A new method for state estimation of dynamic system based on Dempster Shafer theory

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***Abstract*— The goal of state estimation method is to compute an accurate estimation of the state of the system based on the measurement given by different sensors and a mathematical representation of the system. In this paper a new state estimation method based on Dampster-Shafer theory and interval analysis is presented. This method uses belief structures composed of a finite number of axis-aligned boxes with associated masses. Such belief structures can model partial information on model and measurement uncertainties, more accurately than the bounded error approach alone. Focal sets are propagated in the system equations using tools from interval arithmetics and constraint satisfaction techniques, thus generalizing pure interval analysis. The results of applying the proposed method on a vehicle localization problem show the usefulness of the proposed method.**

I. INTRODUCTION

State estimating of dynamic systems is an important task in many applications such as mobile localization or target tracking. The goal of the state estimation methods is to combine the measurement given by different sensors in order to compute an estimate value of the state vector that gives a complete description of the system. If several sensors or information sources are used, then, a data fusion algorithm should be used. Based on the representation of the uncertainty, two categories of data fusion methods can be identified. Methods in the first category are based on a probabilistic description of uncertainty and assume the measurement noise and state perturbations to be realizations of random variables with known statistical properties [2][3]. These methods are affected by systematic measurement errors such as bias and drift, as well as by partial or total conflicts between the sources of information [7].

The second category of methods corresponds to state bounding. It is known as bounded error approach (BEE). In this approach, all variables are assumed to belong to known compact sets and we attempt to built simple sets, such as ellipsoids are boxes, that guaranteed to contain all state vectors consistent with given constraints [9][13][14]. The major implementation problem of this approach is to determine correctly the bounds of the noises. Indeed, if these bounds are underestimated, the contractor may lead to no solution. On the contrary, if the bounds are overestimated, the estimated sets can be very large (the estimates are then very pessimistic) [6].

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The theory of belief functions also known as Dempster Shafer theory (DS) is considered as formal tools suitable for representing the inaccuracy, uncertainty and unavailable knowledge. The use of belief function theory steadily spreads out, mostly because of its ability to model various states of knowledge ranging from complete ignorance to probabilistic uncertainty [5][15][16].

In this paper, we propose to replace the set based representation of uncertainty in the BEE by a more general formalism based on belief functions. Thus, a new data fusion method, based on Dempster shafer theory will be presented. In this method the measurement noises are represented by DS mass functions which assign belief masses to a finite number of focal sets, chosen to be axis-aligned boxes. Such mass functions can be seen as generalized boxes composed of a collection of boxes with associated weights. In order to compute an estimate state of the system, focal sets are propagated and updated respecting to the system equations using tools of intervals theory and DS theory. This approach can be seen as an extension of the pure bounded error approach. It is more robust than exiting bounded error method as it uses a Dempster Shafer mass function with axis aligned boxes as focal set for representing the measurement noises. Also, it retains the essential property of interval analysis to provide guaranteed computations (as the set provided by the bounded error methods is always one of the focal sets computed by the proposed method).

The article is organized as follows. Section II first presents the background on interval analysis and bounded error state estimation. The necessary concepts of Dempstet-Shafer theory are then recalled in Section III. Our approach is introduced in Section IV, and Section V presents an application of our method to dynamic land vehicle localization using GPS, gyrometer and odometer measurements. Finally, we conclude and discuss the main contributions of the paper in Section VI.

II. INTERVAL ANALYSIS

In this section we briefly introduce some notions of interval analysis [9]. A real interval, denoted $[x]$, is defined as a closed and connected subset of $\mathbb{R}$: $[x] = [\underline{x}, \overline{x}] = \{x \in \mathbb{R} / \underline{x} \leq x \leq \overline{x}\}$, where $\underline{x}$ and $\overline{x}$ are the lower and upper bound of $[x]$. Set-theoretic operations such as intersection or union can be applied to intervals. The four classical arithmetic operations can be extended to intervals. For any such binary operator $\diamond \in \{+, -, *, \backslash\}$, the interval $[x] \diamond [y]$ is defined for any intervals $[x]$ and $[y]$ as:

$$[x] \diamond [y] = [\{x \diamond y \in \mathbb{R} | x \in [x], y \in [y]\}] . \quad (1)$$

