Workshop on belief functions

Lecture 1 – Representation and Combination of Evidence

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Topic of this workshop

1. This workshop is about the theory of belief functions and its applications to Computational Statistics and Econometrics.
2. What is the Theory of Belief Functions?
   - A formal framework for reasoning and making decisions under uncertainty.
   - Originates from Arthur Dempster's seminal work on statistical inference with lower and upper probabilities.
   - It was then further developed by Glenn Shafer who showed that belief functions can be used as a general framework for representing and reasoning with uncertain information.
   - Also known as Evidence theory or Dempster-Shafer theory.
3. Many applications in computer science (artificial intelligence, information fusion, pattern recognition, etc.).
4. Recently, there has been a revived interested in its application to Statistical Inference and Computational Statistics (classification, clustering).

Outline

1. Representation of evidence
   - Mass functions
   - Belief and plausibility functions
2. Relations with alternative theories
   - Possibility theory
   - Imprecise probabilities
3. Combination of evidence
   - Dempster's rule
   - Some other rules
   - Marginalization, extension

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Mass function

Definition

- Let X be a variable taking values in a finite set Ω (frame of discernment)
- Evidence about X may be represented by a mass function $m : 2^{\Omega} \to [0, 1]$ such that

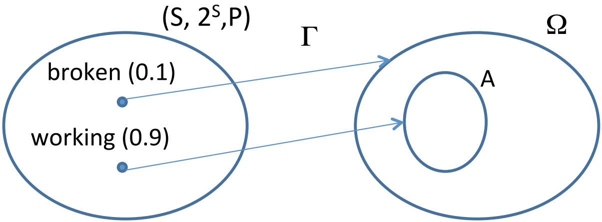
$$\sum_{A \subseteq \Omega} m(A) = 1$$

- Every A of Ω such that $m(A) > 0$ is a focal set of m
- m is said to be normalized if $m(\emptyset) = 0$. This property will be assumed hereafter, unless otherwise specified

Example: the broken sensor

- Let X be some physical quantity (e.g., a temperature), talking values in Ω.
- A sensor returns a set of values $A \subset \Omega$, for instance, $A = [20, 22]$.
- However, the sensor may be broken, in which case the value it returns is completely arbitrary.
- There is a probability $p = 0.1$ that the sensor is broken.
- What can we say about X? How to represent the available information (evidence)?

Analysis



- Here, the probability p is not about X, but about the state of a sensor.
- Let $S = \{\text{working}, \text{broken}\}$ the set of possible sensor states.
  - If the state is "working", we know that $X \in A$.
  - If the state is "broken", we just know that $X \in \Omega$, and nothing more.
- This uncertain evidence can be represented by a mass function m on Ω, such that

$$m(A) = 0.9, \quad m(\Omega) = 0.1$$

Source

- A mass function m on Ω may be viewed as arising from
  - A set $S = \{s_1, \ldots, s_r\}$ of states (interpretations)
  - A probability measure P on S
  - A multi-valued mapping $\Gamma : S \rightarrow 2^{\Omega}$
- The four-tuple $(S, 2^S, P, \Gamma)$ is called a source for m
- Meaning: under interpretation s_i, the evidence tells us that $X \in \Gamma(s_i)$, and nothing more. The probability $P(\{s_i\})$ is transferred to $A_i = \Gamma(s_i)$
- $m(A)$ is the probability of knowing that $X \in A$, and nothing more, given the available evidence

Special cases

- If the evidence tells us that $X \in A$ for sure and nothing more, for some $A \subseteq \Omega$, then we have a logical mass function m_A such that $m_A(A) = 1$
  - m_A is equivalent to A
  - Special case: $m_?$, the vacuous mass function, represents total ignorance
- If each interpretation s_i of the evidence points to a single value of X, then all focal sets are singletons and m is said to be Bayesian. It is equivalent to a probability distribution
- A Dempster-Shafer mass function can thus be seen as
  - a generalized set
  - a generalized probability distribution
- Total ignorance is represented by the vacuous mass function $m_?$ such that $m_?(\Omega) = 1$

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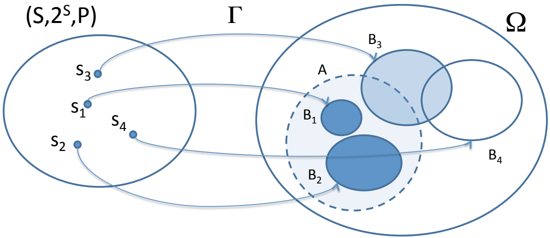
Degrees of support and consistency

- Let m be a normalized mass function on Ω induced by a source $(S, 2^S, P, \Gamma)$.
- Let A be a subset of Ω.
- One may ask:
  1. To what extent does the evidence support the proposition $\omega \in A$?
  2. To what extent is the evidence consistent with this proposition?

