Complements on belief functions

Thierry Denœux

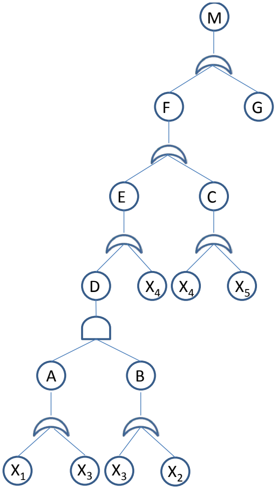
Université de Technologie de Compiègne
HEUDIASYC (UMR CNRS 6599)
http://www.hds.utc.fr/˜tdenoeux

Chiang Mai University
August 2016

Outline

Belief functions on product spaces

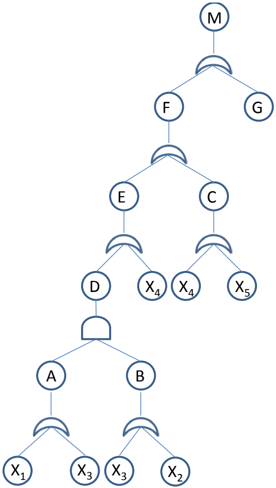
Motivation

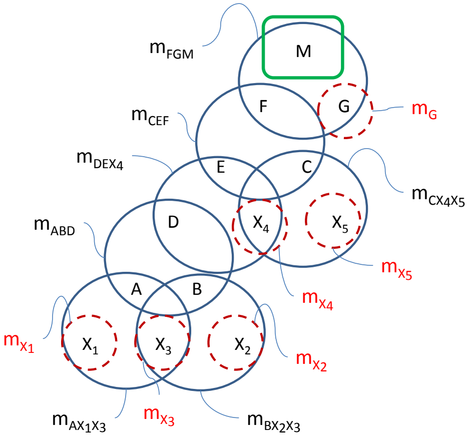


- In many applications, we need to express uncertain information about several variables taking values in different domains
- Example: fault tree (logical relations between Boolean variables and probabilistic or evidential information about elementary events)

Fault tree example

(Dempster & Kong, 1988)





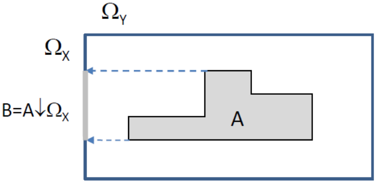
Hypergraph

Multidimensional belief functions

Marginalization, vacuous extension

- Let X and Y be two variables defined on frames Ω_X and Ω_Y
- Let $\Omega_{XY} = \Omega_X \times \Omega_Y$ be the product frame
- A mass function m_{XY} on Ω_{XY} can be seen as an uncertain relation between variables X and Y
- Two basic operations on product frames
    1. Express a joint mass function m_{XY} in the coarser frame Ω_X or Ω_Y (marginalization)
    2. Express a marginal mass function m_X on Ω_X in the finer frame Ω_{XY} (vacuous extension)

Marginalization



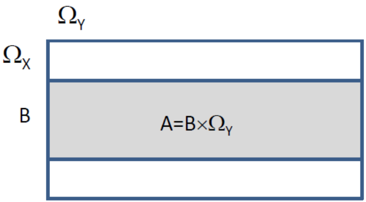
- Problem: express m_{XY} in Ω_X
- Solution: transfer each mass $m_{XY}(A)$ to the projection of A on Ω_X

- Marginal mass function

$$m_{XY\downarrow X}(B) = \sum_{\{A\subseteq\Omega_{XY}, A\downarrow\Omega_X = B\}} m_{XY}(A) \quad \forall B \subseteq \Omega_X$$

- Generalizes both set projection and probabilistic marginalization

Vacuous extension



- Problem: express m_X in Ω_{XY}
- Solution: transfer each mass $m_X(B)$ to the cylindrical extension of B: $B \times \Omega_Y$

- Vacuous extension:

$$m_{X\uparrow XY}(A) = \begin{cases} m_X(B) & \text{if } A = B \times \Omega_Y \\ 0 & \text{otherwise} \end{cases}$$

Operations in product frames

Application to approximate reasoning

- Assume that we have:
  - Partial knowledge of X formalized as a mass function m_X
  - A joint mass function m_{XY} representing an uncertain relation between X and Y
- What can we say about Y?
- Solution:

$$m_Y = \left(m_{X \uparrow XY} \oplus m_{XY}\right)_{\downarrow Y}$$

- Infeasible with many variables and large frames of discernment, but efficient algorithms exist to carry out the operations in frames of minimal dimensions