A box $[\mathbf{x}]$ of $\mathbb{R}^{n_x}$ is defined as a Cartesian product of n_x intervals: $[\mathbf{x}] = [x_1] \times [x_2] \cdots \times [x_{n_x}] = \times_{i=1}^{n_x} [x_i]$. The set of n-dimensional interval real vectors will be denoted $\mathbb{IR}^n$. The notions recalled above may be easily extended to boxes. In general, the image of a box $[\mathbf{x}] \in \mathbb{R}^n$ by a function $\mathbf{f}$ is not a box. An inclusion function $[\mathbf{f}]$ defined as:

$$\forall [\mathbf{x}] \in \mathbb{R}^n, \mathbf{f}([\mathbf{x}]) \subset [\mathbf{f}]([\mathbf{x}]), \tag{2}$$

computes a box containing $\mathbf{f}([\mathbf{x}])$. This function should be calculated such that the box enclosing $\mathbf{f}([\mathbf{x}])$ is optimal. Different algorithms exist in order to reduce the size of boxes enclosing $\mathbf{f}([\mathbf{x}])$. In this paper we use the *Waltz algorithm* [9] which is based on the propagation of primitive constraints[1]. This method is independent of the non linearity of the constraints and give accurate results when the system present a great redundancy of data and equations [8][9][11].

III. Dempster-Shafer theory

In this section, we introduce the main concepts of Dempster-Shafer theory that are used in section IV.

A. Basic definitions

Let Ω denote the domain of some variable, called the frame of discernment. Let $A_1, \ldots, A_p$ be p subsets of Ω. A *belief structure* (BS) or discrete mass function m with focal sets $A_1, \ldots, A_p$ is a function from 2^Ω to $[0,1]$ verifying $m(A) > 0$ if $A \in \{A_1, \ldots, A_p\}$, $m(A) = 0$ otherwise, and: $\sum_{i=1}^{p} m(A_i) = 1$. We note $\mathcal{F}(m) = \{A_1, \ldots, A_p\}$. A BS is said to be normal if $\emptyset \notin \mathcal{F}(m)$. The BS on Ω which have $m(\Omega) = 1$ is the vacuous belief function. If focal sets are nested then the BS is said to be consonant. A categorical BS is a BS with $p = 1$. In the following, all BSs will be assumed to be normal, unless otherwise specified. In most presentations of D-S theory, Ω is assumed to be finite. However, the theory remains basically unchanged if Ω is infinite (even uncountable), as long as the number of focal sets remains finite. If $\Omega = \mathbb{R}$, the focals sets are usually assumed to be intervals [12][18]. In the multidimensional case where $\Omega = \mathbb{R}^n$, this approach can be extended by assuming focal sets to be n-dimensional boxes.

Given a normal BS m with focal sets $A_1, \ldots, A_p$, the corresponding *belief* and *plausibility* functions are defined, respectively, as:

$$bel(A) = \sum_{\{i | A_i \subseteq A\}} m(A_i) \tag{3}$$

$$pl(A) = \sum_{\{i | A_i \cap A \neq \emptyset\}} m(A_i), \tag{4}$$

for all $A \subseteq \Omega$.

Let us now consider two BSs m_1 and m_2 defined on the same frame of discernment Ω, with focal sets $\mathcal{F}(m_1) = \{A_1, \ldots, A_p\}$ and $\mathcal{F}(m_2) = \{B_1, \ldots, B_q\}$. Assuming these BSs to be induced by two independent sources of information, they can be combined using the conjunctive rule of combination [16] defined by

$$(m_1 \circledcirc m_2)(C) = \sum_{\{i,j | A_i \cap B_j = C\}} m_1(A_i) m_2(B_j), \quad \forall C \subseteq \Omega. \tag{5}$$

We observe that the above operation may produce a non normal BS, even the combined BSs are normal. The mass $(m_1 \cap m_2)(\emptyset)$ is called the degree of conflict between m_1 and m_2. If the degree of conflict is not equal to 1, a normal BS may be obtained by setting the mass of the emptyset to 0 and renormalizing. This is the definition of *Dempster's rule* of combination [15], denoted as $\oplus$.