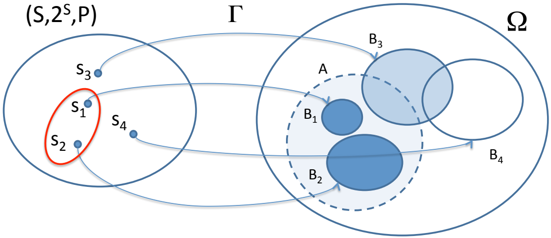


Belief function

Definition and interpretation

- For any $A \subseteq \Omega$, the probability that the evidence implies (supports) the proposition $X \in A$ is

$$Bel(A) = P(\{s \in S | \Gamma(s) \subseteq A\}) = \sum_{B \subseteq A} m(B).$$

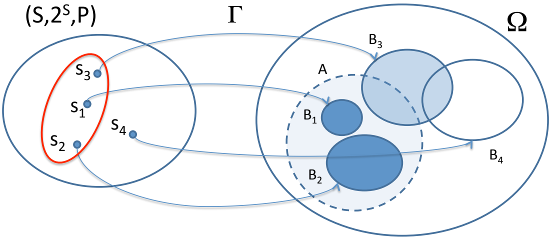


- The function $Bel : A \rightarrow Bel(A)$ is called a belief function.

Plausibility function

- The probability that the evidence is consistent with (does not contradict) the proposition $X \in A$

$$Pl(A) = P(\{s \in S | \Gamma(s) \cap A \neq \emptyset\}) = \sum_{B \cap A \neq \emptyset} m(B) = 1 - Bel(\overline{A})$$



- The function $Pl : A \rightarrow Pl(A)$ is called a plausibility function.
- The function $pl : \omega \rightarrow Pl(\{\omega\})$ is called a contour function.

Two-dimensional representation

- The uncertainty about a proposition A is represented by two numbers: $Bel(A)$ and $Pl(A)$, with $Bel(A) \leq Pl(A)$
- The intervals $[Bel(A), Pl(A)]$ have maximum length when $m = m_?$ is vacuous: then, $Bel(A) = 0$ for all $A \neq \Omega$, and $Pl(A) = 1$ for all $A \neq \emptyset$.
- The intervals $[Bel(A), Pl(A)]$ have minimum length when m is Bayesian. Then, $Bel(A) = Pl(A)$ for all A, and Bel is a probability measure.

Broken sensor example

- From

$$m(A) = 0.9, \quad m(\Omega) = 0.1$$

  we get

$$Bel(A) = m(A) = 0.9, \quad Pl(A) = m(A) + m(\Omega) = 1$$

$$Bel(\overline{A}) = 0, \quad Pl(\overline{A}) = m(\Omega) = 0.1$$

$$Bel(\Omega) = Pl(\Omega) = 1$$

- We observe that

$$Bel(A \cup \overline{A}) \geq Bel(A) + Bel(\overline{A})$$

$$Pl(A \cup \overline{A}) \leq Pl(A) + Pl(\overline{A})$$

- *Bel* and *Pl* are non additive measures.

Characterization of belief functions

- Function $Bel : 2^\Omega \rightarrow [0,1]$ is a completely monotone capacity: it verifies $Bel(\emptyset) = 0$, $Bel(\Omega) = 1$ and

$$Bel\left(\bigcup_{i=1}^{k} A_i\right) \geq \sum_{\emptyset \neq I \subseteq \{1,\ldots,k\}} (-1)^{|I|+1} Bel\left(\bigcap_{i \in I} A_i\right).$$

for any $k \geq 2$ and for any family $A_1, \ldots, A_k$ in 2^Ω.

