On Latent Belief Structures ⋆

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Abstract. Based on the canonical decomposition of belief functions, Smets introduced the concept of a latent belief structure (LBS). This concept is revisited in this article. The study of the combination of LBSs allows us to propose a less committed version of Dempster's rule, resulting in a commutative, associative and idempotent rule of combination for LBSs. This latter property makes it suitable to combine non distinct bodies of evidence. A sound method based on the plausibility transformation is also given to infer decisions from LBSs. In addition, an extension of the new rule is proposed so that it may be used to optimize the combination of imperfect information with respect to the decisions inferred.

1 Introduction

The theory of belief functions [14] is recognized as a rich framework for representing and reasoning with imperfect information. Contrary to probability theory, it allows in particular the representation of different forms of ignorance. However, when decisions have to be made in an uncertain context, rationality principles [13] justify the use of a probability distribution. There exist different methods for the transformation of a belief function to a probability distribution; in particular the pignistic transformation [17] and the plausibility transformation [2]. In this article, two results related to the latter transformation are presented. First, it can be extended to transform a so-called latent belief structure (LBS) [16] into a probability distribution. Second, two ways of modeling negative statements become equivalent with the extension of this transformation.

Equipped with a well-defined means to use LBSs with respect to decision making, this paper deepens their study. The analysis of the combination of LBSs leads to families of conjunctive combination rules. One of these rules is idempotent, a property required for the combination of LBSs obtained from, e.g., belief functions based on non distinct bodies of evidence.

The rest of this paper is organized as follows. The mathematical concept of LBS and Smets's tentative interpretation of a LBS will first be recalled in Section 2. Combination rules for LBSs will then be studied in Section 3. Section 4 will describe decision making from LBSs and Section 5 will conclude the paper.

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2 Latent Belief Structures

2.1 Background Material on Belief Function Theory

The presentation of belief function theory adopted here is in line with the one of the transferable belief model (TBM) [17]. The beliefs held by a rational agent Ag on a finite frame of discernment $\Omega = \{\omega_1, ..., \omega_K\}$ are represented by a basic belief assignment (BBA) m^{Ω}_{Ag} defined as a mapping from 2^{Ω} to $[0, 1]$ verifying $\sum_{A \subseteq \Omega} m(A) = 1$. For $A \subseteq \Omega$, if $m(A) > 0$ holds, then A is called a focal set (FS) of m. A BBA m is called: *normal* if $\emptyset$ is not a FS; *vacuous* if Ω is the only FS; *dogmatic* if Ω is not a FS; *categorical* if is has only one FS different from Ω; *simple* if it has at most two FSs, Ω included. If m is a simple BBA (SBBA) defined by $m(A) = 1 - w$ and $m(\Omega) = w$ for $A \neq \Omega$, it is noted A^w; if $A = \Omega$ then we write Ω if no confusion can occur. Note that normality is not required by the TBM. Equivalent representations of a BBA m exist. In particular the belief, plausibility, and commonality functions are defined, respectively, by:

$$bel(A) = \sum_{\emptyset \neq B \subseteq A} m(B) \quad , \tag{1}$$

$$pl(A) = \sum_{B \cap A \neq \emptyset} m(B) \quad , \tag{2}$$

and

$$q(A) = \sum_{B \supseteq A} m(B) \quad , \tag{3}$$

for all $A \subseteq \Omega$. Two distinct BBAs m_1 and m_2 can be combined using the TBM conjunctive combination rule, noted $\bigcirc\!\!\!\!\!\cap$, or using Dempter's rule [14], noted $\oplus$. Assuming that $m_{1 \bigcirc\!\!\!\!\cap 2}(\emptyset) \neq 1$, those rules are defined by:

$$m_{1 \bigcirc\!\!\!\!\cap 2}(A) = \sum_{B \cap C = A} m_1(B) \, m_2(C), \forall A \subseteq \Omega \quad , \tag{4}$$

$$m_{1 \oplus 2}(A) = \begin{cases} 0 & \text{if } A = \emptyset, \\ m_{1 \bigcirc\!\!\!\!\cap 2}(A) / \left(1 - m_{1 \bigcirc\!\!\!\!\cap 2}(\emptyset)\right) & \text{otherwise.} \end{cases} \tag{5}$$

Under conflicting information, i.e. $m_{1 \bigcirc\!\!\!\!\cap 2}(\emptyset) > 0$, the legitimacy of the normalization operation involved by Dempster's rule has been questioned. Indeed, the conflict may originate from different situations such as unreliable sources of information or a lack of exhaustiveness of Ω, in which cases other normalization operations may be reasonable [6].

The pignistic and the plausibility transformations allow the transformation of a BBA m to probability distributions noted respectively $BetP_m$ and PlP_m. They are defined as follows:

$$BetP_m(\{\omega_k\}) = \sum_{\{A \subseteq \Omega, \omega_k \in A\}} \frac{m(A)}{(1 - m(\emptyset)) \, |A|} \ , \tag{6}$$

$$PlP_m(\{\omega_k\}) = \kappa^{-1} pl(\{\omega_k\}) \,, \tag{7}$$

with $\kappa = \sum_{j=1}^{K} pl(\{\omega_j\})$.

