0$). Under the closed-world assumption, some kind of normalization thus has to be performed. The *Dempster normalization procedure* converts a subnormal BS m into a normal BS m^* by dividing each belief number by $1 - m(\emptyset)$ [7]. To avoid some counterintuitive effects of this rule in case of conflicting evidence, Yager [13] proposed to convert a subnormal BS m into a normal one m° by transferring the mass $m(\emptyset)$ to Ω. However, the association of the conjunctive sum with this normalization procedure (hereafter referred to as *Yager normalization*) defines an operation on BS's which is not associative.

As argued by several authors [7, 9], a distinctive advantage of the TBM as compared to the classical Bayesian approach based on probability measures resides in its ability to represent the state of total ignorance. However, the TBM in its standard form does not allow You to assign degrees of belief to ambiguous propositions such as typically expressed in verbal statements. Nevertheless, the theory may be extended to allow degrees of belief to be assigned to fuzzy subsets of the possibility space.

The idea of extending the concepts of evidence theory to fuzzy sets was first put forward by Zadeh [15], in relation to his work on information granularity and Possibility theory. Zadeh's approach leads to the definition of a *generalized belief structure*[1] (GBS) as a BS with fuzzy focal elements F_i, $1 \leq i \leq n$. The concept of plausibility of a fuzzy subset A may then be generalized as the expectation of the conditional possibility measure of A given that Y is F_i, defined as:

$$\mathrm{pl}_m(A) \triangleq \sum_{i=1}^{n} m(F_i) \Pi(A|F_i) \quad (5)$$

with $\Pi(A|F_i) \triangleq \max_{\omega \in \Omega} \mu_A(\omega) \wedge \mu_{F_i}(\omega)$. Similarly, the credibility of a fuzzy subset A induced by a GBS m may be defined as the expectation of the conditional necessity of A:

$$\mathrm{bel}_m(A) \triangleq \sum_{i=1}^{n} m(F_i) N(A|F_i) \quad (6)$$

with $N(A|F_i) \triangleq 1 - \Pi(\bar{A}|F_i)$.

Remark: The condition that the fuzzy focal elements of a GBS m be normal generalizes the normality condition imposed to classical belief structures under the closed-world assumption. If this condition is relaxed, the definition of the conditional necessity of A given F_i should to be changed to:

$$N(A|F_i) \triangleq \Pi(\Omega|F_i) - \Pi(\bar{A}|F_i), \quad (7)$$

as remarked by Dubois and Prade [4]. Equation (6) is then still a valid generalization of (2).

The next step in the generalization of evidence theory to fuzzy events concerns the combination of GBS's. As proposed by Yager [12], the conjunctive and disjunctive sums may be readily extended to fuzzy belief structures by replacing the crisp intersection and union by fuzzy counterparts, defined for example using the min and max operations. More generally, any binary set operator ∇ defines a corresponding operation on GBS's such that:

$$(m_1 \nabla m_2)(A) \triangleq \sum_{B \nabla C = A} m_1(B) m_2(C) \quad (8)$$

[1] Such a function is called a fuzzy BS by Yager [11]; however, we prefer to avoid this term here in order to avoid any confusion with the notion of fuzzy-valued BS introduced in this paper.

where A is an arbitrary fuzzy subset of Ω [12].

Of course, the combination of two normal GBS's using, for example, the conjunctive sum, may produce a subnormal GBS. If the normality condition is enforced, the conversion of an arbitrary GBS into a normal one may be performed by generalizing either of the Dempster and Yager normalization procedures. Yager [13] proposed to generalize the Dempster procedure as:

$$m^*(A) \triangleq \frac{\sum_{B^*=A} h_B m(B)}{\sum_{B \in \mathcal{F}(m)} h_B m(B)} \tag{9}$$

where $h_B = \max_\omega \mu_B(\omega)$ denotes the height of B, B^* is the normal fuzzy set defined by $\mu_{B^*}(\omega) = \mu_B(\omega)/h_B$, and $\mathcal{F}(m)$ is the set of focal elements of m (this procedure is called *soft normalization* par Yager). In [13], it is also proposed to generalize the Yager procedure as:

$$m^\circ(A) \triangleq \sum_{B^\circ = A} m(B) \tag{10}$$

where B° is a normal fuzzy set defined by $\mu_{B^\circ}(\omega) = \mu_B(\omega) + 1 - h_B$.

3 Interval-valued belief structures

Generalized belief structures as defined above provide a means of representing someone's belief in vague propositions such as produced in natural language. However, they still assign precise real numbers to each focal element, thereby ignoring the uncertainty attached to elicited belief numbers in many realistic situations. In this section, we go one step further in the generalization of evidence theory, by allowing belief masses to be provided in the form of *intervals*. Although the concept of interval-valued belief structure defined in this section is interesting its own right [2], it is mainly seen in this paper as a preliminary step towards the complete fuzzification of the TBM (undertaken in the next section) in which fuzzy belief numbers are allowed to be assigned to fuzzy propositions.