- Conversely, to any completely monotone capacity Bel corresponds a unique mass function m such that:

$$m(A) = \sum_{\emptyset \neq B \subseteq A} (-1)^{|A|-|B|} Bel(B), \quad \forall A \subseteq \Omega.$$

Relations between *m*, *Bel* et *Pl*

- Let m be a mass function, Bel and Pl the corresponding belief and plausibility functions
- For all $A \subseteq \Omega$,

$$Bel(A) = 1 - Pl(\overline{A})$$

$$m(A) = \sum_{\emptyset \neq B \subseteq A} (-1)^{|A|-|B|} Bel(B)$$

$$m(A) = \sum_{B \subseteq A} (-1)^{|A|-|B|+1} Pl(\overline{B})$$

- m, Bel et Pl are thus three equivalent representations of
  - a piece of evidence or, equivalently
  - a state of belief induced by this evidence

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Consonant belief function

- When the focal sets of m are nested: $A_1 \subset A_2 \subset \ldots \subset A_r$, m is said to be consonant
- The following relations then hold, for all $A, B \subseteq \Omega$,

$$Pl(A \cup B) = \max\left(Pl(A), Pl(B)\right)$$

$$Bel(A \cap B) = \min\left(Bel(A), Bel(B)\right)$$

- *Pl* is this a possibility measure, and *Bel* is the dual necessity measure

Contour function

- The contour function of a belief function *Bel* is defined by

$$pl(\omega) = Pl(\{\omega\}), \quad \forall \omega \in \Omega$$

- When *Bel* is consonant, it can be recovered from its contour function,

$$Pl(A) = \max_{\omega \in A} pl(\omega).$$

- The contour function is then a possibility distribution
- The theory of belief function can thus be considered as more expressive than possibility theory

From the contour function to the mass function

- Let pl be a contour on the frame $\Omega = \{\omega_1, \ldots, \omega_n\}$, with elements arranged by decreasing order of plausibility, i.e.,

$$1 = pl(\omega_1) \geq pl(\omega_2) \geq \ldots \geq pl(\omega_n),$$

  and let A_i denote the set $\{\omega_1, \ldots, \omega_i\}$, for $1 \leq i \leq n$.

- Then, the corresponding mass function m is

$$m(A_i) = pl(\omega_i) - pl(\omega_{i+1}), \quad 1 \leq i \leq n-1,$$
$$m(\Omega) = pl(\omega_n).$$

Example

- Consider, for instance, the following contour distribution defined on the frame $\Omega = \{a, b, c, d\}$:

ω	a	b	c	d
$pl(\omega)$	0.3	0.5	1	0.7

- The corresponding mass function is

$$\begin{aligned} m(\{c\}) &= 1 - 0.7 = 0.3 \\ m(\{c, d\}) &= 0.7 - 0.5 = 0.2 \\ m(\{c, d, b\}) &= 0.5 - 0.3 = 0.2 \\ m(\{c, d, b, a\}) &= 0.3. \end{aligned}$$

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Credal set

- A probability measure P on Ω is said to be compatible with *Bel* if

$$Bel(A) \leq P(A)$$

  for all $A \subseteq \Omega$

- Equivalently, $P(A) \leq Pl(A)$ for all $A \subseteq \Omega$
- The set $\mathcal{P}(Bel)$ of probability measures compatible with *Bel* is called the credal set of *Bel*

$$\mathcal{P}(Bel) = \{P : \forall A \subseteq \Omega, Bel(A) \leq P(A)\}$$

Construction of $\mathcal{P}(Bel)$

- An arbitrary element of $\mathcal{P}(Bel)$ can be obtained by distributing each mass $m(A)$ among the elements of A.
- More precisely, let $\alpha(\omega, A)$ be the fraction of $m(A)$ allocated to the element ω. (Function α is called an allocation of probability.) We have

$$\sum_{\omega \in A} \alpha(\omega, A) = m(A).$$

- By summing up the numbers $\alpha(\omega, A)$ for each ω, we get a probability mass function on Ω,

$$p_\alpha(\omega) = \sum_{A \ni \omega} \alpha(\omega, A).$$

- It can be verified that

$$P_\alpha(A) = \sum_{\omega \in A} p_\alpha(\omega) \geq Bel(A),$$

for all $A \subseteq \Omega$.

