# Consonant Belief Function induced by a Confidence Set of Pignistic Probabilities


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**Abstract.** A new method is proposed for building a predictive belief function from statistical data in the Transferable Belief Model framework. The starting point of this method is the assumption that, if the probability distribution $\mathbb{P}_X$ of a random variable $X$ is known, then the belief function quantifying our belief regarding a future realization of $X$ should have its pignistic probability distribution equal to $\mathbb{P}_X$. When $\mathbb{P}_X$ is unknown but a random sample of $X$ is available, it is possible to build a set $\mathscr{P}$ of probability distributions containing $\mathbb{P}_X$ with some confidence level. Following the Least Commitment Principle, we then look for a belief function less committed than all belief functions with pignistic probability distribution in $\mathscr{P}$. Our method selects the most committed consonant belief function verifying this property. This general principle is applied to the case of the normal distribution.


Keywords: Dempster-Shafer theory, Evidence theory, Transferable Belief Model, possibility distribution, statistical data.


## 1 Introduction

The Transferable Belief Model (TBM) is gaining increasing interest as a formal framework for information fusion, decision making under uncertainty and imprecise data analysis [14, 21, 18]. However, it is no always clear how to quantify various uncertainties using belief functions as required in this framework, especially when statistical data are involved. A contribution to this problem will be presented here.

More precisely, the problem considered in this paper can be described as follows. Let $X$ be a random variable with unknown probability distribution $\mathbb{P}_X$. We would like to quantify the beliefs held by an agent about a future realization of $X$ from past independent observations $X_1, \ldots, X_n$ drawn from the same distribution. In [5], it was argued that a belief function $bel(\cdot; X_1, \ldots, X_n)$ solution to this problem should verify two properties: it should be less committed than $\mathbb{P}_X$ with a given probability (i.e., for a given proportion of realizations of the random sample), and it should converge towards $\mathbb{P}_X$ in probability as the size of the sample tends to infinity. Several methods for constructing such belief functions (refered to as *predictive belief functions*) were proposed in [5] in the special case where $X$ is discrete, based on multinomial confidence intervals. This approach was recently extended to the continuous case using confidence bands on the unknown cumulative probability distribution instead of multinomial confidence intervals [1], and a similar approach in the context of Possibility Theory was presented in [12].

In the above approach, the second requirement demanding that, in the long run, the predictive belief function converge towards the probability distribution of $X$ is based on Hacking's frequency principle [11, 17], which equates the degree of belief of an event to its probability (long run frequency), when the latter is known. This principle, however, can be questioned. For instance, consider the result $X$ of a coin-tossing experiment, with $X \in \{H, T\}$, where $H$ and $T$ stand for "Head" and "Tail", respectively. If the coin is known to be perfectly balanced, then $\mathbb{P}_X(\{H\}) = \mathbb{P}_X(\{T\}) = 0.5$. If asked about our opinion regarding the result of the next toss, should we necessarily assign a degree of belief 0.5 to the event that this toss will bring a "Head"? This requirement seems hard to justify. However, if we are forced to bet on the result of this random experiment, then it seems reasonable to assign equal odds to the two elementary events. In the TBM, degrees of chance are not equated with degrees of belief: decision making is assumed to be handled at the *pignistic level*, which is distinguished from the *credal level* at which beliefs are entertained [21, 20]. The pignistic transformation converts each belief function *bel* into a *pignistic* probability distribution *BetP* that is used for decision making. As a consequence, we may replace Hacking's principle by the weaker requirement that the pignistic probability of an event be equal to its long run frequency, when the latter is known. Coming back to the coin example, this requirement leads to the constraint $BetP(\{H\}) = BetP(\{T\}) = 0.5$, which defines a set of admissible belief functions. Among this set, the Least Commitment Principle [16] dictates to choose the least committed one (i.e., the least informative), which is here the vacuous belief function.

In the above example, the probability distribution of $X$ was assumed to be known. In the more realistic situation considered here, we only have partial information about this distribution, in the form of a random sample $X_1, \ldots, X_n$. In that case, it is possible to construct a set $\mathscr{P}$ of probability distributions defined, e.g., by a parametric confidence region. A natural extension of the above line of reasoning is then to require that *bel* be less committed than any belief function with pignistic probability distribution in $\mathscr{P}$. This leads to the definition of a set of admissible belief functions, among which the most committed one can be chosen. This is the principle of the approach presented in this paper.