B. Extending Interval Analysis to Mass Functions

In [17], Yager proposed a simple scheme for extending operations on sets to operations on BSs. Using notations and assumptions similar to those of Yager, let Ω_1, Ω_2 and Ω_3 be three, not necessarily different, finite frames of discernment, and let S be a binary set operator from $2^{\Omega_1} \times 2^{\Omega_2}$ to 2^{Ω_3}. Let m_1 and m_2 be two BSs on Ω_1 and Ω_2, respectively. Following the random set interpretation mentioned in Section III-A, for any $A_i \in \mathcal{F}(m_1)$, $m_1(A_i)$ may be seen as the probability of selecting A_i in some random experiment $\mathcal{E}_1$. Similarly for any $B_j \in \mathcal{F}(m_2)$, $m_2(B_j)$ may be seen as the probability of selecting B_j in some random experiment $\mathcal{E}_2$. Assume that we carry out the two random experiments $\mathcal{E}_1$ and $\mathcal{E}_2$, and the combine the result using function S. If $\mathcal{E}_1$ and $\mathcal{E}_2$ are independent, then the probability of selecting A_i and B_j is $m_1(A_i) \cdot m_2(B_j)$, and the corresponding image under S will be $S(A_i, B_j)$. For any $C \subseteq \Omega_3$, the probability of obtaining C as a result of the above process is thus $m_3(C) = \sum_{\{i,j | S(A_i,B_j)=C\}} m_1(A_i) \cdot m_2(B_j)$.

Let us now assume that m_1 and m_2 are two BSs with focal intervals, i.e., $\mathcal{F}(m_1) \subset \mathbb{IR}$ and $\mathcal{F}(m_2) \subset \mathbb{IR}$. Then, using the above approach and the extension of arithmetic operations to intervals defined by (1), the combination of m_1 and m_2 by any arithmetic operation $\diamond$ may be defined as: $m_3([z]) = \sum_{\{[x] \in \mathcal{F}(m_1), [y] \in \mathcal{F}(m_2) | [x] \diamond [y] = [z]\}} m_1([x]) \cdot m_2([y])$, for all $[z] \in \mathbb{IR}$. Similarly, if f is a function from $\mathbb{R}$ to $\mathbb{R}$ and $[f]$ its interval counterpart defined by (2), the image by $[f]$ of a BS m such that $\mathcal{F}(m) \subset \mathbb{IR}$ is the BS $[f](m)$ defined by

$$([f](m))([y]) = \sum_{\{[x] \in \mathcal{F}(m) | [y] = [f]([x])\}} m([x]). \tag{6}$$

C. Expectations

Let us assume in this subsection that $\Omega = \mathbb{R}$, and our state of knowledge regarding some variable x is represented by a BS m. How can the concept of expected value of x be defined in this context? An answer to this question is more easily found by interpreting m as defining a set of probability measures P such that $bel(A) \leq P(A) \leq pl(A)$, for all measurable subset A of $\mathbb{R}$. A probability measure P verifying the above inequalities is said to be compatible with m; the set of such probability measures will be noted $\mathcal{P}(m)$.

[1] A primitive constraint is a constraint involving a single operator (such as $+, -, *$ or $\backslash$) or a single function (such as $\cos$, $\sin$ or $\sinh$).

The lower and upper expectations [5] of x with respect to m may then be defined, respectively, as: $\mathbb{E}_*(m) = \inf_{P \in \mathcal{P}(m)} \mathbb{E}(P)$ and $\mathbb{E}^*(m) = \sup_{P \in \mathcal{P}(m)} \mathbb{E}(P)$.

Let us assume the focal sets of m to be p real intervals $[x_i] = [\underline{x}_i, \overline{x}_i]$ with masses $m_i = m([x_i])$, $i = 1, \ldots, p$. It may be shown [5] that: $\mathbb{E}_*(m) = \sum_{i=1}^{p} m_i \underline{x}_i$ and $\mathbb{E}^*(m) = \sum_{i=1}^{p} m_i \overline{x}_i$. Denoting $[\mathbb{E}](m) = [\mathbb{E}_*(m), \mathbb{E}^*(m)]$, we may write using interval arithmetics:

$$[\mathbb{E}](m) = \sum_{i=1}^{p} m_i \cdot [x_i]. \tag{7}$$

The interval $[\mathbb{E}](m)$ will be referred to as the interval expectation of m. A particular value in this interval may be selected by averaging the center c_i of each focal interval $[x_i]$, which will be denoted:

$$\mathbb{E}(m) = \sum_{i=1}^{p} m_i c_i. \tag{8}$$

The quantity $\mathbb{E}(m)$ happens to be the expectation with respect to the pignistic probability measure associated to m [16], [12]; it will be refereed to as the pignistic expectation of m. Equations (7) and (8) can easily be extended to the multidimensional case where $\Omega = \mathbb{R}^n$. The interval expectation of m is then an n-dimensional box, and its pignistic expectation is a vector of $\mathbb{R}^n$.