Belief functions are coherent lower probabilities

- It can be shown (Dempster, 1967) that any element of the credal set $\mathcal{P}(Bel)$ can be obtained in that way.
- Furthermore, the bounds in the inequalities $Bel(A) \leq P(A)$ and $P(A) \leq Pl(A)$ are attained. We thus have, for all $A \subseteq \Omega$,

$$Bel(A) = \min_{P \in \mathcal{P}(Bel)} P(A)$$

$$Pl(A) = \max_{P \in \mathcal{P}(Bel)} P(A)$$

- We say that Bel is a coherent lower probability.
- Not all lower envelopes of sets of probability measures are belief functions!

A counterexample

- Suppose a fair coin is tossed twice, in such a way that the outcome of the second toss may depend on the outcome of the first toss.
- The outcome of the experiment can be denoted by $\Omega = \{(H,H),(H,T),(T,H),(T,T)\}$.
- Let $H_1 = \{(H,H),(H,T)\}$ and $H_2 = \{(H,H),(T,H)\}$ the events that we get Heads in the first and second toss, respectively.
- Let $\mathcal{P}$ be the set of probability measures on Ω which assign $P(H_1) = P(H_2) = 1/2$ and have an arbitrary degree of dependence between tosses.
- Let P_* be the lower envelope of $\mathcal{P}$.

A counterexample – continued

- It is clear that $P_*(H_1) = 1/2$, $P_*(H_2) = 1/2$ and $P_*(H_1 \cap H_2) = 0$ (as the occurrence Heads in the first toss may never lead to getting Heads in the second toss).
- Now, in the case of complete positive dependence, $P(H_1 \cup H_2) = P(H_1) = 1/2$, hence $P_*(H_1 \cup H_2) \leq 1/2$.
- We thus have

$$P_*(H_1 \cup H_2) < P_*(H_1) + P_*(H_2) - P_*(H_1 \cap H_2),$$

which violates the complete monotonicity condition for $k = 2$.

Two different theories

- Mathematically, the notion of coherent lower probability is thus more general than that of belief function.
- However, the definition of the credal set associated with a belief function is purely formal, as these probabilities have no particular interpretation in our framework.
- The theory of belief functions is not a theory of imprecise probabilities.

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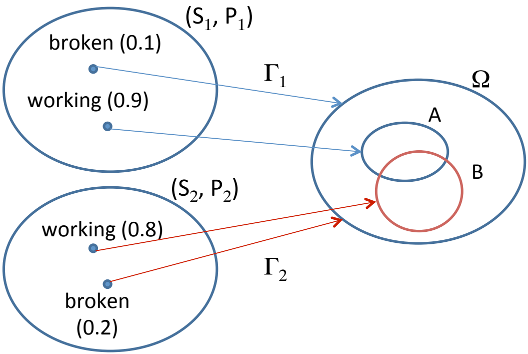
Broken sensor example continued

- The first item of evidence gave us: $m_1(A) = 0.9$, $m_1(\Omega) = 0.1$.
- Another sensor returns another set of values B, and it is in working condition with probability 0.8.
- This second piece if evidence can be represented by the mass function: $m_2(B) = 0.8$, $m_2(\Omega) = 0.2$
- How to combine these two pieces of evidence?

Analysis



- If interpretations $s_1 \in S_1$ and $s_2 \in S_2$ both hold, then $X \in \Gamma_1(s_1) \cap \Gamma_2(s_2)$
- If the two pieces of evidence are independent, then the probability that s_1 and s_2 both hold is $P_1(\{s_1\})P_2(\{s_2\})$

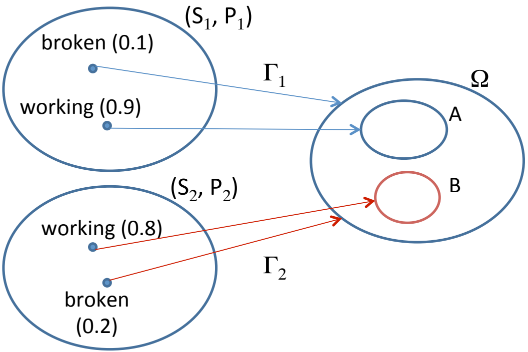
Computation

	S_2 working (0.8)	S_2 broken (0.2)
S_1 working (0.9)	$A \cap B$, 0.72	A, 0.18
S_1 broken (0.1)	B, 0.08	Ω, 0.02