The rest of this paper is organized as follows. The background on the TBM will first be recalled in Section 2. The proposed approach will be formalized in Section 3. It will then be applied to the case of the normal distribution in Section 4. Section 5 will finally conclude the paper.

## 2 Background on the TBM

This section provides a short introduction to the main notions pertaining to the theory of belief functions that will be used throughout the paper, and in particular, its TBM interpretation. We first consider the case of belief functions defined on a finite domain [14], and then address the case of a continuous domain [19].

### 2.1 Belief Functions on a Finite Domain

Let $\mathscr{X} = \{\xi_1, \ldots, \xi_K\}$ be a finite set, and let $X$ be a variable taking values in $\mathscr{X}$. Given some evidential corpus, the knowledge held by a given agent at a given time over the actual value of variable $X$ can be modeled by a so-called *basic belief assignment* (bba) $m$ defined as a mapping from $2^{\mathscr{X}}$ into $[0, 1]$ such that:

$$\sum_{A \subseteq \mathscr{X}} m(A) = 1. \tag{1}$$

Each mass $m(A)$ is interpreted as the part of the agent's belief allocated to the hypothesis that $X$ takes some value in $A$ [14, 21]. The subsets $A \in \mathscr{X}$ such that $m(A) > 0$ are called the focal sets of $A$. When the focal sets are nested, $m$ is said to be consonant.

Equivalent representations of $m$ include the belief, plausibility and commonality functions defined, respectively, as:

$$bel(A) = \sum_{\emptyset \neq B \subseteq A} m(B), \tag{2}$$

$$pl(A) = \sum_{B \cap A \neq \emptyset} m(B), \tag{3}$$

and

$$q(A) = \sum_{B \cap A \neq \emptyset} m(B), \tag{4}$$

for all $A \subseteq \mathscr{X}$. When $m$ is consonant, then the plausibility function is a possibility measure: it verifies $pl(A \cup B) = \max(pl(A), pl(B))$ for all $A, B \subseteq \mathscr{X}$. The corresponding possibility distribution is defined by $\text{poss}(x) = pl(\{x\}) = q(\{x\})$ for all $x \in \mathscr{X}$, and the commonality function verifies $q(A \cup B) = \min(q(A), q(B))$ for all $A, B \subseteq \mathscr{X}$. Conversely, any possibility measure $\Pi$ with possibility distribution $\text{poss}(x) = \Pi(\{x\})$ for all $x \in \mathscr{X}$ is a plausibility function corresponding to a consonant bba $m$ defined as follows [7]. Let $\pi_k = \text{poss}(\xi_k)$, and let us assume that the elements of $\mathscr{X}$ have been arranged in such a way that $\pi_1 \geq \pi_2 \geq \ldots \geq \pi_K$. Then, we have:

$$m(A) = \begin{cases} 1 - \pi_1 & \text{if } A = \emptyset, \\ \pi_k - \pi_{k+1} & \text{if } A = \{\xi_1, \ldots, \xi_k\} \text{ for some } k \in \{1, \ldots, K-1\}, \\ \pi_K & \text{if } A = \mathscr{X}, \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

In the TBM, the *Least commitment Principle* (LCP) plays a role similar to the principle of maximum entropy in Bayesian Probability Theory. As explained in [16], the LCP states that, given two belief functions compatible with a set of constraints, the most appropriate is the least informative. To make this principle operational, it is necessary to define ways of comparing belief functions according to their information content. Several such partial orderings, generalizing set inclusion, have been proposed [22, 8]. Among them, the $q$- and $pl$-ordering relations are defined as follows:

- $m_1$ is said to be $q$-more committed than $m_2$ (noted $m_1 \sqsubseteq_q m_2$) if $q_1(A) \leq q_2(A)$, for all $A \subseteq \mathscr{X}$;

– $m_1$ is said to be *pl*-more committed than $m_2$ (noted $m_1 \sqsubseteq_{pl} m_2$) if $pl_1(A) \leq pl_2(A)$, for all $A \subseteq \mathscr{X}$;

The interpretation of these and other ordering relations is discussed in [8] from a set-theoretical perspective, and in [9] from the point of view of the TBM. In general, *q*- and *pl*-orderings are distinct notions, and none of them implies the other. However, these two orderings are equivalent in the special case of consonant belief functions: if $m_1$ and $m_2$ are consonant, then

$$m_1 \sqsubseteq_q m_2 \Leftrightarrow m_1 \sqsubseteq_{pl} m_2 \Leftrightarrow \text{poss}_1 \leq \text{poss}_2.$$