2.2 Canonical Decomposition of a Belief Function

The canonical decomposition of a belief function, introduced in [16], is based on a generalization of the canonical representation of a *separable* BBA m defined by Shafer [14]. A BBA is called separable if it can be written as the $\bigcirc\!\!\!\!\cap$ combination of SBBAs. For a separable BBA m, one has thus:

$$m = \bigcirc\!\!\!\!\!\cap_{A \subset \Omega} A^{w(A)}, \tag{8}$$

with $w(A) \in [0,1]$ for all $A \subset \Omega$. Through the definition of a *generalized* SBBA, Smets [16] proposed a means to canonically decompose any *non dogmatic* BBA (NDBBA). A generalized SBBA is defined as a function μ from 2^Ω to $\mathbb{R}$ by:

$$\mu(A) = 1 - w,\ \mu(\Omega) = w,\ \mu(B) = 0 \ \ \forall B \in 2^\Omega \setminus \{A, \Omega\}, \tag{9}$$

for $A \neq \Omega$ and $w \in [0, +\infty)$. Extending the SBBA notation, any generalized SBBA can be written A^w; when $w \leq 1$, μ is thus a SBBA. When $w > 1$, μ is called *inverse* SBBA. Smets showed that any NDBBA can be uniquely represented as the $\bigcirc\!\!\!\!\cap$ combination of non categorical generalized SBBAs; the expression for this decomposition is then the same as (8) with $w \in (0, +\infty)$ this time. The weights $w(A)$ for each $A \in 2^\Omega \setminus \{\Omega\}$ are obtained as follows:

$$w(A) = \prod_{B \supseteq A} q(B)^{(-1)^{|B|-|A|+1}}. \tag{10}$$

The weight function, $w : 2^\Omega \setminus \{\Omega\} \to (0, +\infty)$, is thus yet another equivalent representation of a NDBBA m.

If A^{w_1} and A^{w_2} are two SBBAs, their combination by $\bigcirc\!\!\!\!\cap$ is the SBBA $A^{w_1 w_2}$. From the commutativity and associativity of the $\bigcirc\!\!\!\!\cap$ rule, the combination of two NDBBAs m_1 and m_2 with respective weight functions w_1 and w_2 is written:

$$m_{1 \bigcirc\!\!\!\!\cap 2} = \bigcirc\!\!\!\!\!\cap_{A \subset \Omega} A^{w_1(A) \cdot w_2(A)}. \tag{11}$$

Details on normalized versions of those results can be found in a recent exposition of the canonical decomposition [3]. Other combinations of belief functions have been proposed. In particular the cautious rule [3], noted $\bigcirc\!\!\!\!\wedge$, possesses the idempotence property. It is defined as follows ($\wedge$ is the minimum operator):

$$m_{1 \bigcirc\!\!\!\!\wedge 2} = \bigcirc\!\!\!\!\!\cap_{A \subset \Omega} A^{w_1(A) \wedge w_2(A)}. \tag{12}$$

2.3 Decombination Rule

In the area of belief revision [8], the addition of beliefs without retracting others is known as *expansion*; the inverse operation, *contraction*, allows the removal of beliefs. In belief function theory, those operations are performed respectively by the $\bigcirc\!\!\!\!\cap$ and $\bigcirc\!\!\!\!\!\not\cap$ rules. Different authors [16, 15, 10] have studied the $\bigcirc\!\!\!\!\!\not\cap$ rule which is either called the *decombination* [16] or *removal* [15] rule. Let q_1 and q_2 be the commonality functions of two NDBBAs, the decombination is defined by:

$$q_{1 \bigcirc\!\!\!\!\!\not\cap 2}(A) = q_1(A) / q_2(A), \ \ \forall A \subseteq \Omega. \tag{13}$$

The resulting function may not be a belief function. In this case it is called a *pseudo* belief function [16] or *signed* belief function [10].

The interest of this operator is motivated by the following example. Suppose you are in a state of ignorance about the actual state ω_0 of the world Ω. Suppose that you then have some good reasons to believe in A, for $A \subset \Omega$; you perform an *expansion* of your beliefs (here your ignorance) by the SBBA A^x (if the *good reasons* amounts to $(1-x)$, for x small), which is equivalent to the combination $\Omega \bigcirc\!\!\!\!\cap\, A^x$. Later, another information arrives telling you that the first information was not valid. This can be handled in at least three ways, all of them bringing you to the state of indecision that this example intuitively leads to. Note that this situation is illustrated by *the Pravda bias* example [16].

First of all, the second information can be understood as: the negation of the first information holds true. This results in an expansion of belief in favor of $\bar{A}$. It leads to the state of belief $A^x \bigcirc\!\!\!\!\cap\, \bar{A}^x$, yielding $bel\,(A) = bel\,(\bar{A})$ which is indeed a state of indecision. This solution produces however a share of conflict $(m\,(\emptyset) > 0)$ depending on the value of x. This conflict cannot be escaped whatever further information you receive [10], unless an arbitrary normalization operation is used.

Another way of interpreting the second information is that all conclusions that may be drawn from the first information must be cancelled, i.e. you should come back to the state of belief in which you were before receiving the first information [10]. This means here that you should come back to the state of ignorance, i.e. $bel\,(\Omega) = 1$, which is a state of indecision. This interpretation of the example may be treated in two ways. It may be argued that both ways involve a contraction in their development as showed by the following reasoning.