IV. Belief state estimation

As mentioned in section I, state estimation methods may be based on probabilistic representation of uncertainty. They assume that the measurement noise and the state perturbation can be represented by random variables with known statistical properties. Other methods are based on a much weaker assumption. They assume the noise processes to be bounded, and consider that nothing is known except the bounds. Under this assumption, a new state estimation method is presented. This method is called Belief state estimation method (BSE). It is based on more general representation of uncertainty based on belief functions. Thus, Measurement noises are represented by DS mass functions which assign a belief masses to finite number of focal sets, chosen to be axis aligned boxes.

Consider a dynamical system represented in state space by the following discrete-time equations:

$$\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{u}_k, \mathbf{v}_k) \tag{9}$$
$$\mathbf{z}_{k+1} = \mathbf{g}(\mathbf{x}_{k+1}, \mathbf{w}_{k+1}), \tag{10}$$

where $\mathbf{f} : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \times \mathbb{R}^{n_v} \longrightarrow \mathbb{R}^{n_x}$ is a possibly nonlinear function that relates the state $\mathbf{x}_{k+1}$ at time $k+1$ to the previous state $\mathbf{x}_k$ at time k, the input $\mathbf{u}_k$ and an independent identically distributed (i.i.d.) bounded process noise sequence $\mathbf{v}_k$. The function $\mathbf{g} : \mathbb{R}^{n_x} \times \mathbb{R}^{n_w} \longrightarrow \mathbb{R}^{n_z}$ is the observation equation. It defines the relation between the measurement (or the observation) $\mathbf{z}_{k+1}$, the state $\mathbf{x}_{k+1}$ and an i.i.d bounded observation noise sequence $\mathbf{w}_{k+1}$ at time step $k+1$. For simplicity, we will assume, as done in [1], the process and

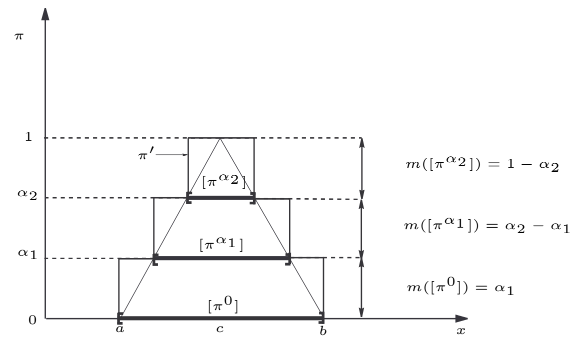

Fig. 1. Approximation of a triangular possibility distribution by a discrete consonant BS.

measurement errors to be additive. This allows us to write a simplified version of the system in the interval setting:

$$[\mathbf{x}_{k+1}] = \mathbf{f}([\mathbf{x}_k], [\mathbf{u}_k]) \tag{11}$$
$$[\mathbf{z}_{k+1}] = \mathbf{g}([\mathbf{x}_{k+1}]), \tag{12}$$

where the noises $\mathbf{v}_k$ and $\mathbf{w}_k$ have been integrated to build the intervals $[\mathbf{u}_k]$ and $[\mathbf{z}_{k+1}]$, respectively. The state estimation problem consists in estimating the state vector $\mathbf{x}_k$ at successive time steps $k = 0, 1, 2, \ldots$ from the sequence of input vectors $\mathbf{u}_k$ and measurement vectors $\mathbf{z}_k$, using system (11)-(12).