The TBM is a two-level mental model in which belief representation and updating take place at a first level termed *credal level*, whereas decision making takes place at a second level called *pignistic level* [21]. To make decisions, any bba $m$ such that $m(\emptyset) < 1$ is mapped into a pignistic probability function $Betp = Bet(m)$ given by

$$Betp(x) = \sum_{A \subseteq \mathscr{X}, A \neq \emptyset} \frac{m(A)}{1 - m(\emptyset)} \frac{1_A(x)}{|A|}, \quad \forall x \in \mathscr{X}, \tag{6}$$

where $1_A$ denotes the indicator function of $A$ defined by $1_A(x) = 1$ if $x \in A$, 0 otherwise.

Conversely, let us assume that we know the pignistic probability function $p_0$ of an agent and we would like to find the *q-least committed* (*q*-LC) belief function associated to $p_0$. As shown in [9, 10], the solution is a consonant belief function, called the *q*-LC *isopignistic* belief function. It is defined by the following possibility distribution:

$$\text{poss}(x) = \sum_{x' \in \mathscr{X}} \min(p_0(x), p_0(x')). \tag{7}$$

If $m$ is the bba associated to poss, we note $m = Bet_{LC}^{-1}(p_0)$.

### 2.2 Continuous Belief Functions on $\mathbb{R}$

Belief functions on $\mathbb{R}$ may be defined by replacing the concept of bba by that of basic belief density (bbd) [4, 15, 19]. A normal bbd $m$ is a function taking values from the set of closed real intervals into $[0, +\infty)$, such that

$$\iint_{x \leq y} m([x, y])\, dx\, dy = 1. \tag{8}$$

The belief, plausibility and commonality functions can be defined in the same way as in the finite case, replacing finite sums by integrals. In particular,

$$bel([x, y]) = \int_x^y \int_u^y m([u, v]) dv du, \tag{9}$$

$$pl([x, y]) = \int_{-\infty}^y \int_{\max(x,u)}^{+\infty} m([u, v]) dv du, \tag{10}$$

$$q([x, y]) = \int_{-\infty}^{x} \int_{y}^{+\infty} m([u, v]) dv du, \tag{11}$$

for all $x \leq y$. The domains of these integrals may be represented as in Figure 1, where each point in the triangle corresponds to an interval with upper and lower bounds indicated on the horizontal and vertical axes, respectively.

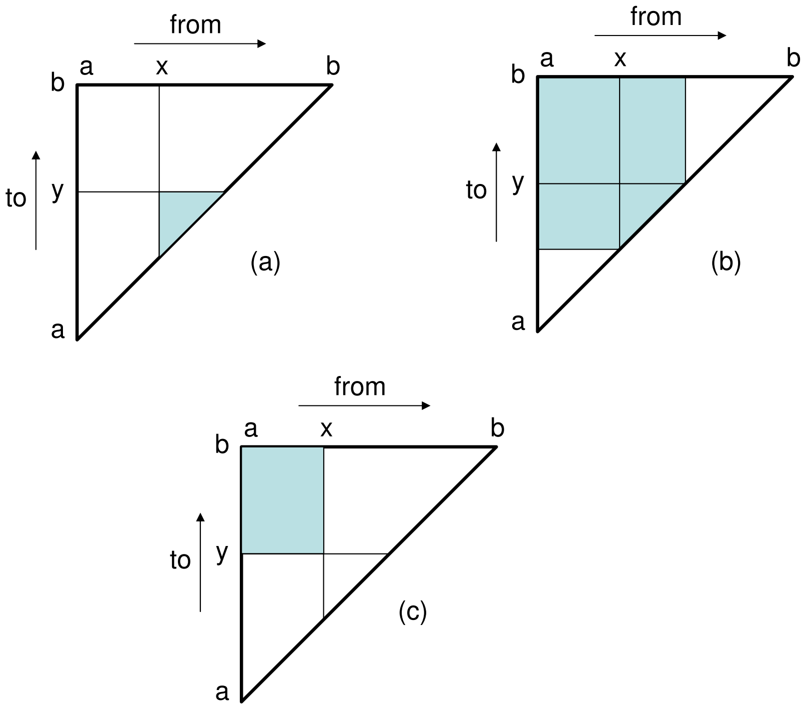


**Fig. 1.** The belief, plausibility and commonality functions are defined as integrals of the bbd with support $[a, b]$ on the shaded area of triangles (a), (b) and (c), respectively.