Outline

Outline

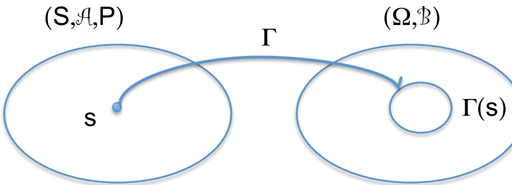
Belief function: general definition

- Let Ω be a set (finite or not) and $\mathcal{B}$ be an algebra of subsets of Ω
- A belief function (BF) on $\mathcal{B}$ is a mapping $Bel : \mathcal{B} \rightarrow [0, 1]$ verifying $Bel(\emptyset) = 0$, $Bel(\Omega) = 1$ and the complete monotonicity property: for any $k \geq 2$ and any collection $B_1, \ldots, B_k$ of elements of $\mathcal{B}$,

$$Bel\left(\bigcup_{i=1}^{k} B_i\right) \geq \sum_{\emptyset \neq I \subseteq \{1,\ldots,k\}} (-1)^{|I|+1} Bel\left(\bigcap_{i \in I} B_i\right)$$

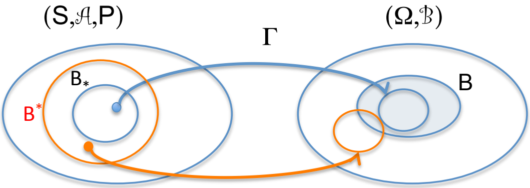
- A function $Pl : \mathcal{B} \rightarrow [0, 1]$ is a plausibility function iff $Bel : B \rightarrow 1 - Pl(\overline{B})$ is a belief function

Source



- Let S be a state space, $\mathcal{A}$ an algebra of subsets of S, $\mathbb{P}$ a finitely additive probability on $(S, \mathcal{A})$
- Let Ω be a set and $\mathcal{B}$ an algebra of subsets of Ω
- Γ a multivalued mapping from S to 2^{Ω}
- The four-tuple $(S, \mathcal{A}, \mathbb{P}, \Gamma)$ is called a source
- Under some conditions, it induces a belief function on $(\Omega, \mathcal{B})$

Strong measurability



- Lower and upper inverses: for all $B \in \mathcal{B}$,

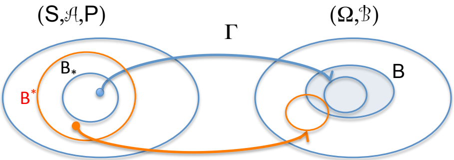
$$\Gamma_*(B) = B_* = \{s \in S | \Gamma(s) \neq \emptyset, \Gamma(s) \subseteq B\}$$

$$\Gamma^*(B) = B^* = \{s \in S | \Gamma(s) \cap B \neq \emptyset\}$$

- Γ is strongly measurable wrt $\mathcal{A}$ and $\mathcal{B}$ if, for all $B \in \mathcal{B}$, $B^* \in \mathcal{A}$
- $(\forall B \in \mathcal{B}, B^* \in \mathcal{A}) \Leftrightarrow (\forall B \in \mathcal{B}, B_* \in \mathcal{A})$
- A strongly measurable multi-valued mapping Γ is called a random set

Belief function induced by a source

Lower and upper probabilities

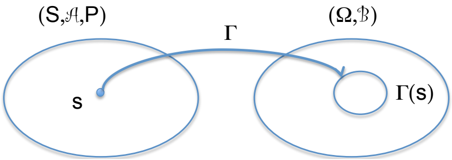


- Lower and upper probabilities:

$$\forall B \in \mathcal{B}, \quad \mathbb{P}_*(B) = \frac{\mathbb{P}(B_*)}{\mathbb{P}(\Omega^*)}, \quad \mathbb{P}^*(B) = \frac{\mathbb{P}(B^*)}{\mathbb{P}(\Omega^*)} = 1 - Bel(\overline{B})$$

- $\mathbb{P}_*$ is a BF, and $\mathbb{P}^*$ is the dual plausibility function
- Conversely, for any belief function, there is a source that induces it (Shafer's thesis, 1973)