The first way of treating this second interpretation of the example consists in contracting the belief you had in favor of A, which amounts to do $A^x \oslash A^x = \Omega$. The ignorance state is thus recovered. The second method uses the discounting operation [14]. This operation is based on the use of a second frame $\mathcal{R} = \{R, NR\}$ which represents a meta-knowledge $m^{\mathcal{R}}$ on the reliability of the information that is given to you. If the first source of information is reliable then your belief on Ω is the one given by this source; this is noted $m^{\Omega}_{Ag}\,[\{R\}] = m^{\Omega}_{S}$ where S denotes the first source of information. If the source of information is not reliable then your belief is vacuous: $m^{\Omega}_{Ag}\,[\{NR\}]\,(\Omega) = 1$. Let us suppose that before you receive the second information, you are a priori almost certain that S is reliable: $m^{\mathcal{R}}_{Ag}\,(\{R\}) = 1 - \epsilon$, and $m^{\mathcal{R}}_{Ag}\,(\mathcal{R}) = \epsilon$, with ϵ a small positive real number. Your belief on Ω is computed by $\bigcirc\!\!\!\!\cap$ combining $m^{\mathcal{R}}_{Ag}$ with $m^{\Omega}_{Ag}\,[\{R\}]$ and then marginalizing this belief on Ω. To be in a state of indecision after receiving the second information, $m^{\mathcal{R}}_{Ag}\,(\{R\}) = 0$ must hold. It is possible by a contraction of your initial $m^{\mathcal{R}}_{Ag}$. It is also possible through the $\bigcirc\!\!\!\!\cap$ combination of your initial $m^{\mathcal{R}}_{Ag}$ with a BBA $m^{\mathcal{R}}\,(\{NR\}) = 1$; this solution implies however the use of a categorical belief (see [3] for a discussion on dogmatic beliefs).

More generally, the $\oslash$ rule allows a form of non monotonic reasoning in the belief function theory. Indeed if for $A \subseteq \Omega$ you have a belief $bel\,(A) > 0$ then it will be impossible without this operator to obtain later $bel\,(\bar{A}) = 1$ by an expansion with other beliefs, unless normalization is used.

Smets [16] goes further than the interpretation of removal of beliefs that is given to the $\oslash$ operator. He introduces states of debt of belief (also called *diffident* beliefs). Indeed, Smets reformulates the example so that the second source of information gives good reasons *not to believe* in A. He argues that if the weights of the good reasons to believe A and not to believe A counterbalance each other, then you should be in a state of ignorance. Those states of debt of belief are used by Smets to introduce the LBSs recalled in the next subsection.

Note that the existence of positive and negative information is generally coined under the term *bipolarity*. Other authors have tried to model such dual information in the belief function theory; in particular we can cite the work of Dubois et al. [7], and of Labreuche et al. [11]. The question of the relevance of debt of belief remains open. Nonetheless, the next subsection shows that decombination is useful at least mathematically and thus deserves attention.

2.4 Confidence and Diffidence

Let A^{w_1} and A^{w_2} be two non categorical SBBAs, hence A^{1/w_2} is an inverse SBBA. The decombination of A^{w_1} by A^{w_2}, i.e. $A^{w_1} \oslash A^{w_2}$, is equal to the $\bigcirc\!\!\!\!\cap$ combination of A^{w_1} with A^{1/w_2} [16]. Let w be the weight function of a NDBBA m. Partition 2^{Ω} into two (disjoint) subsets: $C = \{A : A \subset \Omega, w(A) \in (0,1]\}$, and $D = \{A : A \subset \Omega, w(A) \in (1,\infty)\}$. A NDBBA m can then be written:

$$m = \left(\bigcirc\!\!\!\!\!\cap_{A \in C} A^{w(A)}\right) \oslash \left(\bigcirc\!\!\!\!\!\cap_{A \in D} A^{\frac{1}{w(A)}}\right) \tag{14}$$

Any NDBBA is thus the result of combinations and decombinations of non categorical SBBAs or, equivalently, any NDBBA is equal to the decombination of a separable NDBBA by a separable NDBBA. Smets called the separable NDBBA, noted m^c and obtained from the set C, the *confidence* component and the separable NDBBA, noted m^d and obtained from the set D, the *diffidence* component. We can thus write: $m = m^c \oslash m^d$. The weight functions of m^c and m^d, defined from $2^{\Omega} \setminus \{\Omega\}$ to $(0,1]$ and called the confidence and diffidence weight functions, are noted w^c and w^d. They can easily be found from the original weight function w of a NDBBA m as follows: $w^c(A) = 1 \wedge w(A)$, and $w^d(A) = 1 \wedge \frac{1}{w(A)}$, for all $A \subset \Omega$.

From the canonical decomposition of a belief function, Smets defined a LBS as a pair of BBAs $\left(m^c, m^d\right)$ allowing the representation of belief states in which positive and negative items of evidence (reasons *to believe and not to believe* [16]) occur. Definition 1 is more specific, in that it imposes that this pair be made of separable NDBBAs. Definition 2 defines a concept also introduced in [16].