We then get the following combined mass function,

$$
\begin{aligned}
m(A \cap B) &= 0.72 \\
m(A) &= 0.18 \\
m(B) &= 0.08 \\
m(\Omega) &= 0.02
\end{aligned}
$$

Case of conflicting pieces of evidence



- If $\Gamma_1(s_1) \cap \Gamma_2(s_2) = \emptyset$, we know that s_1 and s_2 cannot hold simultaneously
- The joint probability distribution on $S_1 \times S_2$ must be conditioned to eliminate such pairs

Computation

	S_2 working (0.8)	S_2 broken (0.2)
S_1 working (0.9)	$\emptyset$, 0.72	A, 0.18
S_1 broken (0.1)	B, 0.08	Ω, 0.02

We then get the following combined mass function,

$$m(\emptyset) = 0$$
$$m(A) = 0.18/0.28 \approx 0.64$$
$$m(B) = 0.08/0.28 \approx 0.29$$
$$m(\Omega) = 0.02/0.28 \approx 0.07$$

Dempster's rule

- Let m_1 and m_2 be two mass functions and

$$\kappa = \sum_{B \cap C = \emptyset} m_1(B) m_2(C)$$

their degree of conflict

- If $\kappa < 1$, then m_1 and m_2 can be combined as

$$(m_1 \oplus m_2)(A) = \frac{1}{1-\kappa} \sum_{B \cap C = A} m_1(B) m_2(C), \quad \forall A \neq \emptyset$$

and $(m_1 \oplus m_2)(\emptyset) = 0$

Another example

A	$\emptyset$	$\{a\}$	$\{b\}$	$\{a,b\}$	$\{c\}$	$\{a,c\}$	$\{b,c\}$	$\{a,b,c\}$
$m_1(A)$	0	0	0.5	0.2	0	0.3	0	0
$m_2(A)$	0	0.1	0	0.4	0.5	0	0	0

		m_2		
		$\{a\}, 0.1$	$\{a,b\}, 0.4$	$\{c\}, 0.5$
m_1	$\{b\}, 0.5$	$\emptyset, 0.05$	$\{b\}, 0.2$	$\emptyset, 0.25$
	$\{a,b\}, 0.2$	$\{a\}, 0.02$	$\{a,b\}, 0.08$	$\emptyset, 0.1$
	$\{a,c\}, 0.3$	$\{a\}, 0.03$	$\{a\}, 0.12$	$\{c\}, 0.15$

The degree of conflict is $\kappa = 0.05 + 0.25 + 0.1 = 0.4$. The combined mass function is

$$(m_1 \oplus m_2)(\{a\}) = (0.02 + 0.03 + 0.12)/0.6 = 0.17/0.6$$
$$(m_1 \oplus m_2)(\{b\}) = 0.2/0.6$$
$$(m_1 \oplus m_2)(\{a,b\}) = 0.08/0.6$$
$$(m_1 \oplus m_2)(\{c\}) = 0.15/0.6.$$

Dempster's rule

Properties

- Commutativity, associativity. Neutral element: $m_?$
- Generalization of intersection: if m_A and m_B are logical mass functions and $A \cap B \neq \emptyset$, then
$$m_A \oplus m_B = m_{A \cap B}$$
- If either m_1 or m_2 is Bayesian, then so is $m_1 \oplus m_2$ (as the intersection of a singleton with another subset is either a singleton, or the empty set).

Dempster's conditioning

- Conditioning is a special case, where a mass function m is combined with a logical mass function m_A. Notation:

$$m \oplus m_A = m(\cdot|A)$$

- It can be shown that

$$Pl(B|A) = \frac{Pl(A \cap B)}{Pl(A)}.$$

- Generalization of Bayes' conditioning: if m is a Bayesian mass function and m_A is a logical mass function, then $m \oplus m_A$ is a Bayesian mass function corresponding to the conditioning of m by A

Commonality function

- Commonality function: let $Q : 2^{\Omega} \to [0,1]$ be defined as

$$Q(A) = \sum_{B \supseteq A} m(B), \quad \forall A \subseteq \Omega$$

- Conversely,

$$m(A) = \sum_{B \supseteq A} (-1)^{|B \setminus A|} Q(B)$$

- Q is another equivalent representation of a belief function.