A pignistic probability distribution $Betf = Bet(m)$ can be defined as in the discrete case. It is a continuous distribution with the following probability density [19]:

$$Betf(x) = \lim_{\epsilon \to 0} \int_{-\infty}^{x} \int_{x+\epsilon}^{+\infty} \frac{m([u, v])}{v - u} dv du. \tag{12}$$

The expression of the $q$-LC isopignistic bbd $m = Bet_{LC}^{-1}(f_0)$ associated with a unimodal probability density $f_0$ with mode $\nu$ was also derived in [19]. The focal sets of $m$ are the level sets of the density function $f_0$. They are intervals $I_b = [a, b]$ such that $f_0(a) = f_0(b)$. Given the upper bound $b$ of any such interval, the lower bound is uniquely defined by $a = \gamma(b)$ for all $b \geq \nu$. The bbd is defined by

$$m([a, b]) = \theta(b)\delta(a - \gamma(b)),$$

with

$$\theta(b) = (\gamma(b) - b) f_0'(b),$$

where $f_0'$ is the derivative of $f_0$ and $\delta$ is the Dirac delta function. Note that $m$ is consonant. Consequently, the associated plausibility function is a possibility measure. The corresponding possibility distribution poss is given by:

$$\text{poss}(x) = pl(\{x\}) = \begin{cases} \int_x^{+\infty} (\gamma(t) - t) f_0'(t) dt & \text{if } x \geq \nu \\ \int_{\gamma^{-1}(x)}^{+\infty} (\gamma(t) - t) f_0'(t) dt & \text{otherwise.} \end{cases}$$

If $f_0$ is symmetrical, then $\gamma(x) = 2\nu - x$, and the above equation simplifies to

$$\text{poss}(x) = \begin{cases} 2(x - \nu) f_0(x) + 2 \int_x^{+\infty} f_0(t) dt & \text{if } x \geq \nu \\ 2(\nu - x) f_0(x) + 2 \int_{-\infty}^{x} f_0(t) dt & \text{otherwise.} \end{cases} \tag{13}$$

## 3 Consonant Belief Function Induced by a Set of Pignistic Probabilities

Let us now assume that the pignistic probability distribution $p_0$ of an agent is only known to belong to a set $\mathscr{P}$ of probability distributions and, as before, we seek to approximate the agent's bba $m_0$. The problem is again underdetermined, as we can only say that $m_0$ belongs to the set $\mathscr{M}(\mathscr{P}) = Bet^{-1}(\mathscr{P})$ defined by

$$\begin{aligned} \mathscr{M}(\mathscr{P}) &= \{m \mid Bet(m) \in \mathscr{P}\} \\ &= \bigcup_{p \in \mathscr{P}} \mathscr{M}(p), \end{aligned}$$

where $\mathscr{M}(p) = Bet^{-1}(p)$ denotes the set of bbas whose pignistic probability distribution is equal to $p$ (see Figure 2).

According to the LCP, $m_0$ should be approximated by a bba $m^*$ less committed than $m_0$, with respect to some ordering $\sqsubseteq$. In general, the set $\mathscr{M}(\mathscr{P})$ does not contain a LC element. However, we may define the *admissible* set $\mathscr{M}^*(\mathscr{P})$ as the set of bbas *dominating* (i.e., less committed than) all bbas in $\mathscr{M}(\mathscr{P})$:

$$\mathscr{M}^*(\mathscr{P}) = \{m' \mid m \sqsubseteq m', \forall m \in \mathscr{M}(\mathscr{P})\}.$$

It is then natural to choose $m^*$ as the *most committed* element in $\mathscr{M}^*(\mathscr{P})$, if this element exists. The solution of this problem is not obvious in the general case. However, a simple solution can be found if we restrict the search to the subset $\mathscr{C}^*(\mathscr{P}) \subset \mathscr{M}^*(\mathscr{P})$ of *consonant* bbas less committed than all bbas in $\mathscr{M}(\mathscr{P})$, and we consider the $q$-ordering.