Interpretation

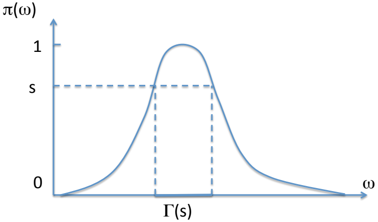


- Typically, Ω is the domain of an unknown quantity ω, and S is a set of interpretations of a given piece of evidence about ω
- If $s \in S$ holds, then the evidence tells us that $\omega \in \Gamma(s)$, and nothing more
- Then
  - $Bel(B)$ is the probability that the evidence supports B
  - $Pl(B)$ is the probability that the evidence is consistent with B

Outline

Consonant belief function

Source



- Let π be a mapping from $\Omega = \mathbb{R}^p$ to $S = [0,1]$ s.t. $\sup \pi = 1$
- Let Γ be the multi-valued mapping from S to 2^Ω defined by

$$\forall s \in [0,1], \quad \Gamma(s) = \{\omega \in \Omega | \pi(\omega) \geq s\}$$

- Let $\mathcal{B}([0,1])$ be the Borel σ-field on $[0,1]$, and P the uniform probability measure on $[0,1]$
- We consider the source $([0,1], \mathcal{B}([0,1]), P, \Gamma)$

Consonant belief function

Properties

- Let *Bel* and *Pl* be the belief and plausibility functions induced by $([0,1], \mathcal{B}([0,1]), P, \Gamma)$
- The focal sets $\Gamma(s)$ are nested, i.e., for any s and s',

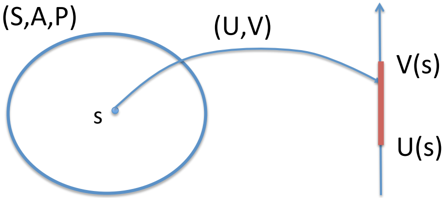
$$s \geq s' \Rightarrow \Gamma(s) \subseteq \Gamma(s')$$

  The belief function is said to be consonant.
- The corresponding contour function *pl* is equal to π
- The corresponding plausibility function is a possibility measure: for any $B \subseteq \Omega$,

$$Pl(B) = \sup_{\omega \in B} pl(\omega)$$

$$Bel(B) = \inf_{\omega \notin B} (1 - pl(\omega))$$

Random closed interval



- Let (U, V) be a bi-dimensional random vector from a probability space $(S, \mathcal{A}, \mathbb{P})$ to $\mathbb{R}^2$ such that $U \leq V$ a.s.
- Multi-valued mapping:

$$\Gamma : s \rightarrow \Gamma(s) = [U(s), V(s)]$$

- The source $(S, \mathcal{A}, \mathbb{P}, \Gamma)$ is a random closed interval. It defines a BF on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$

Random closed interval

Properties

- Lower/upper cdfs:

$$Bel((-\infty, x]) = \mathbb{P}([U, V] \subseteq (-\infty, x]) = \mathbb{P}(V \leq x) = F_V(x)$$
$$Pl((-\infty, x]) = \mathbb{P}([U, V] \cap (-\infty, x] \neq \emptyset) = \mathbb{P}(U \leq x) = F_U(x)$$

- Lower/upper expectation:

$$\mathbb{E}_*(\Gamma) = \mathbb{E}(U)$$
$$\mathbb{E}^*(\Gamma) = \mathbb{E}(V)$$

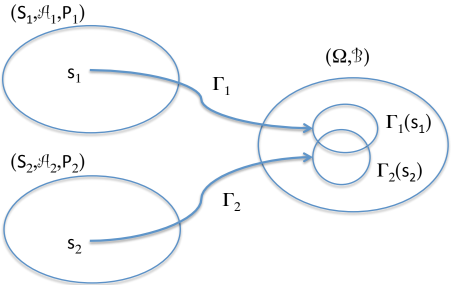
- Lower/upper quantiles

$$q_*(\alpha) = F_U^{-1}(\alpha),$$
$$q^*(\alpha) = F_V^{-1}(\alpha).$$

Outline

Dempster's rule

Definition



- Let $(S_i, \mathcal{A}_i, \mathbb{P}_i, \Gamma_i)$, $i = 1, 2$ be two sources representing independent items of evidence, inducing BF Bel_1 and Bel_2
- The combined BF $Bel = Bel_1 \oplus Bel_2$ is induced by the source $(S_1 \times S_2, \mathcal{A}_1 \otimes \mathcal{A}_2, \mathbb{P}_1 \otimes \mathbb{P}_2, \Gamma_\cap)$ with