Commonality function and Dempster's rule

- Let Q_1 and Q_2 be the commonality functions associated to m_1 and m_2.
- Let $Q_1 \oplus Q_2$ be the commonality function associated to $m_1 \oplus m_2$.
- We have

$$(Q_1 \oplus Q_2)(A) = \frac{1}{1-\kappa} Q_1(A) \cdot Q_2(A), \quad \forall A \subseteq \Omega, A \neq \emptyset$$

$$(Q_1 \oplus Q_2)(\emptyset) = 1$$

- In particular, $pl(\omega) = Q(\{\omega\})$. Consequently,

$$pl_1 \oplus pl_2 \propto (1-\kappa)^{-1} pl_1 pl_2.$$

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Disjunctive rule

Definition and justification

- Let (S_1, P_1, Γ_1) and (S_2, P_2, Γ_2) be sources associated to two pieces of evidence
- If interpretation $s_k \in S_k$ holds and piece of evidence k is reliable, then we can conclude that $X \in \Gamma_k(s_k)$
- If interpretation $s \in S_1$ and $s_2 \in S_2$ both hold and we assume that at least one of the two pieces of evidence is reliable, then we can conclude that $X \in \Gamma_1(s_1) \cup \Gamma_2(s_2)$
- This leads to the TBM disjunctive rule:

$$(m_1 \cup m_2)(A) = \sum_{B \cup C = A} m_1(B) m_2(C), \quad \forall A \subseteq \Omega$$

Disjunctive rule

Example

A	$\emptyset$	$\{a\}$	$\{b\}$	$\{a,b\}$	$\{c\}$	$\{a,c\}$	$\{b,c\}$	$\{a,b,c\}$
$m_1(A)$	0	0	0.5	0.2	0	0.3	0	0
$m_2(A)$	0	0.1	0	0.4	0.5	0	0	0

		m_2		
		$\{a\}, 0.1$	$\{a,b\}, 0.4$	$\{c\}, 0.5$
	$\{b\}, 0.5$	$\{a,b\}, 0.05$	$\{a,b\}, 0.2$	$\{b,c\}, 0.25$
m_1	$\{a,b\}, 0.2$	$\{a,b\}, 0.02$	$\{a,b\}, 0.08$	$\{a,b,c\}, 0.1$
	$\{a,c\}, 0.3$	$\{a,c\}, 0.03$	$\{a,b,c\}, 0.12$	$\{a,c\}, 0.15$

The resulting mass function is

$$(m_1 \cup m_2)(\{a,b\}) = 0.05 + 0.2 + 0.02 + 0.08 = 0.35$$
$$(m_1 \cup m_2)(\{b,c\}) = 0.25$$
$$(m_1 \cup m_2)(\{a,c\}) = 0.03 + 0.15 = 0.18$$
$$(m_1 \cup m_2)(\Omega) = 0.1 + 0.12 = 0.22.$$

Disjunctive rule

Properties

- Commutativity, associativity.
- No neutral element.
- $m_?$ is an absorbing element.
- Expression using belief functions:

$$Bel_1 \cup Bel_2 = Bel_1 \cdot Bel_2$$

Definition

- In general, the disjunctive rule may be preferred in case of heavy conflict between the different pieces of evidence.
- An alternative rule, which is somehow intermediate between the disjunctive and conjunctive rules, has been proposed by Dubois and Prade (1988). It is defined as follows:

$$(m_1 \uplus m_2)(A) = \sum_{B\cap C=A} m_1(B)m_2(C) + \sum_{\{B\cap C=\emptyset, B\cup C=A\}} m_1(B)m_2(C),$$

for all $A \subseteq \Omega$, $A \neq \emptyset$, and $(m_1 \uplus m_2)(\emptyset) = 0$.