For all $p \in \mathscr{P}$, let $m_p = Bet_{LC}^{-1}(p)$ be the $q$-LC isopignistic bba induced by $p$. It is consonant. Let $\text{poss}_p$ denote the corresponding possibility distribution. Bba $m_p$ is the $q$-least committed bba in the set $\mathscr{M}(p)$ of bbas whose pignistic probability distribution is $p$. Consequently, a consonant bba $m$ belongs to $\mathscr{C}^*(\mathscr{P})$ if and only if it is $q$-less committed than $m_p$, for all $p \in \mathscr{P}$, ie, if and only if

$$\text{poss}_p \leq \text{poss}, \quad \forall p \in \mathscr{P},$$

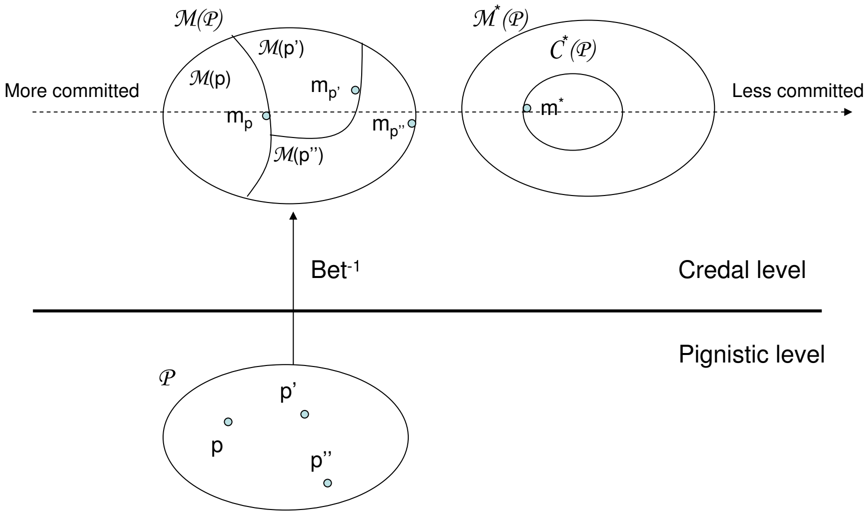


**Fig. 2.** Definition of the $q$-most committed dominating ($q$-MCD) bba $m^*$ associated to a set $\mathscr{P}$ of probability distribution. The set $\mathscr{M}(\mathscr{P})$ contains all bbas with pignistic probability function in $\mathscr{P}$. The set $\mathscr{M}^*(\mathscr{P})$ contains all bbas dominating (i.e., less committed than) all bbas in $\mathscr{M}(\mathscr{P})$. The $q$-MCD bba $m^*$ is the $q$-most committed consonant bba in $\mathscr{M}^*(\mathscr{P})$.

where poss is the possibility distribution associated to $m$. It follows that the $q$-most committed element in $\mathscr{C}^*(\mathscr{P})$ is defined by the following possibility distribution

$$\text{poss}^*(x) = \sup_{p\in\mathscr{P}} \text{poss}_p(x), \quad \forall x \in \mathscr{X}. \tag{14}$$

Possibility distribution poss$^*$ will be referred to as the *$q$-most committed dominating* ($q$-MCD) possibility distribution associated to $\mathscr{P}$. The corresponding bba will be noted $m^*$.

*Example 1.* Let us consider a frame $\mathscr{X} = \{\xi_1, \xi_2, \xi_3\}$ with three elements, and a set $\mathscr{P} = \{p, p', p''\}$ of three probability distributions shown in the first three columns of Table 1. The possibility distributions poss, poss$'$, poss$''$ associated with the corresponding $q$-LC isopignistic bbas are displayed in Table 1. Note that there is no $q$-LC element among these three bbas. Possibility distribution poss$^*$ is shown in the last column of Table 1. Using (5), we obtain the corresponding bba as

$$m^*(\{\xi_1\}) = 0.35, \quad m^*(\{\xi_1, \xi_2\}) = 0.05, \quad m^*(\mathscr{X}) = 0.6.$$

*Remark 1.* By definition, the $q$-MCD bba $m^*$ is the $q$-most committed element among all *consonant* bbas that are $q$-less committed than all bbas in $\mathscr{M}(\mathscr{P})$. The restriction to consonant bbas is justified by the existence and unicity of a solution in $\mathscr{C}^*(\mathscr{P})$, whereas the existence of a $q$-most committed element in $\mathscr{M}^*(\mathscr{P})$ is not guaranteed