Definition 1 (Latent Belief Structure). *A latent belief structure is defined as a pair of separable NDBBAs m^c and m^d called respectively the confidence and diffidence components. A LBS is noted using a upper-case L.*

Definition 2 (Apparent Belief Structure). *The apparent belief structure associated with a LBS $L = (m^c, m^d)$ is the signed belief function obtained from the decombination $m^c \oslash m^d$ of the confidence and diffidence components of L.*

The motivation for Definition 1 is due to the following observation: if m^c and m^d are two NDBBAs, then we can always find two separable NDBBAs m'^c and m'^d such that $m^c \oslash m^d = m'^c \oslash m'^d$, hence a LBS can be merely defined as a pair of separable NDBBAs. A LBS is thus a generalization of a NDBBA.

The properties linking these definitions are the following. By definition, the apparent belief structure associated to a LBS may or may not be a belief function. Further, an infinity of LBSs correspond to the same apparent belief structure. Besides, among the infinity of LBSs corresponding to the same apparent belief structure, one LBS has a particular structure: it is then called a *canonical* LBS (see Definition 3 below). In particular, an infinity of LBSs can yield the same NDBBA, for instance the LBSs $(A^{0.2}, A^{0.3})$ and $(A^{0.6}, A^{0.9})$ correspond to the same NDBBA $A^{2/3}$ and the canonical LBS of this NDBBA is $\left(A^{2/3}, \Omega\right)$. Remark that the canonical decomposition of a NDBBA m yields the canonical LBS of m. Example 1 shows how a CLBS can be generated from expert opinions. Note that L_Ω will be used to denote the LBS obtained from the vacuous BBA.

Definition 3 (Canonical Latent Belief Structure). *A CLBS is a LBS verifying:* $\forall A \subset \Omega$, $w^c(A) \vee w^d(A) = 1$ *where* $\vee$ *denotes the maximum operator.*

Example 1. Suppose $\Omega = \{a, b, c\}$ and the sets $A = \{a, b\}, B = \{b, c\}, C = \{a, c\}$. Now, a first expert gives the opinion $A \sim B$ which, according to the elicitation technique proposed in [1], means that he believes equivalently in A and B, i.e. $bel_1(A) = bel_1(B)$. A second expert gives the opinion: $C \sim B$, i.e $bel_2(C) = bel_2(B)$. Given those constraints on bel_1 and bel_2, the BBAs m_1 and m_2 of Table 1 on page 7 may be produced using the method proposed in [1] for a certain set of parameters required by the method.

3 Combination of LBSs

This section studies mathematical operations on LBSs. Let us first express two known operations of belief function theory using LBSs.

Let $\left(m_1^c, m_1^d\right)$ and $\left(m_2^c, m_2^d\right)$ be the CLBSs associated with two NDBBAs m_1 and m_2. Then $\left(m_1^c \bigcirc\!\!\!\!\cap\, m_2^c, m_1^d \bigcirc\!\!\!\!\cap\, m_2^d\right)$ is a LBS associated with $m_1 \bigcirc\!\!\!\!\cap\, m_2$. This lead Smets to define the conjunctive combination of two LBSs as follows.

Definition 4. *The conjunctive combination of two LBSs L_1 and L_2 is a LBS noted $L_{1 \bigcirc\!\!\!\!\cap\, 2}$. It is defined by the weight functions (15) and (16):*

$$w^c_{1 \bigcirc\!\!\!\!\cap\, 2}(A) = w^c_1(A) \cdot w^c_2(A)\,, \tag{15}$$

$$w^d_{1 \bigcirc\!\!\!\!\cap\, 2}(A) = w^d_1(A) \cdot w^d_2(A)\,. \tag{16}$$

The vacuous LBS L_Ω is a neutral element for $\bigcirc\!\!\!\!\cap$, i.e. $L \bigcirc\!\!\!\!\cap\, L_\Omega = L$ for all LBSs L. The cautious rule of combination [3] can also be expressed in terms of LBSs.

Definition 5. *([4, Proposition 6]) The cautious combination of two LBSs L_1 and L_2 is a LBS noted $L_{1 \bigcirc\!\!\!\!\wedge\, 2}$. It is defined by the following weight functions:*

$$w^c_{1 \bigcirc\!\!\!\!\wedge\, 2}(A) = w^c_1(A) \wedge w^c_2(A)\,, \tag{17}$$

$$w^d_{1 \bigcirc\!\!\!\!\wedge\, 2}(A) = w^d_1(A) \vee w^d_2(A)\,. \tag{18}$$

Table 1. Two NDBBAs and their CLBSs.

A	$m_1(A)$	$w_1^c(A)$	$w_1^d(A)$	$m_1^c(A)$	$m_1^d(A)$	$m_2(A)$	$w_2^c(A)$	$w_2^d(A)$	$m_2^c(A)$	$m_2^d(A)$
$\emptyset$	0	1	1	0	0	0	1	1	0	0
$\{a\}$	0	1	1	0	0	0	1	1	0	0
$\{b\}$	0	1	5/9	4/9	4/9	0	1	1	0	0
$\{a,b\}$	0.4	1/3	1	2/9	0	0	1	1	0	0
$\{c\}$	0	1	1	0	0	0	1	5/9	4/9	4/9
$\{a,c\}$	0	1	1	0	0	0.4	1/3	1	2/9	0
$\{b,c\}$	0.4	1/3	1	2/9	0	0.4	1/3	1	2/9	0
Ω	0.2			1/9	5/9	0.2			1/9	5/9

It is clear that those two rules belong to different families of combination rules: the $\textcircled{\cap}$ rule is purely conjunctive whereas the $\textcircled{\wedge}$ rule is both conjunctive and disjunctive [4], hence they treat the diffidence component in different ways. The remainder of this section is devoted to the proposal of other purely conjunctive rules. One of those rules is particularly interesting since it is idempotent; the motivation for its definition relies on the *least commitment principle* of the TBM.

