Computational Statistics
Chapter 4: Classical simulation

1. Suppose we ten observations (8,3,4,3,1,7,2,6,2,7) are observed from a Poisson distribution $\mathcal{P}(\lambda)$. A lognorm prior distribution for λ is assumed: $\log \lambda \sim \mathcal{N}(\log(4), 0.5^2)$. Denote the likelihood as $L(\lambda; \mathbf{x})$ and the prior as $\pi(\lambda)$. The MLE of λ is $\widehat{\lambda} = \overline{x}$.

   (a) Plot $q(\lambda|\mathbf{x}) = \pi(\lambda)L(\lambda; \mathbf{x})$ and $e(\lambda) = \pi(\lambda)L(\widehat{\lambda}; \mathbf{x})$ as a function of λ.

   (b) Generate a sample of size $n = 1000$ from the posterior distribution $f(\lambda|\mathbf{x})$ using the rejection sampling method. Draw a histogram of this sample and plot the prior distribution on the same graph.

   (c) Compute a 95% confidence interval on the posterior expectation $\mathbb{E}(\lambda|\mathbf{x})$.

   (d) Repeat the same operations as in the two previous questions, using now the SIR method.

   (e) Compare graphically the two samples using the functions `qqplot` and `boxplot`.

2. Suppose we want to approximate $\mu = \mathbb{E}[(1 + X^2)^{-1}]$, where X has an exponential distribution $\mathcal{E}(1)$ truncated to $[0, 1]$; that is, we want to approximate the integral

$$\mu = \frac{1}{1 - 1/e} \int_0^1 \frac{1}{1 + x^2} e^{-x} dx.$$

   (a) Plot function $h(x) = (1 + x^2)^{-1}$ and the probability density function of X on the interval $[0, 1]$.

   (b) Method 1: generate a sample $X_1, \ldots, X_n$ from X using the probability integral transformation method and compute the Monte Carlo approximation $\widehat{\mu}_1$ of X and its standard error.

   (c) Method 2: generate a sample $Y_1, \ldots, Y_n$ from the uniform distribution $\mathcal{U}([0, 1])$ and compute the importance sampling approximation $\widehat{\mu}_2$ of X and its standard error.

   (d) Repeat the previous operations 100 times and compare the distributions of $\widehat{\mu}_1$ and $\widehat{\mu}_2$ using boxplots.