Example

A	$\emptyset$	$\{a\}$	$\{b\}$	$\{a,b\}$	$\{c\}$	$\{a,c\}$	$\{b,c\}$	$\{a,b,c\}$
$m_1(A)$	0	0	0.5	0.2	0	0.3	0	0
$m_2(A)$	0	0.1	0	0.4	0.5	0	0	0

		m_2		
		$\{a\}, 0.1$	$\{a,b\}, 0.4$	$\{c\}, 0.5$
m_1	$\{b\}, 0.5$	$\{a,b\}, 0.05$	$\{b\}, 0.2$	$\{b,c\}, 0.25$
	$\{a,b\}, 0.2$	$\{a\}, 0.02$	$\{a,b\}, 0.08$	$\{a,b,c\}, 0.1$
	$\{a,c\}, 0.3$	$\{a\}, 0.03$	$\{a\}, 0.12$	$\{c\}, 0.15$

$$(m_1 \uplus m_2)(\{a,b\}) = 0.05 + 0.08 = 0.13$$
$$(m_1 \uplus m_2)(\{b\}) = 0.2$$
$$(m_1 \uplus m_2)(\{b,c\}) = 0.25$$
$$(m_1 \uplus m_2)(\{a\}) = 0.02 + 0.03 + 0.12 = 0.17$$
$$(m_1 \uplus m_2)(\{c\}) = 0.15$$
$$(m_1 \uplus m_2)(\Omega) = 0.1.$$

Properties

- The DP rule boils down to the conjunctive and disjunctive rules when, respectively, the degree of conflict is equal to zero and one.
- In other cases, it has some intermediate behavior.
- It is not associative. If several pieces of evidence are available, they should be combined at once using an obvious n-ary extension of the above formula.

Outline

1. Representation of evidence
   - Mass functions
   - Belief and plausibility functions

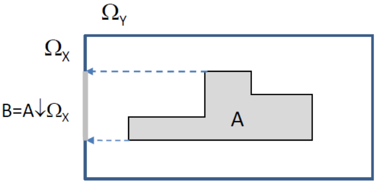
2. Relations with alternative theories
   - Possibility theory
   - Imprecise probabilities

3. Combination of evidence
   - Dempster's rule
   - Some other rules
   - Marginalization, extension

Multidimensional belief functions

- Let X and Y be two variables defined on frames Ω_X and Ω_Y
- Let $\Omega_{XY} = \Omega_X \times \Omega_Y$ be the product frame
- A mass function m_{XY} on Ω_{XY} can be seen as an generalized relation between variables X and Y
- Two basic operations on product frames
  1. Express a joint mass function m_{XY} in the coarser frame Ω_X or Ω_Y (marginalization)
  2. Express a marginal mass function m_X on Ω_X in the finer frame Ω_{XY} (vacuous extension)

Marginalization



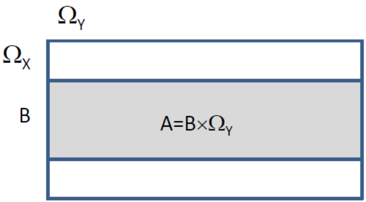
- Problem: express m_{XY} in Ω_X
- Solution: transfer each mass $m_{XY}(A)$ to the projection of A on Ω_X

- Marginal mass function

$$m_{XY\downarrow X}(B) = \sum_{\{A\subseteq\Omega_{XY}, A\downarrow\Omega_X = B\}} m_{XY}(A) \quad \forall B \subseteq \Omega_X$$

- Generalizes both set projection and probabilistic marginalization

Vacuous extension



- Problem: express m_X in Ω_{XY}
- Solution: transfer each mass $m_X(B)$ to the cylindrical extension of B: $B \times \Omega_Y$

- Vacuous extension:

$$m_{X\uparrow XY}(A) = \begin{cases} m_X(B) & \text{if } A = B \times \Omega_Y \\ 0 & \text{otherwise} \end{cases}$$

Application to approximate reasoning

- Assume that we have:
  - Partial knowledge of X formalized as a mass function m_X
  - A joint mass function m_{XY} representing an uncertain relation between X and Y
- What can we say about Y?
- Solution:

$$m_Y = \left(m_{X\uparrow XY} \oplus m_{XY}\right)_{\downarrow Y}$$

- Simpler notation:

$$m_Y = \left(m_X \oplus m_{XY}\right)_{\downarrow Y}$$

- Infeasible with many variables and large frames of discernment, but efficient algorithms exist to carry out the operations in frames of minimal dimensions