In the BSE method, we assume that the initial state $\mathbf{x}_0$ as well as the input vector $\mathbf{u}_k$ and the observation vector $\mathbf{z}_k$ at each time step k to be described by three mass functions with interval focal elements, denoted by $m_0^{\mathbf{x}}$, $m_k^{\mathbf{u}}$ and $m_k^{\mathbf{z}}$. Let p_k^x, p^u and p^z denote respectively, the number of focal elements of $m_k^{\mathbf{x}}$, $m_k^{\mathbf{u}}$ and $m_k^{\mathbf{z}}$. The i-th focal element of $m_k^{\mathbf{x}}$ will be noted $[\mathbf{x}_k^i]$, and similarly for the other mass functions. The goal of the BSE method is to provide at each time step k, a guaranteed estimate of the state $\mathbf{x}_k$, in the form of a mass function $m_k^{\mathbf{x}}$, assuming the system equations (11-12) to be correct and using all available information, $m_{k-1}^{\mathbf{x}}$, $m_{k-1}^{\mathbf{u}}$ and $m_k^{\mathbf{z}}$. We will present first the construction of the mass function in section IV-A, after that the sketch of the BSE method is described in section IV-B.

A. Construction of Belief Structures

As recalled in Section IV, in the BEE methods the noises is assumed to be bounded with known bounds. However, in many cases, additional realistic assumptions about the error distributions can be made: for instance, one can assume some measure of central tendency such as the mode, the mean or the median, to be equal to zero. As will be shown below, the BS formalism allows to us to make a more robust representation of the noises by using such extra assumptions.

Let us consider a continuous real random variable X characterized by a probability distribution P_X with known support $[a, b]$ and mode $c \in (a, b)$. Let π denote the triangular possibility distribution

$$\pi(x) = \begin{cases} \frac{x-a}{c-a} & \text{if } a \leq x < c, \\ \frac{b-x}{b-c} & \text{if } c \leq x < b, \\ 0 & \text{otherwise.} \end{cases} \tag{13}$$

As shown in [4], the following inequalities hold for every measurable set A: $\Pi(A) \leq P_X(A) \leq 1 - \Pi(\overline{A})$, where $\overline{A}$ denotes the complement of A, and Π is the possibility measure associated to π: $\Pi(A) = \sup_{x \in A} \pi(A)$. The possibility distribution π can thus be seen as an approximation of the set of probability measures with support $[a, b]$ and mode c. Such a continuous possibility measure can be itself approximated by a discrete BS m with p focal intervals defined as follows.

For any $\alpha \in (0, 1]$, the α-cut of π is the set of values x such that $\pi(x) \geq \alpha$. It is the interval $[\pi^\alpha]$ with bounds:

$$\underline{\pi}^\alpha = a + \alpha(c - a) \tag{14}$$

$$\overline{\pi}^\alpha = c + (1 - \alpha)(b - c). \tag{15}$$

Let $[\pi^0] = [a, b]$. Let us consider $p-1$ distinct values for α: $0 = \alpha_0 < \alpha_1 < \alpha_2 < \ldots < \alpha_{p-1} < 1$, and let m be the consonant BS with focal intervals $[\pi^{\alpha_k}]$, $k = 0, \ldots, p-1$, defined by:

$$\begin{aligned} m([\pi^0]) &= \alpha_1, \\ m([\pi^{\alpha_1}]) &= \alpha_2 - \alpha_1, \\ &\vdots \\ m([\pi^{\alpha_{p-1}}]) &= 1 - \alpha_{p-1}. \end{aligned}$$

The principle of this construction is illustrated in Figure 1. It is clear that m corresponds to a possibility distribution π' that approximates π, and such that $\pi' \geq \pi$. Note here that m can be considered as an approximation of the set of probability measures with known support $[a, b]$ and mode c.

B. Sketch of the BSE

The BSE method consists, at each time step $k+1$, in combining all available information ($m_k^{\mathbf{x}}$, $m_k^{\mathbf{u}}$ and $m_{k+1}^{\mathbf{z}}$) in order to compute an estimation of the state assuming the system equations (11-12) to be correct and using tools of DS and interval analysis theory. First, $m_k^{\mathbf{x}}$ is combined with $m_k^{\mathbf{u}}$ in order to compute a mass function on the state using equations (11) and (6). When measurement vector is available, the mass function of the state are updated using DS tools and $m_{k+1}^{\mathbf{z}}$, according to the system equations (11-12). The result of this process is $m_{k+1}^{\mathbf{x}}$. In order to compute a state using $m_{k+1}^{\mathbf{x}}$, a decision rule of belief function theory should be used. Note that, the numbers of focal elements of $m_{k+1}^{\mathbf{x}}$ can be augmented exponentially during the combination of different mass functions. Thereby, the computational complexity of the BSE method augment. In order to solve this problem, a combination method is used for decrease the number of focal elements of $m_k^{\mathbf{x}}$. In this paper we used the summarization algorithm [10]. In this algorithm the $q-1$ focal sets with highest masses are saved. The other focal sets (focal sets with the smallest masses) are aggregated and the sum of their masses are transferred to their union.

