Computational Statistics – Classical simulation

1. Suppose that ten observations (8,3,4,3,1,7,2,6,2,7) are observed from a Poisson distribution $\mathcal{P}(\lambda)$. A lognormal prior distribution for λ is assumed: $\log \lambda \sim \mathcal{N}(\log(4), 0.5^2)$. Denote the likelihood as $L(\lambda; \mathbf{x})$ and the prior as $\pi(\lambda)$. The MLE of λ is $\widehat{\lambda} = \overline{x}$.

   (a) Plot $q(\lambda|\mathbf{x}) = \pi(\lambda)L(\lambda; \mathbf{x})$ and $e(\lambda) = \pi(\lambda)L(\widehat{\lambda}; \mathbf{x})$ as a function of λ.

   (b) Generate a sample of size $n = 1000$ from the posterior distribution $f(\lambda|\mathbf{x})$ using the rejection sampling method. Draw a histogram of this sample and plot the prior distribution on the same graph.

   (c) Compute a 95% confidence interval on the posterior expectation $\mathbb{E}(\lambda|\mathbf{x})$.

   (d) Repeat the same operations as in the two previous questions, using now the SIR method.

   (e) Compare graphically the two samples using the functions `qqplot` and `boxplot`.

2. Consider the following regression model,

$$y_i = \beta x_i + u_i, \quad i = 1, \ldots, n,$$

   where the error has a Student distribution with $\nu = 2$ degrees of freedom. Assume a Cauchy prior on β, with location parameter $\beta_0 = 0$ and scale parameter $\gamma = 2$.

   (a) Generate a dataset by choosing $n = 50$, x_i from $\mathcal{N}(0, 1)$ and a value of β from its prior distribution. Plot the data and the regression line.

   (b) Plot the log-likelihood function $\log L(\beta)$, the prior density $f(\beta)$, and function $q(\beta) = L(\beta)f(\beta)$.

   (c) Compute the MLE of β.

   (d) Generate samples of size $N = 1000$ from the posteriori distribution of β, using (1) rejection sampling, and (2) the SIR method. Compare these two samples graphically.

   (e) Give 95% credibility on β by the two methods.

(f) Generate samples from the posterior predictive distribution of $y_0 = \beta x_0 + u$ with $x_0 = 2$.