3.1 Definitions

In this section, *interval-valued belief structures* (IBS's) are introduced as convex sets of GBS's verifying certain constraints. In the rest of this paper, we denote by $[0,1]^\Omega$ the set of fuzzy subsets of Ω, and by $\mathcal{S}_\Omega$ the set of GBS's[2] on Ω. The reader is referred to a longer paper [2] for detailed proofs of most results presented in this section.

[2] Unless explicitly stated, no distinction shall be made between BS's with crisp and fuzzy focal elements, neither shall we assume the BS's to be normalized.

Definition 1 (Interval-valued BS)
An interval-valued belief structure (IBS) $\mathbf{m}$ is a non empty subset of $\mathcal{S}_\Omega$ such that there exist n crisp or fuzzy subsets $F_1, \ldots, F_n$ of Ω, and n intervals $([a_i, b_i])_{1 \le i \le n}$ of $\mathbb{R}$, such that $m \in \mathbf{m}$ iif

- $a_i \le m(F_i) \le b_i \quad \forall i \in \{1, \ldots, n\}$, *and*
- $\sum_{i=1}^{n} m(F_i) = 1.$ □

An IBS is thus completely specified by a set of focal elements, and a corresponding set of intervals. However, it is important to note that this representation is not unique: since both b_i and $1 - \sum_{j \ne i} a_i$ are upper bounds of $m(F_i)$, it is clear that, whenever $b_i \ge 1 - \sum_{j \ne i} a_i$, b_i may be replaced by a higher bound $b'_i \ge b_i$. To obtain a unique characterization of $\mathbf{m}$, we thus introduce the concepts of *tightest lower and upper bounds* of $\mathbf{m}$, defined for all $A \in [0,1]^\Omega$ as, respectively:

$$m^-(A) \triangleq \min_{m \in \mathbf{m}} m(A) \tag{11}$$

$$m^+(A) \triangleq \max_{m \in \mathbf{m}} m(A). \tag{12}$$

We may then define the set $\mathcal{F}(\mathbf{m})$ of focal elements of $\mathbf{m}$ as

$$\mathcal{F}(\mathbf{m}) \triangleq \{A \in [0,1]^\Omega | m^+(A) > 0\}.$$

The tightest bounds may easily obtained from any set of intervals $[a_i, b_i]$ defining $\mathbf{m}$ by:

$$m^-(F_i) = \max\left[a_i, 1 - \sum_{j \ne i} b_j\right]$$

$$m^+(F_i) = \min\left[b_i, 1 - \sum_{j \ne i} a_j\right]$$

for all $1 \le i \le n$, and $m^-(A) = m^+(A) = 0$, for all $A \notin \mathcal{F}(\mathbf{m})$.

BS's with at most three focal elements may be conveniently represented as points of the two-dimensional probability simplex [10]. This is an equilateral triangle with unit height, in which the masses assigned to each of the three focal elements are identified with perpendicular distances to each side of the triangle. In this representation, each constraint of the form $m(F) \le m^+(F)$ or $m(F) \ge m^-(F)$ for some $F \in \mathcal{F}(\mathbf{m})$ is identified with a line parallel to one side of the triangle, and dividing the simplex in two parts. An IBS is thus represented as a convex polyhedron with sides parallel to sides of the triangle (Figure 1).

3.2 Interval-valued belief functions

Given an IBS $\mathbf{m}$, and a crisp or fuzzy subset A of Ω, let us now consider the problem of determining the

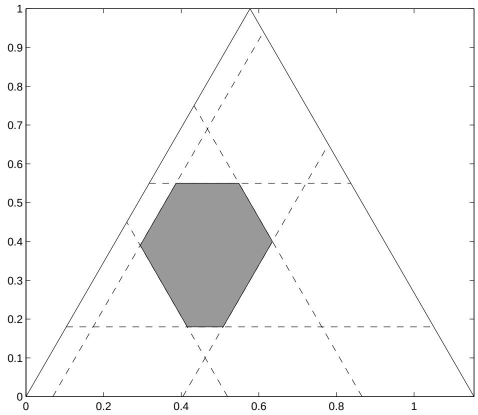

Figure 1: Representation of an IBS in the probability simplex.

possible values of $\mathrm{bel}_m(A)$ (defined by (6)), where m ranges over $\mathbf{m}$. Since this quantity is a linear combination of belief numbers constrained to lie in closed intervals, its range is itself a closed interval. Let $\mathrm{bel}_m^-(A)$ and $\mathrm{bel}_m^+(A)$ denote, respectively, the minimum and maximum of $\mathrm{bel}_m(A)$ for all $m \in \mathbf{m}$. The interval $[\mathrm{bel}_m^-(A), \mathrm{bel}_m^+(A)]$ will be called the *credibility interval* of A induced by $\mathbf{m}$, and will be noted $\mathbf{bel_m}(A)$. The function $\mathbf{bel_m} : A \mapsto \mathbf{bel_m}(A)$ will be called the interval-valued belief function induced by $\mathbf{m}$ [3].