The detailed procedure is described in Algorithm 1. It can be summarized as follows. The algorithm depends on three parameters: the numbers of focal elements $p^{\mathbf{u}}$ and $p^{\mathbf{z}}$ for the input and the output, respectively, and the maximum number q of focal elements for $m_{k+1}^{\mathbf{x}}$ used by the summarization algorithm. In lines 1 and 2, the mass functions on the input at time k and on the measurement at time $k+1$ are constructed using the method described in Section IV-A. After that, for each focal interval $[\mathbf{x}_k^i]$ of $m_k^{\mathbf{x}}$ and each focal interval $[\mathbf{u}_k^j]$ of $m_k^{\mathbf{u}}$, a predicted state $[\mathbf{x}_{k+1}]$ and a predicted measurement $[\widetilde{\mathbf{z}}_{k+1}]$ are computed (lines 7-8). The predicted measurement is then intersected with each focal set $[\mathbf{z}_{k+1}^\ell]$ of $m_{k+1}^{\mathbf{z}}$ to compute an innovation $[\mathbf{I}_{k+1}^r]$, which is then used to compute a new interval state estimate $[\mathbf{x}_{k+1}^r]$ by inverting the observation equation (lines 12-13). This interval is then contracted using the Waltz algorithm (line 14). The belief mass assigned to $[\mathbf{x}_{k+1}^r]$ is the product of the masses assigned to $[\mathbf{x}_k^i]$, $[\mathbf{u}_k^j]$ and $[\mathbf{z}_{k+1}^\ell]$ (line 15). Once the whole BS $m_{k+1}^{\mathbf{x}}$ has been computed, it is normalized and summarized (lines 19-20). Finally, interval and pignistic expectations are computed using (7) and (8) (lines 22-23).

Algorithm 1 Belief State Estimation (BSE) algorithm.

Require: $m_k^{\mathbf{x}}$, $m_k^{\mathbf{u}}$, $m_{k+1}^{\mathbf{z}}$, $p^{\mathbf{u}}$, $p^{\mathbf{z}}$, q
Ensure: $m_{k+1}^{\mathbf{x}}$, $[\widehat{\mathbf{x}}_{k+1}]$, $\widehat{\mathbf{x}}_{k+1}$

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1:  Read the input u_k and its error. Deduce m_k^u with p^u focal elements
2:  Read the output z_{k+1} and its error. Deduce m_{k+1}^z with p^z focal elements
3:  r ← 0
4:  for i = 1 to p_k^x do
5:    for j = 1 to p^u do
6:      % Prediction %
7:      [x_{k+1}] ← [f]([x_k^i], [u_k^j])
8:      [z~_{k+1}] ← [g]([x_{k+1}])
9:      for ℓ = 1 to p^z do
10:       r ← r + 1
11:       % Correction %
12:       [I_{k+1}^r] ← [z~_{k+1}] ∩ [z_{k+1}^ℓ]
13:       [x_{k+1}^r] ← [g^{-1}]([I_{k+1}^r])
14:       [x_{k+1}^r] ← Waltz([x_k^i], [x_{k+1}^r], [u_k^j], [I_{k+1}^r], f, g)
          % Waltz algorithm %
15:       m_{k+1}^x([x_{k+1}^r]) ← m_k^x([x_k^i]) · m_k^u([u_k^j]) · m_{k+1}^z([z_{k+1}^ℓ])
16:     end for
17:   end for
18: end for
19: Normalize m_{k+1}^x
20: Summarize m_{k+1}^x to keep at most q focal elements.
21: % Computation of interval and point estimates %
22: [x^_{k+1}] ← [E](m_{k+1}^x)
23: x^_{k+1} ← E(m_{k+1}^x)
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V. Application

In this section, we apply the belief state estimation (BSE) algorithm introduced above to dynamic localization of a

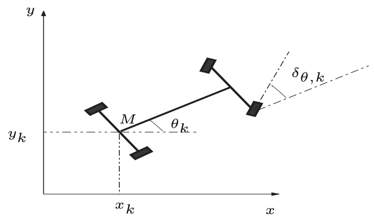

Fig. 2. Vehicle representation.