**Table 1.** Pignistic probabilities and corresponding $q$-LC isopignistic possibility distributions of Example 1.

| $x$ | $p(x)$ | $p'(x)$ | $p''(x)$ | $\text{poss}(x)$ | $\text{poss}'(x)$ | $\text{poss}''(x)$ | $\text{poss}^*(x)$ |
|---|---|---|---|---|---|---|---|
| $\xi_1$ | 0.7 | 0.6 | 0.65 | 1 | 1 | 1 | 1 |
| $\xi_2$ | 0.2 | 0.25 | 0.1 | 0.5 | 0.65 | 0.3 | 0.65 |
| $\xi_3$ | 0.1 | 0.15 | 0.25 | 0.3 | 0.45 | 0.6 | 0.6 |

in general. Additionally, finding the solution in $\mathscr{C}^*(\mathscr{P})$ is computationally tractable in several cases of practical interest, as will be shown below, and the result usually has a very simple expression. It may happen, however, that a $q$-most committed element in $\mathscr{M}^*(\mathscr{P})$ exists, and that it is strictly more committed than $m^*$. This is the case, in particular, when function $q_{max}$ defined by

$$q_{max}(A) = \max_{p \in \mathscr{P}} q_p(A), \quad \forall A \subseteq \mathscr{X}$$

is a commonality function, $q_p$ being the commonality function associated to $m_p$, . In that case, the corresponding bba $m_{max}$ is obviously the $q$-most committed element in $\mathscr{M}^*(\mathscr{P})$. This is the case in Example 1: it may be shown that $q_{max} = \max(q, q', q'')$ is a commonality function, and the corresponding bba $m_{max}$ is strictly $q$-more committed than $m^*$.

*Remark 2.* The approach presented here is different from that introduced in [5] and [2], in which we searched for the *pl*-most committed bba $m^\circ$, in the set $\mathscr{M}^\circ(\mathscr{P})$ of bbas that are less committed than *all probability measures* in $\mathscr{P}$. In this alternative approach, the solution is obtained as the lower envelope $P_*$ of $\mathscr{P}$, when it is a belief function. This is the case, in particular, when $\mathscr{P}$ is a $p$-box [2], or when it is constructed from a multinomial confidence region with $K \leq 3$ [5]. Different heuristics were introduced in [5] for constructing a belief function less committed than $P_*$ when $P_*$ is not a belief function. The approach adopted here usually yields a simpler result as it produces consonant belief functions. Additionally, it may be argued to be more in line with the two-level structure of the TBM, as it does not directly compare probabilities at the pignistic level with belief functions at the credal level.

## 4 Application to the normal distribution

Let us now assume that $X$ has a normal distribution with mean $\mu$ and variance $\sigma^2$. If these two parameters are known, then the possibility distribution poss associated with the $q$-LC isopignistic bbd is given by (13):

$$\text{poss}(x; \mu, \sigma) = \begin{cases} \frac{2(x-\mu)}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) + 2\left(1 - \Phi\left(\frac{x-\mu}{\sigma}\right)\right) & \text{if } x \geq \mu \\ \frac{2(\mu-x)}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) + 2\Phi\left(\frac{x-\mu}{\sigma}\right) & \text{otherwise,} \end{cases} \quad (15)$$

where $\Phi$ is the standard normal cumulative distribution function.

When $\mu$ and $\sigma^2$ are unknown but an iid sample $X_1, \ldots, X_n$ is available, then it is possible to define a joint confidence region for $\mu$ and $\sigma^2$ [3]. In particular, the Mood exact confidence region at level $1 - \alpha = (1 - \alpha_1)(1 - \alpha_2)$ is defined by

$$\mathscr{R}(X_1, \ldots, X_n) = \Bigg\{(\mu; \sigma^2) : \overline{X} - u_{1-\alpha_1/2}\frac{\sigma}{\sqrt{n}} \leq \mu \leq \overline{X} + u_{1-\alpha_1/2}\frac{\sigma}{\sqrt{n}}, \quad \frac{nS^2}{\chi^2_{n-1;1-\alpha_2/2}} \leq \sigma^2 \leq \frac{nS^2}{\chi^2_{n-1;\alpha_2/2}}\Bigg\}, \quad (16)$$

where $\overline{X}$ is the sample mean, $S^2 = (1/n)\sum_{i=1}^{n}(X_i - \overline{X})^2$ is the sample variance, $u_{1-\alpha_1/2}$ is the upper $\alpha_1/2$ percentile of a standard normal distribution, and $\chi^2_{n-1;\alpha_2/2}$ and $\chi^2_{n-1;1-\alpha_2/2}$ are the lower and upper $\alpha_2/2$ percentiles of a $\chi^2_{n-1}$ distribution. The shape of that region is illustrated in Figure 3. Values of $\alpha_1$ and $\alpha_2$ yielding a region of smallest possible size for a fixed confidence level are given in [3].