3.1 Least Commitment Principle (LCP)

The LCP is similar to the principle of maximum entropy in Bayesian Probability Theory. It postulates that given a set $\mathcal{M}$ of BBAs compatible with a set of constraints, the most appropriate BBA is the least informative. This principle becomes operational through the definition of partial orders allowing the informational comparison of BBAs. Such orders, generalizing set inclusion, are [5]:

- *pl*-order: for all $A \subseteq \Omega$, iff $pl_1(A) \leq pl_2(A)$ then $m_1 \sqsubseteq_{pl} m_2$;
- *q*-order: for all $A \subseteq \Omega$, iff $q_1(A) \leq q_2(A)$ then $m_1 \sqsubseteq_q m_2$;
- *s*-order: $m_1 \sqsubseteq_s m_2$, i.e. m_1 is a *specialization* of m_2, iff there exists a square matrix S with general term $S(A,B)$, $A, B \subseteq \Omega$ such that $\sum_{B \subseteq \Omega} S(A,B) = 1$, $\forall A \subseteq \Omega$, and $S(A,B) > 0 \Rightarrow A \subseteq B, \forall A, B \subseteq \Omega$, and $m_1(A) = \sum_{B \subseteq \Omega} S(A,B)\, m_2(A)$, $\forall A \subseteq \Omega$.

A BBA m_1 is said to be *x-more committed* than m_2, with $x \in \{pl, q, s\}$, if $m_1 \sqsubseteq_x m_2$. A particular case of specialization is the dempsterian specialization [9], noted $\sqsubseteq_d$: $m_1 \sqsubseteq_d m_2$, iff there exists a BBA m such that $m_1 = m \textcircled{\cap} m_2$. This condition is stronger than specialization, i.e. $m_1 \sqsubseteq_d m_2 \Rightarrow m_1 \sqsubseteq_s m_2$.

It is reasonable to say that a SBBA A^{w_1} is more committed than a SBBA A^{w_2}, if $w_1 \leq w_2$. Hence a BBA m_1 obtained from the combination by $\textcircled{\cap}$ of SBBAs, i.e. a separable BBA, will be more committed than another separable BBA m_2 if $w_1(A) \leq w_2(A)$ for all $A \in 2^{\Omega} \setminus \{\Omega\}$; this is equivalent to the existence of a separable BBA m such that $m_1 = m \textcircled{\cap} m_2$. This new partial order, defined for separable BBAs and noted $m_1 \sqsubseteq_w m_2$ with m_1 and m_2 two separable BBAs, is consequently stricly stronger than d-ordering as a non-separable BBA m such that $m_1 = m \textcircled{\cap} m_2$, i.e. $m_1 \sqsubseteq_d m_2$, can easily be found. Let us also remark that, using a special representation of categorical BBAs, Denoeux [4, Proposition 3] has shown that $\sqsubseteq_w$ may be seen as generalizing set inclusion, much as the x-orderings, with $x \in \{pl, q, s\}$, do.

All those partial orders can be extended to LBSs. In particular, the LBS $L_{1\circledcap 2} = (m^c_{1\circledcap 2}, m^d_{1\circledcap 2})$ obtained from the combination by $\circledcap$ of two LBSs $L_1 = (m^c_1, m^d_1)$ and $L_2 = (m^c_2, m^d_2)$ has the following properties: for $i \in \{1,2\}$, $m^c_{1\circledcap 2} \sqsubseteq_w m^c_i$, and $m^d_{1\circledcap 2} \sqsubseteq_w m^d_i$, i.e. the $\circledcap$ rule produces a LBS $L_{1\circledcap 2}$ which is both w-more committed in confidence and in diffidence than the LBSs L_1 and L_2. To simplify the presentation, a LBS L which is both w-more committed in confidence and in diffidence than a LBS $L^{'}$ will be noted $L \sqsubseteq_l L^{'}$ (l stands for *latent*). Informally, the l-order seems natural as it exhibits properties similar to those of classical orderings. To see that, simply replace l by x with $x \in \{d, s, q, pl\}$ and L by m in the following expressions: $\forall L, L \sqsubseteq_l L_\Omega$ and $L_{1\circledcap 2} \sqsubseteq_l L_1$, $L_{1\circledcap 2} \sqsubseteq_l L_2$.

3.2 Combination of non-distinct LBSs

As remarked in [5], it is possible to think of $\sqsubseteq_x$ as generalizing set inclusion. This reasoning can be used to see conjunctive combination rules as generalizing set intersection. Denœux [3] considers thus the following situation. Suppose we get two reliable sources of information. One states that ω is in $A \subseteq \Omega$, whereas the other states that it is in $B \subseteq \Omega$. It is then certain that ω is in C such that $C \subseteq A$ and $C \subseteq B$. The largest subset C satisfying those constraints is $A \cap B$.