$$\Gamma_\cap(s_1, s_2) = \Gamma_1(s_1) \cap \Gamma_2(s_2)$$

Dempster's rule

Definition

- For each $B \in \mathcal{B}$, $Bel(B)$ is the conditional probability that $\Gamma_\cap(s) \subseteq B$, given that $\Gamma_\cap(s) \neq \emptyset$:

$$Bel(B) = \frac{\mathbb{P}\left(\{(s_1, s_2) \in S_1 \times S_2 | \Gamma_\cap(s_1, s_2) \neq \emptyset, \Gamma_\cap(s_1, s_2) \subseteq B\}\right)}{\mathbb{P}(\{(s_1, s_2) \in S_1 \times S_2 | \Gamma_\cap(s_1, s_2) \neq \emptyset\})}$$

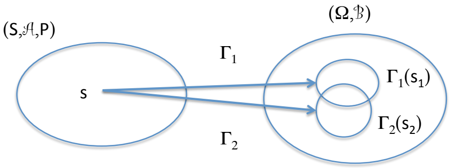
- It is well defined iff the denominator is non null
- As in the finite case, the degree of conflict between the belief functions can be defined as one minus the denominator in the above equation.

Approximate computation

Monte Carlo simulation

Require: Desired number of focal sets N
  $i \leftarrow 0$
  while $i < N$ **do**
    Draw s_1 in S_1 from $\mathbb{P}_1$
    Draw s_2 in S_2 from $\mathbb{P}_2$
    $\Gamma_\cap(s_1, s_2) \leftarrow \Gamma_1(s_1) \cap \Gamma_2(s_2)$
    if $\Gamma_\cap(s_1, s_2) \neq \emptyset$ **then**
      $i \leftarrow i + 1$
      $B_i \leftarrow \Gamma_\cap(s_1, s_2)$
    end if
  end while
  $\widehat{Bel}(B) \leftarrow \frac{1}{N}\#\{i \in \{1, \ldots, N\} | B_i \subseteq B\}$
  $\widehat{Pl}(B) \leftarrow \frac{1}{N}\#\{i \in \{1, \ldots, N\} | B_i \cap B \neq \emptyset\}$

Combination of dependent evidence



- The case of complete dependence between two pieces of evidence can be modeled by two sources formed by different multivalued mappings Γ_1 and Γ_2 from the same probability space.
- The combined BF is induced by the source $(S, \mathcal{A}, \mathbb{P}, \Gamma_\cap)$
- This combination rule preserves consonance: the combination of two consonant BFs is still consonant.
- This is the rule used in Possibility Theory.

Propagation of belief functions

- Assume that a quantity Z is defined as function of two other quantities X and Y
$$Z = \varphi(X, Y)$$
- Given BFs Bel_X and Bel_Y on X and Y, what is the BF Bel_Z on Z?
- Solution:
$$Bel_Z = (Bel_{X\uparrow XYZ} \oplus Bel_{Y\uparrow XYZ} \oplus Bel_\varphi)_{\downarrow Z}$$
- For any $A \subseteq \Omega_X$ and $B \subseteq \Omega_Y$,
$$(A \uparrow \Omega_{XYZ}) \cap (B \uparrow \Omega_{XYZ}) \cap R_\varphi = \varphi(A, B)$$
- Consequently, if Bel_X and Bel_Y are induced by random sets $\Gamma(U)$ and $\Lambda(V)$, where U and V are independent rvs, then Bel_Z is induced by the RS
$$\varphi(\Gamma(U), \Lambda(V))$$

Exercise

- In R, we can represent (an approximation of) a random interval (RI) by a matrix B of size $N \times 2$, where $B[i,]$ is a realization of the random interval.
- Write a function in R that generates a RI representation for the consonant belief function with contour function $\pi : \mathbb{R} \to [0, 1]$ (assumed to be continuous and unimodal)
- Write a function that computes the RI representation of $Z = \varphi(X, Y)$, as a function of φ, and the RI representations of X and Y.
- Run some examples. Draw the lower and upper cdfs of the RIs obtained, and compute their lower and upper expectations.