The practical determination of credibility intervals involves the resolution of a particular class of linear programming (LP) problems, in which the goal is to find the minimum and maximum of a linear function of n variables $x_1, \ldots, x_n$, under one linear equality constraint and a set of box constraints. A general solution to this problem was provided by Dubois and Prade [3, 5] who proved the following theorem:

Theorem 1 (Dubois and Prade, 1981)
Let $x_1, \ldots, x_n$ be n variables linked by the following constraints:

$$\sum_{i=1}^{n} x_i = 1$$

$$a_i \leq x_i \leq b_i \quad 1 \leq i \leq n$$

and let f be a function defined by $f(x_1, \ldots, x_n) = \sum_{i=1}^{n} c_i x_i$ with $0 \leq c_1 \leq c_2 \leq \ldots \leq c_n$. Then

$$\min f = \max_{k=1,n} \left(\sum_{j=1}^{k-1} b_j c_j + \alpha_k c_k + \sum_{j=k+1}^{n} a_j c_j \right)$$

$$\max f = \min_{k=1,n} \left(\sum_{j=1}^{k-1} a_j c_j + \beta_k c_k + \sum_{j=k+1}^{n} b_j c_j \right)$$

with $\alpha_k = 1 - \sum_{j=1}^{k-1} b_j - \sum_{j=k+1}^{n} a_j$ and $\beta_k = 1 - \sum_{j=1}^{k-1} a_j - \sum_{j=k+1}^{n} b_j$. □

Hence, an exact determination of $\mathrm{bel}_{\mathbf{m}}^-(A)$ and $\mathrm{bel}_{\mathbf{m}}^+(A)$ may be obtained without resorting to an iterative procedure. In particular, when both A and the focal elements of $\mathbf{m}$ are crisp, the coefficients c_i in Theorem 1 are all equal to 0 or 1; one then obtains without difficulty the following expressions for the bounds of $\mathbf{bel_m}(A)$:

$$\mathrm{bel}^-(A) = \max \left(\sum_{B \in \mathcal{I}_A} m^-(B), 1 - \sum_{B \notin \mathcal{I}_A} m^+(B) \right)$$

$$\mathrm{bel}^+(A) = \min \left(\sum_{B \in \mathcal{I}_A} m^+(B), 1 - \sum_{B \notin \mathcal{I}_A} m^-(B) \right),$$

where $\mathcal{I}_A$ denotes the set of non empty subsets of A. Needless to say, the approach adopted above to define the credibility interval of a fuzzy event A may easily be transposed to the definition of the plausibility interval of A, denoted as $\mathbf{pl_m}(A)$:

$$\mathbf{pl_m}(A) \triangleq [\min_{m \in \mathbf{m}} \mathrm{pl}_m(A), \max_{m \in \mathbf{m}} \mathrm{pl}_m(A)].$$

The bounds of $\mathbf{pl_m}(A)$ are easily obtained using Theorem 1, in exactly the same way as explained above.

3.3 Combination of IBS's

As already mentioned, any binary set operation ∇ induces a binary operation on GBS's (also denoted ∇ for simplicity) through (8). In this section, we go one step further in the generalization process by extending any binary operation in $\mathcal{S}_\Omega$ to IBS's. This will be achieved by considering the lower and upper bounds of $(m_1 \nabla m_2)(A)$, for all $A \in [0,1]^\Omega$.

Definition 2 (Combination of two IBS's)
Let $\mathbf{m}_1$ and $\mathbf{m}_2$ be two IBS's on the same frame Ω, and let ∇ be a binary operation on BS's. The combination of $\mathbf{m}_1$ and $\mathbf{m}_2$ by ∇ is defined as the IBS $\mathbf{m} = \mathbf{m}_1 \nabla \mathbf{m}_2$ with bounds:

$$m^-(A) = \min_{(m_1,m_2) \in \mathbf{m}_1 \times \mathbf{m}_2} (m_1 \nabla m_2)(A)$$

$$m^+(A) = \max_{(m_1,m_2) \in \mathbf{m}_1 \times \mathbf{m}_2} (m_1 \nabla m_2)(A)$$

[3] It must be noted that the set $\mathbf{bel_m}$ *is not* the set $\mathcal{B}_\mathbf{m}$ of belief functions induced by some IBS in $\mathbf{m}$. However, we obviously have the inclusion $\mathcal{B}_\mathbf{m} \subseteq \mathbf{bel_m}$, which allows to regard $\mathbf{bel_m}$ as an approximation to $\mathcal{B}_\mathbf{m}$ (it is in fact the smallest interval-valued belief function containing $\mathcal{B}_\mathbf{m}$).