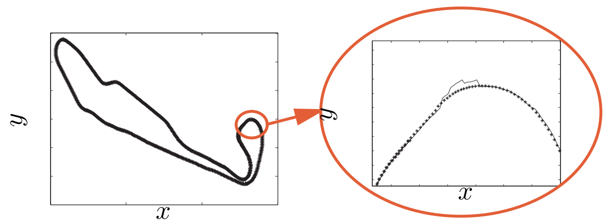

Fig. 3. Test trajectory (left), with a zoom on part of the trajectory (right) showing the GPS positions (solid line) and the estimated positions (*).

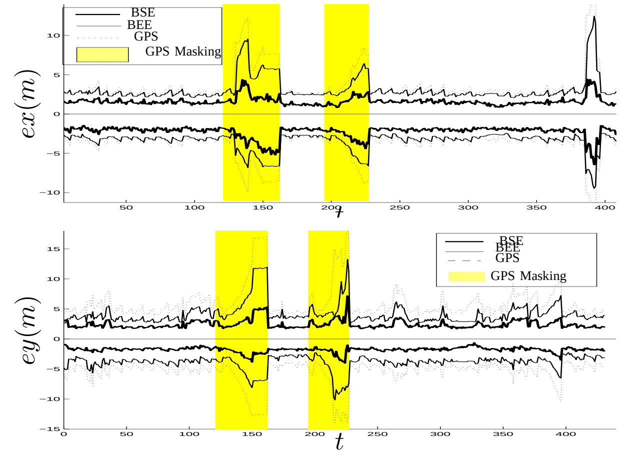

Fig. 4. Interval errors on x and y for GPS (dashed line), BSE (bold line) and BEE (thin line).

land vehicle. The application and the data are identical to those used in [1] and [6]. In this application, three kinds of sensors are used: odometers, gyrometers and Global Positioning System (GPS). A major issue is to exploit the complementarity and redundancy between these sensors in order to achieve higher precision, availability and integrity. For instance, the quality of GPS positioning depends on the configuration of visible satellites, and GPS information can even become unavailable because of masking effect occurring in forests, tunnels, cities, etc. Fusing GPS information with dead reckoning sensor measurements makes it possible to filter the GPS estimates, thus increasing the performances of the localizer. The conventions and notations used in this section are illustrated in Figure 2. The mobile frame origin M is chosen at the middle of the rear axle. Let (x_k, y_k) be the position of the vehicle and θ_k its heading angle at time step k. Let $\mathbf{x}_k = (x_k, y_k, \theta_k)^T$ be the state of the vehicle. As shown in [6], its evolution can be described by the following state equations:

$$\begin{cases} x_{k+1} = & x_k + \delta_{S,k}\cos(\theta_k + \frac{\delta_{\theta,k}}{2}) \\ y_{k+1} = & y_k + \delta_{S,k}\sin(\theta_k + \frac{\delta_{\theta,k}}{2}) \\ \theta_{k+1} = & \theta_k + \delta_{\theta,k}, \end{cases} \quad (16)$$

where $\delta_{S,k}$ and $\delta_{\theta,k}$ denote, respectively, the elementary displacement and rotation at time step k. These quantities can be obtained with good precision with two rear wheels ABS sensors and a fiber optic gyrometer, respectively. They are considered as the input to the system. With previous notations, we can thus denote $\mathbf{u}_k = (\delta_{S,k}, \delta_{\theta,k})^T$. The measure of the position $\mathbf{z}_k = (x_{GPS}, y_{GPS})$ is given by a GPS receiver, after converting each (longitude, latitude) estimated point in a Cartesian local frame [6]. We note that the heading angle θ_k is not observed. The results reported here were obtained using real sensor measurements collected using an experimental car [6]. In order to be able to compute estimation errors, the vehicle was equipped with a Thales Navigation GPS receiver used in a post-processed kinematic mode working with a local base (a Trimble 7400). This system was able to give reference positions at 1 Hz sampling rate. Since the constellation of the satellites was good enough during all the trials, all the kinematics ambiguities were fixed, and a few centimeter accuracy was reached. Experiments have been carried out on a test track in Versailles (France). We report hereafter the analysis of a 4.7 km trajectory with a mean speed of 50 km/h (see the left-hand side of Figure 3). The zoomed part of the trajectory (right-hand side of Figure 3) shows the GPS and estimated positions.