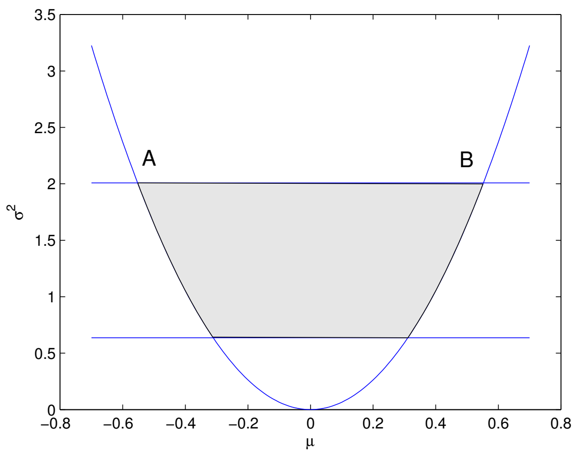


**Fig. 3.** Shape of Mood's exact region: the Mood Exact Region for $\alpha = 0.1$, $\alpha_1 = \alpha_2$ and $n = 25$. Without loss of generality, $\overline{x} = 0$ and $s^2 = 1$. The points with coordinates $(\widehat{\mu}^-, (\widehat{\sigma}^+)^2)$ and $(\widehat{\mu}^+, (\widehat{\sigma}^+)^2)$ are denoted $A$ and $B$, respectively.

Let $\mathscr{P}$ denote the set of Gaussian distributions with parameters contained in confidence region $\mathscr{R}$. Applying the principle outlined in Section 3, we may obtain the $q$-MCD possibility distribution $\text{poss}^*$ for any $x$ by maximizing $\text{poss}(x; \mu, \sigma)$ given by (15) with respect to $\mu$ and $\sigma$, under the constraint $(\mu, \sigma^2) \in \mathscr{R}$. The result is given by the following proposition.

**Proposition 1.** *The q-MCD possibility distribution poss$^*$ associated with the Mood confidence confidence region $\mathscr{R}$ at level* $(1-\alpha_1)(1-\alpha_2)$ *is*

$$poss^*(x) = \begin{cases} poss(x; \widehat{\mu}^-, \widehat{\sigma}^+) & if\ x < \widehat{\mu}^- \\ 1 & if\ \widehat{\mu}^- \leq x \leq \widehat{\mu}^+ \\ poss(x; \widehat{\mu}^+, \widehat{\sigma}^+) & if\ x > \widehat{\mu}^+, \end{cases} \tag{17}$$

*with*

$$\widehat{\sigma}^+ = \left( \frac{nS^2}{\chi^2_{n-1;\alpha_2/2}} \right)^{1/2},$$

$$\widehat{\mu}^- = \overline{X} - u_{1-\alpha_1/2} \frac{\widehat{\sigma}^+}{\sqrt{n}}, \quad \widehat{\mu}^+ = \overline{X} + u_{1-\alpha_1/2} \frac{\widehat{\sigma}^+}{\sqrt{n}}.$$

*Proof.* We have by definition

$$\text{poss}^*(x) = \sup_{(\mu,\sigma^2)\in\mathscr{R}} \text{poss}(x; \mu, \sigma).$$

If $x \in [\widehat{\mu}^-, \widehat{\mu}^+]$, then we can get $\text{poss}(x, \mu, \sigma) = 1$ by setting $\mu = x$ and $\sigma = \widehat{\sigma}^+$. If $x < \widehat{\mu}^-$, then the value 1 cannot be reached. However, we obtain using standard calculus for $x < \mu$:

$$\frac{\partial \text{poss}(x; \mu, \sigma)}{\partial \mu} = -\frac{(x-\mu)^2}{\sigma^3 \sqrt{2\pi}} \exp\left( -\frac{(x-\mu)^2}{2\sigma^2} \right) < 0$$

and

$$\frac{\partial \text{poss}(x; \mu, \sigma)}{\partial \sigma} = \frac{(\mu-x)^3}{\sigma^4 \sqrt{2\pi}} \exp\left( -\frac{(x-\mu)^2}{2\sigma^2} \right) > 0.$$