for all $A \in [0,1]^\Omega$. □

It may be shown by counterexamples [2] that the extension of the ∇ operation from BS's to IBS's performed according to Definition 2 does not, in general, preserve the associativity property, i.e., we may have

$$(\mathbf{m}_1 \nabla \mathbf{m}_2) \nabla \mathbf{m}_3 \neq \mathbf{m}_1 \nabla (\mathbf{m}_2 \nabla \mathbf{m}_3)$$

for some $\mathbf{m}_1$, $\mathbf{m}_2$ and $\mathbf{m}_3$. To avoid any influence of the order in which n IBS's are combined, it is therefore necessary to combine them at once using an n-ary combination operator introduced in the following definition.

Definition 3 (Combination of n IBS's)
Let $\mathbf{m}_1, \ldots, \mathbf{m}_n$ *be* n *IBS's on the same frame* Ω, *and let* ∇ *be an associative operation on BS's. The combination of* $\mathbf{m}_1, \ldots, \mathbf{m}_n$ *by* ∇ *is defined as the IBS* $\mathbf{m} = \mathbf{m}_1 \nabla \ldots \nabla \mathbf{m}_n$ *with bounds:*

$$\begin{aligned} m^-(A) &= \min_{(m_1,\ldots,m_n) \in \mathbf{m}_1 \times \ldots \times \mathbf{m}_n} (m_1 \nabla \ldots \nabla m_n)(A) \\ m^+(A) &= \max_{(m_1,\ldots,m_n) \in \mathbf{m}_1 \times \ldots \times \mathbf{m}_n} (m_1 \nabla \ldots \nabla m_n)(A) \end{aligned}$$

for all $A \in [0,1]^\Omega$. □

It may be shown [2] that, for any IBS's $\mathbf{m}_1$, $\mathbf{m}_2$ and $\mathbf{m}_3$, we have:

$$(\mathbf{m}_1 \nabla \mathbf{m}_2) \nabla \mathbf{m}_3 \supseteq \mathbf{m}_1 \nabla \mathbf{m}_2 \nabla \mathbf{m}_3.$$

Hence, given a sequence of n IBS $\mathbf{m}_1, \ldots, \mathbf{m}_n$ the strategy of combining them one by one using the binary operator introduced in Definition 2 leads to pessimistic lower and upper bounds for the belief intervals introduced more rigorously in Definition 3.

The practical determination of $\mathbf{m} = \mathbf{m}_1 \nabla \mathbf{m}_2$ for some IBS $\mathbf{m}_1$ and $\mathbf{m}_2$ and some set operation ∇ requires to search for the extrema of a quadratic function

$$\varphi_A(m_1, m_2) = \sum_{B \nabla C = A} m_1(B) m_2(C), \qquad (13)$$

under linear and box constraints. The solution of this problem is trivial when the sum in the right-hand side of (13) contains only one term, since we then simply have a product of two non interactive variables. A result with considerably higher level of generality was obtained in [2], in which we derived an analytical expression for $(\mathbf{m}_1 \cap \mathbf{m}_2)(A)$ in the case where $\mathbf{m}_2$ is a *simple* IBS, i.e., $\mathcal{F}(\mathbf{m}_2) = \{F, \Omega\}$ for some $F \in [0,1]^\Omega$. In the most general case, however, an explicit solution to the above quadratic programming problem is difficult to obtain, and one has to resort to numerical procedures. A very efficient algorithm based on an alternate directions scheme is described in [2].

3.4 Normalization of an IBS

As already mentioned in Section 2, the normality condition to be imposed on BS's under the closed-world assumption may be defined, in a fuzzy environment, as $h_F = 1$ for all $F \in \mathcal{F}(m)$. By analogy, an IBS $\mathbf{m}$ will be said to be normal if it contains only normal BS's, which can be expressed as $h_F = 1$ for all $F \in \mathcal{F}(\mathbf{m})$. The aim of this section is to introduce extensions of the Dempster and Yager normalization procedures, allowing to convert subnormal IBS's into normal ones. As it involves only linear transformations, the Yager procedure is considerably simpler, and will therefore be examined first.