As in [1] and [6], the GPS error bounds were taken to be plus or minus three times the estimated standard deviation computed in real time by the receiver. Assuming the mode on the GPS error to be zero, BSs $m_0^{\mathbf{x}}$ and $m_k^{\mathbf{z}}$ were constructed to approximate triangular possibility distributions, as explained in Section IV-A, with $p_0^{\mathbf{x}} = 6$ and $p_k^{\mathbf{z}} = 4$ focal intervals. The heading angle θ was initialized as $[\theta_0] = [0, 2\pi]$. The error bounds on the input vector $\mathbf{u}_k$ were computed from characteristics of the ABS sensor and the gyrometer (± 3 degrees for $\delta_{\theta,k}$). To quantify the uncertainty on $\mathbf{u}_k$, a categorical BS $m_k^{\mathbf{u}}$ with only one focal interval was considered, as increasing the number of focal sets was not found to significantly improve the performances of the system. The maximum number q of focal elements on $\mathbf{x}_k$ was fixed to 20, e.g, if $p_k^{\mathbf{x}} > 20$, the summarization algorithm was used in order to reduce the number of focal intervals of $m_k^{\mathbf{x}}$.

Figure 4 shows the interval errors on x and y for the GPS as well as the BEE and BSE methods. We can see that the BSE method provides narrower intervals that still contain the true positions along both coordinates. The BSE method is also much less affected than the BEE method by a degradation of the GPS signal due to masking effects, as occurred around time steps 150, 210 and 400.

Table I reports the mean squared errors (MSE) on both coordinates x and y as well as the mean one-step running time (on a PC with Matlab) for the GPS, the PF (with 1000,

TABLE I

MEAN SQUARED ERRORS (MSE) ON x AND y, AND ONE-STEP RUNNING TIMES FOR THE GPS, AND THE FOLLOWING METHODS: PARTICLE FILTER (PF) WITH 3000, 2000, 1000 PARTICLES, BOUNDED ERROR ESTIMATION (BEE) AND BELIEF STATE ESTIMATION (BSE) WITH $q = 20$.

	GPS	PF(3000)	PF(2000)	PF(1000)	BEE	BSE (20)
MSE on x (m^2)	0.134	0.119	0.121	0.125	0.123	0.118
MSE on y (m^2)	0.374	0.215	0.232	0.243	0.249	0.199
Running time (ms)	-	639	526	401	136	409

2000 and 3000 particles), bounded error estimator (BEE) and the belief state estimator (BSE) with the default parameter values ($p^{\mathbf{z}} = 4$, $p^{\mathbf{u}} = 1$ and $q = 20$). The implementation and parametrization of the PF method were the same as reported in [1]. The PF was run with 1000 particles. We can see that the BSE method significantly outperforms the BEE method, and is also slightly more accurate than the PF method, with comparable running time. Increasing the number of particles in the PF improves its performance, at the cost of higher computation cost.

VI. CONCLUSION

In this paper, a new approach to non linear state estimation based on belief function theory and interval analysis has been proposed. This method uses belief structures composed of a finite number of axis-aligned boxes with associated masses in order to represent model and measurement uncertainties. Focal sets are propagated in the system equations using tools from interval arithmetics and constraint satisfaction techniques, thus generalizing the pure interval analysis approach. Applied to state estimation in dynamical systems, this approach makes it possible to compute a belief function on the system state at each time step. This approach has been applied to the localization of a land vehicle based on the dynamic fusion of GPS measurements with ABS sensors and a gyrometer. The method has been shown to provide more accurate estimates of the vehicle position than the bounded error method, while retaining the essential property of interval analysis to provide guaranteed computations. The performances of the BSE approach are also slightly better than those of the particle filter method, with comparable running time. This suggests that our method might be a viable alternative to both bounded error and probabilistic Monte-Carlo approaches to non linear state estimation, at least for the kind of applications considered here. A further advantage of the belief function approach is the possibility to combine the computed belief structures with additional information such as digital road network data, which can also be represented in the belief function framework[11].

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