Suppose now that the sources provide the NDBBAs m_1 and m_2 and let L_1 and L_2 be the equivalent CLBSs of those NDBBAs. Upon receiving those two pieces of information, the agent's state of belief should be represented by a LBS L_{12}, i.e. $\left(m^c_{12}, m^d_{12}\right)$, more informative than L_1 and L_2. Let $\mathcal{S}_x(L)$ be the set of LBSs L' such that $L' \sqsubseteq_x L$. Hence $L_{12} \in \mathcal{S}_x(L_1)$ and $L_{12} \in \mathcal{S}_x(L_2)$, or equivalently $L_{12} \in \mathcal{S}_x(L_1) \cap \mathcal{S}_x(L_2)$. According to the LCP, the x-least committed LBS should be chosen in $\mathcal{S}_x(L_1) \cap \mathcal{S}_x(L_2)$. This defines a conjunctive combination rule if the x-least committed LBS exists and is unique. Proposition 1 shows that the l-order may be an interesting solution for this problem.

Proposition 1. *Let L_1 and L_2 be two LBSs. The l-least committed element in $\mathcal{S}_l(L_1) \cap \mathcal{S}_l(L_2)$ exists and is unique (the proof is trivial by Proposition 1 of [3]). It is defined by the following confidence and diffidence weight functions:*

$$w^c_{1\circledwedge 2}(A) = w^c_1(A) \wedge w^c_2(A), \quad A \in 2^\Omega \setminus \{\Omega\}, \tag{19}$$

$$w^d_{1\circledwedge 2}(A) = w^d_1(A) \wedge w^d_2(A), \quad A \in 2^\Omega \setminus \{\Omega\}. \tag{20}$$

Definition 6 (Weak Rule). *Let L_1 and L_2 be two LBSs. Their combination with the weak rule is defined as the LBS whose weight functions are given by (19) and (20). It is noted: $L_{1\circledwedge 2}$.*

This rule is commutative, associative and idempotent. In addition, $\circledcap$ is distributive with respect to $\circledwedge$. Those properties originate from the properties of the $\owedge$ rule [3] since there is a connection between the partial orders on which those two rules are built. We can thus see that the combination by the $\circledwedge$ rule consists in combining the confidence and diffidence components by the $\owedge$ rule.

The $\circledwedge$ rule exhibits other properties: L_Ω is a neutral element and if $L_1 \sqsubseteq_l L_2$, the result of the least committed combination of those LBSs is $L_1 \circledwedge L_2 = L_1$. Further, using the l-order in the derivation of the rule allows the construction of a 'weaker', or l-less committed, version of Dempster's rule, i.e. $L_1 \circledcap L_2 \sqsubseteq_l L_1 \circledwedge L_2$.

Note that the apparent form of a LBS $L_{1 \circledwedge 2}$, produced by the $\circledwedge$ combination of two CLBSs L_1 and L_2 obtained from two NDBBAs m_1 and m_2, may not be a BBA. However, if m_1 and m_2 are separable BBAs then the apparent form of the LBS $L_{1 \circledwedge 2}$ is a BBA since a separable BBA yields a LBS whose diffidence component is vacuous and the $\circledwedge$ combination consists in combining the confidence component of L_1 and L_2 by the $\owedge$ rule. It can also be shown that the combination by the $\owedge$ rule of two consonant BBAs does not always yield a consonant BBA. A consonant BBA is separable [4, Proposition 2], hence the $\circledwedge$ rule applied to two CLBSs obtained from two consonant BBAs will yield a LBS whose apparent form is a separable BBA which is not necessarily consonant.

Example 2. The left-hand part of Table 2 shows the weight functions resulting from the weak ($\circledwedge$), the conjunctive ($\circledcap$), and the cautious ($\owedge$) combinations of the expert opinions of Example 1. Note that $w^d_{1 \owedge 2}(A) = 1$, for all $A \subset \Omega$.

Table 2. Weight functions obtained from different combinations (left). Plausibility transformations of the LBSs obtained with those combinations (right, see Section 4).

A	$w^c_{1 \circledwedge 2}(A)$	$w^d_{1 \circledwedge 2}(A)$	$w^c_{1 \circledcap 2}(A)$	$w^d_{1 \circledcap 2}(A)$	$w^c_{1 \owedge 2}(A)$	$PlP_{1 \circledwedge 2}$	$PlP_{1 \circledcap 2}$	$PlP_{1 \owedge 2}$
$\emptyset$	1	1	1	1	1			
$\{a\}$	1	1	1	1	1	9/19	0.23	1/3
$\{b\}$	1	5/9	1	5/9	1	5/19	0.385	1/3
$\{a,b\}$	1/3	1	1/3	1	1/3			
$\{c\}$	1	5/9	1	5/9	1	5/19	0.385	1/3
$\{a,c\}$	1/3	1	1/3	1	1/3			
$\{b,c\}$	1/3	1	1/9	1	1/3			

Interestingly, the idea of distinctness conveyed by the derivation of the $\circledwedge$ rule relates, in part, to the foci of the SBBAs underlying a complex belief state. Indeed let us assume that two bodies of evidence, yielding the LBSs L_1 and L_2, are non distinct, then $L_{1 \circledwedge 2} \neq L_{1 \circledcap 2}$ iff $C_1 \cap C_2 \neq \emptyset$ or $D_1 \cap D_2 \neq \emptyset$, with $C_i = \{A : A \subset \Omega, w^c_i(A) < 1\}$ and $D_i = \left\{A : A \subset \Omega, w^d_i(A) < 1\right\}$. The effect of this view of distinctness is illustrated in Example 3 of Section 4, where double counting the SBBA implicitly shared by two agents is avoided.