Let $\mathbf{m}$ be an IBS with crisp or fuzzy focal elements. The normalized form of $\mathbf{m}$, according to the Yager procedure, will be defined as the IBS $\mathbf{m}^\circ$ with bounds:

$$\begin{aligned} m^{\circ -}(A) &\triangleq \min_{m \in \mathbf{m}} m^\circ(A) \qquad (14) \\ m^{\circ +}(A) &\triangleq \max_{m \in \mathbf{m}} m^\circ(A), \qquad (15) \end{aligned}$$

for all A in $[0,1]^\Omega$, with $m^\circ(A)$ defined by (10). The bounds of $\mathbf{m}^\circ$ may therefore be found as the solutions to very simple linear programming problems. It is easy to show that:

$$\begin{aligned} m^{\circ -}(A) &= \max\left(\sum_{F^\circ = A} m^-(F), 1 - \sum_{F^\circ \neq A} m^+(F) \right) \\ m^{\circ +}(A) &= \min\left(\sum_{F^\circ = A} m^+(F), 1 - \sum_{F^\circ \neq A} m^-(F) \right). \end{aligned}$$

Conceptually, the extension of the Dempster normalization procedure to IBS's may be performed in exactly the same way as for Yager's procedure. Given an arbitrary IBS $\mathbf{m}$, its normalized form, according to the Dempster procedure, will be defined as the IBS $\mathbf{m}^*$ with bounds:

$$\begin{aligned} m^{*-}(A) &\triangleq \min_{m \in \mathbf{m}} m^*(A) \qquad (16) \\ m^{*+}(A) &\triangleq \max_{m \in \mathbf{m}} m^*(A), \qquad (17) \end{aligned}$$

for all A in $[0,1]^\Omega$, with $m^*(A)$ defined by (9). However, because of the non linearity of (9), the practical determination of $\mathbf{m}^*$ is significantly more difficult than that of $\mathbf{m}^\circ$. This problem was solved exactly in [2] for the case where all focal elements of $\mathbf{m}$ are crisp. We showed that:

$$m^{*-}(A) = \frac{m^-(A)}{1 - m^-(\emptyset) \vee \left(1 - \sum_{\substack{B \neq A \\ B \neq \emptyset}} m^+(B) - m^-(A)\right)}$$

$$m^{*+}(A) = \frac{m^+(A)}{1 - m^+(\emptyset) \wedge \left(1 - \sum_{\substack{B \neq A \\ B \neq \emptyset}} m^-(B) - m^+(A)\right)}$$

where $\vee$ and $\wedge$ denote, respectively, the max and min operations, and A is a non empty focal element of $\mathbf{m}$.

In the more general case in which some focal elements of $\mathbf{m}$ are fuzzy, the bounds of $\mathbf{m}^*$ are the solutions of non linear programming problems for which no analytic solution is, to our knowledge, available. These values thus have to be computed numerically using an iterative non linear optimization procedure.

4 Fuzzy-valued belief structures

4.1 Definition

In many applications, the degrees of belief in various hypotheses are either directly obtained through verbal statements such as "high", "very low", "around 0.8", or are inferred from "vague" evidence expressed linguistically in a similar way. In such situations, it is difficult to avoid arbitrariness in assigning a precise number, or even an interval, to each hypothesis. *Fuzzy numbers* have been proposed as a suitable formalism for handling such kind of ambiguity in modeling subjective probability judgments [5]. A fuzzy number may be viewed as an elastic constraint acting on a certain variable which is only known to lie "around" a certain value. It generalizes both concepts of real number and closed interval.

In this section, we introduce the new concept of a *fuzzy-valued belief structure* (FBS), which will be defined a fuzzy set of GBS's on Ω, whose belief masses are restricted by fuzzy numbers.

Definition 4
A fuzzy-valued belief structure (FBS) is a normal fuzzy subset $\widetilde{\mathbf{m}}$ *of* $\mathcal{S}_\Omega$ *such that there exist n elements* $F_1, \ldots, F_n$ *of* $[0,1]^\Omega$, *and n non null fuzzy numbers* $\widetilde{m}_i, 1 \leq i \leq n$ *such that, for every* $m \in \mathcal{S}_\Omega$,

$$\mu_{\widetilde{\mathbf{m}}}(m) \triangleq \begin{cases} \min_{1 \leq i \leq n} \mu_{\widetilde{m}_i}(m(F_i)) & \text{if } \sum_{i=1}^n m(F_i) = 1 \\ 0 & \text{otherwise} \end{cases}$$ □

Remark: The assumption that $\widetilde{\mathbf{m}}$ is a *normal* fuzzy set imposes certain conditions on fuzzy numbers $\widetilde{m}_i$. More precisely, the fact that $\mu_{\widetilde{\mathbf{m}}}(m) = 1$ for some m implies that $\mu_{\widetilde{m}_i}(m(F_i)) = 1$ for every $i \in \{1, \ldots, n\}$. Hence, for all i, $m(F_i)$ belongs to the core ${}^1\widetilde{m}_i$ of $\widetilde{m}_i$. The BS m thus belongs to an IBS with bounds $[{}^1\widetilde{m}_i^-, {}^1\widetilde{m}_i^+]$. In particular, this implies that

$$\sum_{i=1}^n {}^1\widetilde{m}_i^- \leq 1 \quad \text{and} \quad \sum_{i=1}^n {}^1\widetilde{m}_i^+ \geq 1.$$