Consequently, $\text{poss}(x; \mu, \sigma)$ is maximized by jointly minimizing $\mu$ and maximizing $\sigma$, and the maximum is reached for $(\mu, \sigma) = (\widehat{\mu}^-, \widehat{\sigma}^+)$. Similarly, we get for $x > \widehat{\mu}^+$:

$$\frac{\partial \text{poss}(x; \mu, \sigma)}{\partial \mu} = \frac{(x-\mu)^2}{\sigma^3 \sqrt{2\pi}} \exp\left( -\frac{(x-\mu)^2}{2\sigma^2} \right) > 0$$

and

$$\frac{\partial \text{poss}(x; \mu, \sigma)}{\partial \sigma} = \frac{(x-\mu)^3}{\sigma^4 \sqrt{2\pi}} \exp\left( -\frac{(x-\mu)^2}{2\sigma^2} \right) > 0.$$

Consequently, the maximum of $\text{poss}(x, \mu, \sigma)$ for $x > \widehat{\mu}^+$ is reached for $(\mu, \sigma) = (\widehat{\mu}^+, \widehat{\sigma}^+)$. □

Figure 4 shows the possibility distribution $\text{poss}^*(x)$ for $\overline{x} = 0$, $s^2 = 1$, $\alpha = 0.1$ and various values of $n$. The case $n = \infty$ corresponds to the situation where parameters $\mu$ and $\sigma^2$ are known: in that case, poss$^*$ is simply que $q$-LC isopignistic possibility distribution induced by the normal pignistic distribution with $\mu = 0$ and $\sigma^2 = 1$.

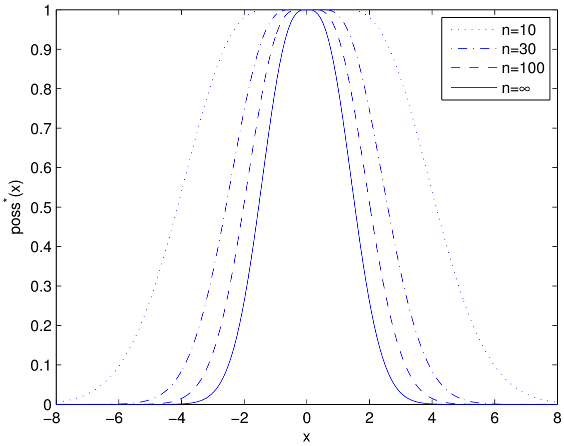


**Fig. 4.** Plot of $\text{poss}^*(x)$ for $\overline{x} = 0$, $s^2 = 1$, $\alpha = 0.1$, $\alpha_1 = \alpha_2$, and $n = 10$, 30, 100 and $\infty$.

## 5 Conclusion

A new method for generating a belief function from statistical data in the TBM framework has been presented. The starting point of this method is the assumption that, if the probability distribution $\mathbb{P}_X$ of a random variable is known, then the belief function quantifying our belief regarding a future realization of $X$ should be such that its pignistic probability distribution equals $\mathbb{P}_X$. In the realistic situation where $\mathbb{P}_X$ is unknown but a random sample of $X$ is available, it is possible to build a set $\mathscr{P}$ of probability distributions containing $\mathbb{P}$ with some confidence level. Following the LCP, it is then reasonable to impose that the sought belief function be $q$-less committed than all belief functions whose pignistic probability distribution is in $\mathscr{P}$. Our method selects the $q$-most committed consonant belief function verifying this property, referred to as the $q$-MCD possibility distribution induced by $\mathscr{P}$. This general principle has been illustrated in the case of the normal distribution.

In conjunction with the General Bayesian Theorem [16, 6], the $q$-LC isopignistic transformation has proved useful to tackle classification problems using the TBM [13]. In this approach, the parameters of the pignistic distributions were assumed to be given by experts or estimated using large samples. Using the tools presented in this paper, it will be possible to apply this methodology to a wider range of problems where only small datasets are available. Future work in this direction will be reported in forthcoming papers.

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