3.3 Generalizing the Weak Rule

In the same vein as Denœux [3], it is possible to derive infinite families of conjunctive combination rules for LBSs. The $\circledcap$ and $\circledwedge$ rules are then merely instances of these families. This extension is based on the observation that the $\circledcap$ rule uses the product, whereas the $\circledwedge$ rule uses the minimum of weights belonging to the unit interval. Now, these two operations on this interval are binary operators known as triangular norms (t-norms). Replacing them by any positive t-norm $\top$ yields $\circledtop$ operators, which possess the following properties: commutativity, associativity, neutral element L_Ω and monotonicity with respect to $\sqsubseteq_l$, i.e. $\forall L_1, L_2$ and L_3, $L_1 \sqsubseteq_l L_2 \Rightarrow L_1 \circledtop L_3 \sqsubseteq_l L_2 \circledtop L_3$. Only the $\circledwedge$ rule is idempotent. Operators exhibiting a behavior between $\circledcap$ and $\circledwedge$ can be obtained using parameterized families of t-norms such as the Dubois and Prade family defined by:

$$x \top^{DP}_{\gamma} y = (xy) / (\max(x, y, \gamma)) \text{ for } x, y, \text{ and } \gamma \in [0, 1]. \tag{21}$$

Note that the $\textcircled{$\cap$}$ and $\textcircled{$\wedge\!\!\wedge$}$ rules are recovered for $\gamma = 1$ and $\gamma = 0$ respectively. The parameterization is what make those rules attractive: they allow the fine-tuning of the behavior of a system. Indeed, the γ parameter may be related to some subjective judgment on the distinctness of the items of evidence. It can also be learnt from data as done in [12] through the use of the plausibility transformation extended to LBSs (see Section 4): the conjunctive combination of two LBSs is then optimized with respect to the decisions inferred.

4 Decision Making with LBSs

This section provides a means to transform a LBS into a probability distribution. The plausibility transformation is of particular interest here due to one of its properties: it is invariant with respect to the combination by $\textcircled{$\cap$}$ [18], which is not the case of the pignistic transformation. Proposition 2 reformulates this property for the $\textcircled{$\not\cap$}$ rule using the decombination operator in probability theory, noted $\oslash$ and defined in [15] as follows. Let P_1 and P_2 be two probability distributions:

$$P_1 \oslash P_2 \left(\{\omega_k\}\right) = \kappa^{-1} P_1 \left(\{\omega_k\}\right) / P_2 \left(\{\omega_k\}\right) \quad , \forall \omega_k \in \mathit{\Omega} \tag{22}$$

with $\kappa = \sum_{j=1}^{K} P_1 \left(\{\omega_j\}\right) / P_2 \left(\{\omega_j\}\right)$.

Proposition 2 (*PlP* is invariant with respect to $\textcircled{$\not\cap$}$). *Let m_1 and m_2 be two NDBBAs:*

$$PlP_{m_1 \textcircled{$\not\cap$} m_2} = PlP_{m_1} \oslash PlP_{m_2} \quad . \tag{23}$$

Proof. For all $\omega_k \in \mathit{\Omega}$, let us denote $\alpha_k = pl_1 \left(\{\omega_k\}\right) = q_1 \left(\{\omega_k\}\right),\ \beta_k = pl_2 \left(\{\omega_k\}\right)$. From Equation (13) we have:

$$PlP_{m_1 \textcircled{$\not\cap$} m_2} \left(\{\omega_k\}\right) = \left(\alpha_k / \beta_k\right) / (\sum_{i=1}^{K} \left(\alpha_i / \beta_i\right)) \quad . \tag{24}$$

Besides,

$$PlP_{m_1} \oslash PlP_{m_2} \left(\{\omega_k\}\right) = \left(\left(\frac{\alpha_k}{\sum_{i=1}^{K} \alpha_i} \right) / \left(\frac{\beta_k}{\sum_{i=1}^{K} \beta_i} \right) \right) / \left(\sum_{j=1}^{K} \frac{\left(\frac{\alpha_j}{\sum_{i=1}^{K} \alpha_i} \right)}{\left(\frac{\beta_j}{\sum_{i=1}^{K} \beta_i} \right)} \right) \tag{25}$$