As suggested in the above remark, each BS m belonging to the *core* of a FBS $\widetilde{\mathbf{m}}$ constrained by fuzzy numbers $\widetilde{m}_i$, belongs to an IBS bounded by the cores of the $\widetilde{m}_i$. Conversely, it is obvious that a BS m such that $m(F_i) \in {}^1\widetilde{m}_i$ for all i has full membership to $\widetilde{\mathbf{m}}$. Hence, we may deduce that the core of a FBS $\widetilde{\mathbf{m}}$ constrained by fuzzy numbers $\widetilde{m}_i$ is an IBS ${}^1\widetilde{\mathbf{m}}$ bounded by the cores of the $\widetilde{m}_i$. This result may be extended to any α-cut of $\widetilde{\mathbf{m}}$, which may be shown to have a very simple characterization in terms of the α-cuts of the fuzzy numbers constraining $\widetilde{\mathbf{m}}$, as stated in the following proposition.

Proposition 1
Let $\widetilde{\mathbf{m}}$ *be a FBS defined by n elements* $F_1, \ldots, F_n$ *of* $[0,1]^\Omega$ *and n fuzzy numbers* $\widetilde{m}_1, \ldots, \widetilde{m}_n$. *For any* $\alpha \in]0,1]$, *the* α*-cut of* $\widetilde{\mathbf{m}}$ *is an IBS* ${}^\alpha\widetilde{\mathbf{m}}$ *with bounds* ${}^\alpha\widetilde{m}_i$ *for all* $i \in \{1, \ldots, n\}$. □

Proof: Let α be any real number in $]0,1]$, and ${}^\alpha\widetilde{\mathbf{m}}$ the α-cut of $\widetilde{\mathbf{m}}$. By definition,

$${}^\alpha\widetilde{\mathbf{m}} = \{m \in \mathcal{S}_\Omega | \mu_{\widetilde{\mathbf{m}}}(m) \geq \alpha\}.$$

The condition $\mu_{\widetilde{\mathbf{m}}}(m) \geq \alpha$ may be translated to

$$\min_{1 \leq i \leq n} \mu_{\widetilde{m}_i}(m(F_i)) \geq \alpha \text{ and } \sum_{i=1}^n m(F_i) = 1$$

which is equivalent to

$$m(F_i)) \in {}^\alpha\widetilde{m}_i \ \forall i \text{ and } \sum_{i=1}^n m(F_i) = 1.$$

Since the $\widetilde{m}_i$ are fuzzy numbers, their α-cuts are closed intervals. Hence, ${}^\alpha\widetilde{\mathbf{m}}$ is an IBS. □

As in the case of IBS's, it is useful to define a unique representation of a FBS $\widetilde{\mathbf{m}}$, in the form of fuzzy numbers assigned to each focal elements. This may be achieved by considering the upper and lower bounds of all its α-cuts. More precisely, let us denote:

$${}^\alpha m^-(F_i) \triangleq \min_{m \in {}^\alpha\widetilde{\mathbf{m}}} m(F_i)$$

$${}^\alpha m^+(F_i) \triangleq \max_{m \in {}^\alpha\widetilde{\mathbf{m}}} m(F_i).$$

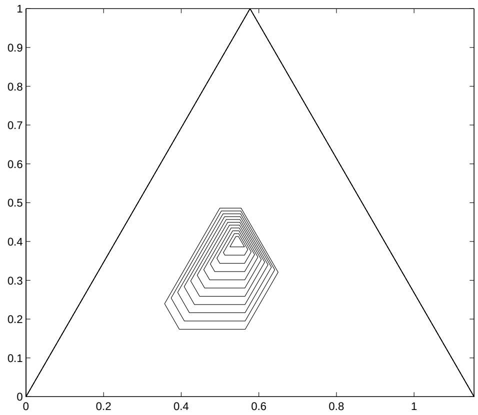

Figure 2: Representation of a FBS in the probability simplex. Each polyhedron corresponds to an α-cut of the FBS.

The fuzzy set $\widetilde{\mathbf{m}}(F_i)$ with α-cuts ${}^\alpha\widetilde{\mathbf{m}}(F_i) = [{}^\alpha m^-(F_i), {}^\alpha m^+(F_i)]$ satisfies all the axioms of a fuzzy number. Hence, a FBS may be seen a fuzzy mapping assigning a fuzzy number to each $A \in [0,1]^\Omega$ (with $\widetilde{\mathbf{m}}(A) = 0$ for all $A \notin \{F_1, \dots, F_n\}$).

As for IBS's, we may define a focal element of a FBS $\widetilde{\mathbf{m}}$ as a crisp or fuzzy subset of Ω that receives a positive mass of belief from at least one BS with non zero membership to $\widetilde{\mathbf{m}}$. The set $\mathcal{F}(\widetilde{\mathbf{m}})$ of focal elements of $\widetilde{\mathbf{m}}$ is thus identical to $\mathcal{F}({}^{0+}\widetilde{\mathbf{m}})$, the set of focal elements of the support of $\widetilde{\mathbf{m}}$.

The membership function $\mu_{\widetilde{\mathbf{m}}}$ of a FBS with at most three focal elements may be visualized in the probability simplex as a surface, the contours of which are the polyhedrons corresponding to the α-cuts of $\widetilde{\mathbf{m}}$. Such a representation is shown in Figure 2.

4.2 Fuzzy credibility and plausibility

The fuzzy credibility and plausibility of a crisp or fuzzy subset A of Ω induced by a FBS may be defined by applying the extension principle to (6) and (5), respectively. Generally speaking, the extension principle provides a canonical way of finding the range of a function f whose arguments are restricted by a certain possibility distribution [14, 3]. In the case where each variable $\widetilde{x}_i$ is restricted by a possibility distribution $\mu_{\widetilde{x}_i}$, and where the variables are constrained to lie within a domain D, their image $\widetilde{z} = f(\widetilde{x}_1, \dots, \widetilde{x}_n)$ under f is defined as a fuzzy set with membership function:

$$\mu_{\widetilde{z}}(w) \triangleq \sup_{u_1,\dots,u_n} \min_i \mu_{\widetilde{x}_i}(u_i) \qquad (18)$$

under the constraints $w = f(u_1, \dots, u_n)$ and $(u_1, \dots, u_n) \in D$.

By applying this principle to (6), we may define the fuzzy credibility of A as a fuzzy set $\widetilde{\mathbf{bel}}(A)$ with membership function:

$$\mu_{\widetilde{\mathbf{bel}}(A)}(w) \triangleq \sup_{\{m \in D | \mathrm{bel}_m(A) = w\}} \min_{1 \le i \le n} \mu_{\widetilde{\mathbf{m}}(F_i)}(m(F_i))$$

where D is the set of GBS's m such that $\sum_{i=1}^n m(F_i) = 1$. This equation may be written more simply as:

$$\mu_{\widetilde{\mathbf{bel}}(A)}(w) \triangleq \sup_{\{m | \mathrm{bel}_m(A) = w\}} \mu_{\widetilde{\mathbf{m}}}(m) \qquad (19)$$

As shown by Dubois and Prade [3], (18) defines a fuzzy number when the $\widetilde{x}_i$ are fuzzy numbers and the domain D is defined by linear equality constraints. Hence, $\widetilde{\mathbf{bel}}(A)$ defined by (19) is a fuzzy number. Its α-cut is defined by:

$${}^\alpha\widetilde{\mathbf{bel}}(A) \triangleq \min_{m \in {}^\alpha\widetilde{\mathbf{m}}} \mathrm{bel}_m(A),$$

which is nothing but the credibility interval induced by the IBS ${}^\alpha\widetilde{\mathbf{m}}$. Similarly, using (5) as a definition for the plausibility $\mathrm{pl}_m(A)$ of a crisp or fuzzy subset A of Ω, induced by a BS m, the fuzzy plausibility of A may be defined as a fuzzy number with membership function:

$$\mu_{\widetilde{\mathbf{pl}}(A)}(w) \triangleq \sup_{\{m | \mathrm{pl}_m(A) = w\}} \mu_{\widetilde{\mathbf{m}}}(m). \qquad (20)$$

Its α-cut is the plausibility interval induced by ${}^\alpha\widetilde{\mathbf{m}}$.

Remark: The manipulation of fuzzy numbers may be considerably simplified by using the LL parameterization introduced by Dubois and Prade [5]. When the fuzzy masses $\widetilde{\mathbf{m}}(F)$ assigned by $\widetilde{\mathbf{m}}$ are fuzzy numbers of type LL, then $\widetilde{\mathbf{bel}}(A)$ and $\widetilde{\mathbf{pl}}(A)$ defined by (19) and (20) are also LL fuzzy numbers [5]. Their parameters may be calculated by applying the formula given by Theorem 1 to the core and support of $\widetilde{\mathbf{m}}$.

4.3 Combination of FBS's

A binary operation ∇ on BS's may be generalized to FBS's by applying the extension principle to (8). Given two FBS's $\widetilde{\mathbf{m}}_1$ and $\widetilde{\mathbf{m}}_2$, their combination by ∇ may be defined as a FBS $\widetilde{\mathbf{m}}$ assigning to each $A \in [0,1]^\Omega$ a fuzzy number $\widetilde{\mathbf{m}}(A) = (\widetilde{\mathbf{m}}_1 \nabla \widetilde{\mathbf{m}}_2)(A)$ with membership function

$$\mu_{\widetilde{\mathbf{m}}(A)}(w) = \sup_{\{m_1, m_2\}} \min[\mu_{\widetilde{\mathbf{m}}_1}(m_1), \mu_{\widetilde{\mathbf{m}}_2}(m_2)]$$

under the constraint $(m_1 \nabla m_2)(A) = w$. The α-cut of $\widetilde{\mathbf{m}}(A)$ is an interval ${}^\alpha\widetilde{\mathbf{m}}(A) = [{}^\alpha\widetilde{\mathbf{m}}(A)^-, {}^\alpha\widetilde{\mathbf{m}}(A)^+]$ with

$$\begin{aligned} {}^\alpha\widetilde{\mathbf{m}}(A)^- &= \min_{(m_1, m_2) \in {}^\alpha\widetilde{\mathbf{m}}_1 \times {}^\alpha\widetilde{\mathbf{m}}_2} (m_1 \nabla m_2)(A) \\ {}^\alpha\widetilde{\mathbf{m}}(A)^+ &= \max_{(m_1, m_2) \in {}^\alpha\widetilde{\mathbf{m}}_1 \times {}^\alpha\widetilde{\mathbf{m}}_2} (m_1 \nabla m_2)(A) \end{aligned}$$

It is therefore equal to $({}^{\alpha}\widetilde{\mathbf{m}}_1 \nabla {}^{\alpha}\widetilde{\mathbf{m}}_2)(A)$, which may be computed as described in Section 3.3.

Note that, because the calculation of $(m_1 \nabla m_2)(A)$ involves multiplications, $\widetilde{\mathbf{m}}(A)$ is not, in general, of type LL. Therefore, its membership function has to be reconstructed, up to a certain accuracy, using a finite set of α-cuts. However, when the precise form of the membership function of $\widetilde{\mathbf{m}}(A)$ is not regarded as important, it may be sufficient to approximate it by an LL fuzzy number with the same core and support, as proposed by Dubois and Prade [5].

4.4 Normalization of a FBS

The Dempster and Yager normalization procedures that were extended to IBS's in Section 3.4 may be further generalized to FBS's using, once again, the extension principle.

For example, let $\widetilde{\mathbf{m}}$ be a FBS with crisp focal elements. Its normalization using the Dempster procedure yields a normal FBS $\widetilde{\mathbf{m}}^*$ with focal elements $\mathcal{F}(\widetilde{\mathbf{m}}^*) = \mathcal{F}(\widetilde{\mathbf{m}}) \setminus \emptyset$, such that

$$\mu_{\widetilde{\mathbf{m}}^*(A)}(w) \triangleq \sup_{\{m | m(A)/(1-m(\emptyset))=w\}} \mu_{\widetilde{\mathbf{m}}}(m).$$

The α-cut of $\widetilde{\mathbf{m}}^*(A)$ is obviously an interval $[{}^{\alpha}\widetilde{\mathbf{m}}^*(A)^-, {}^{\alpha}\widetilde{\mathbf{m}}^*(A)^+]$, with

$$\begin{aligned} {}^{\alpha}\widetilde{\mathbf{m}}^*(A)^- &= \min_{m \in {}^{\alpha}\widetilde{\mathbf{m}}} \frac{m(A)}{1-m(\emptyset)} \\ {}^{\alpha}\widetilde{\mathbf{m}}^*(A)^+ &= \max_{m \in {}^{\alpha}\widetilde{\mathbf{m}}} \frac{m(A)}{1-m(\emptyset)}. \end{aligned}$$

These bounds may be computed as explained in Section 3.4. Note that, even when the masses $\widetilde{\mathbf{m}}(B)$ for $B \in \mathcal{F}(\widetilde{\mathbf{m}})$ are LL fuzzy numbers, $\widetilde{\mathbf{m}}^*(A)$ is not because its calculation involves a division. However, an LL fuzzy number with the same core and support as $\widetilde{\mathbf{m}}^*(A)$ may, here again, easily be computed as an approximation.

The same approach may be used to extend the soft and Yager normalization procedures to FBS's. Yager normalization has the computational advantage of being based only on additions and subtractions, which allows to perform exact computations with LL parameterization of fuzzy numbers.

5 Conclusion

In this paper, the TBM has been extended to model the situation in which the beliefs held by a rational agent may only be expressed (or are only known) with some imprecision. Our approach relies on the concepts of interval and fuzzy-valued belief structures, defined, respectively, as crisp and fuzzy sets of generalized belief structures verifying crisp or fuzzy constraints. This framework is expected to be useful in situations involving the elicitation of degrees of belief through verbal statements. Another application concerns the extension of the evidence-based classification method introduced in [1], to the processing of patterns with imprecise features.

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