(24) and (25) are equal. □

Using Proposition 2, a LBS $L = \left(m^c, m^d\right)$ can be transformed into a probability distribution as follows:

$$PlP_L = PlP_{m^c} \oslash PlP_{m^d} \quad . \tag{26}$$

Example 3. The right side of Table 2 shows three qualitatively different probability distributions computed using (26). They are obtained from the expert opinions of Example 1 combined with different combination rules. It is interesting to note that the application of the $\textcircled{\cap}$ and $\textcircled{\wedge\!\!\wedge}$ rule yields opposite decisions. This is easily explained through the observation of the SBBAs underlying the two opinions. We see in particular that both opinions share a SBBA focused on the set $\{b, c\}$: if we think of the opinions as based on distinct bodies of evidence, then the reasons for which expert 1 believes in $\{b, c\}$ are different from the reasons of expert 2, hence the combined belief in favor of the set $\{b, c\}$ should be stronger than the individual beliefs. On the other hand, if the experts base their beliefs in $\{b, c\}$ on the same items of evidence then the combined belief in favor of the set $\{b, c\}$ should not be stronger than the individual beliefs. Consequently with the $\textcircled{\cap}$ rule we have $w(\{b, c\}) < w(\{a, c\}) = w(\{a, b\})$ which makes the two singletons b and c more probable, actually much more probable than a. This difference is then only partially moderated by the diffidence in b and c so that eventually b and c remain more probable than a. However, with the $\textcircled{\wedge\!\!\wedge}$ and the $\textcircled{\wedge}$ rules, we have $w(\{b, c\}) = w(\{a, c\}) = w(\{a, b\})$, which yields equiprobability for the three singletons. Besides, the $\textcircled{\wedge\!\!\wedge}$ rule keeps the information relating to the diffidence in b and c, hence a is more probable than b and c with this rule.

Proposition 3 shows that two ways of modeling negative statements become equivalent when PlP is used. Indeed, according to Smets's vocabulary [16], for $A \subset \Omega$, *having good reasons to believe in not* A is equivalent to *having good reasons not to believe (or having a debt of belief) in* A. It can also be formulated using the terminology used in belief revision: the expansion by $\bar{A}^{\alpha}$ is equivalent to the contraction by A^{α}, for $\alpha \in (0, 1]$. Let $\overset{PlP}{\sim}$ denote the equivalence relation between LBSs defined by $L_1 \overset{PlP}{\sim} L_2$ iff $PlP_{L_1}(\{\omega_k\}) = PlP_{L_2}(\{\omega_k\})$, $\forall \omega_k \in \Omega$.

Proposition 3. $\bar{A}^{\alpha} \overset{PlP}{\sim} A^{\frac{1}{\alpha}}$, *for* $\alpha \in (0, 1]$.

Proof. $\forall \omega_k \in A, A \subset \Omega$,

$$PlP_{\bar{A}^{\alpha}}(\{\omega_k\}) = \frac{\alpha}{|\bar{A}| + |A|\,\alpha} \enspace , \tag{27}$$

$$PlP_{A^{1/\alpha}}(\{\omega_k\}) = \frac{1}{|A| + |\bar{A}|\,\frac{1}{\alpha}} \enspace . \tag{28}$$

(27) and (28) are equal. □

Propositions 2 and 3 define equivalence classes with respect to the plausibility transformation in which there is at least one separable BBA; for instance we have: $\left(\bar{A}^{0.6}, A^{0.5}\right) \overset{PlP}{\sim} \left(\bar{A}^{0.3}, \Omega\right)$. Note also that the combination by $\textcircled{\cap}$ of any two LBSs belonging to two different equivalence classes always falls in the same equivalence class, for instance if $L_1 \overset{PlP}{\sim} L_2$ and $L_3 \overset{PlP}{\sim} L_4$, then e.g. $L_1 \textcircled{\cap} L_3 \overset{PlP}{\sim} L_2 \textcircled{\cap} L_4$. It can easily be shown that this is not true for the $\textcircled{\wedge}$ and $\textcircled{\wedge\!\!\wedge}$ rules.

Eventually, from Proposition 3, remark that $A^{\alpha} \textcircled{\cap} \bar{A}^{\alpha} \overset{PlP}{\sim} A^{\alpha} \textcircled{\not\cap} A^{\alpha}, \forall A \subseteq \Omega$ with $\alpha \in (0, 1]$. Now, let $\overset{BetP}{\sim}$ denote the equivalence relation between BBAs defined by $m_1 \overset{BetP}{\sim} m_2$ iff $BetP_{m_1}(\{\omega_k\}) = BetP_{m_2}(\{\omega_k\})$, for all $\omega_k \in \Omega$. The two ways of modeling negative statements will yield the same probability distribution, i.e. $A^{\alpha} \textcircled{\cap} \bar{A}^{\alpha} \overset{BetP}{\sim} A^{\alpha} \textcircled{\not\cap} A^{\alpha}$, with the pignistic transformation iff $|A| = |\bar{A}|$; a stricter condition than the one of the plausibility transformation.

5 Conclusion

In this article, latent belief structures have been revisited. The mathematical simplicity of this generalization of non dogmatic belief functions has allowed the analysis of the unnormalized version of Dempster's rule, which resulted in the introduction of infinite families of conjunctive combination rules. Two potential uses of these rules have been proposed. First they may permit to relax the hypothesis of distinctness inherent to the use of Dempster's rule. Second, they may be used to optimize the combination of imperfect information with respect to the decisions inferred. An extension of the plausibility transformation has been also provided to transform a LBS into a probability distribution and two ways of modeling negative statements were proved equivalent under this extension. The interest of this formalism in concrete applications is currently being investigated.

References

1. A. Ben Yaghlane, T. Denoeux and K. Mellouli. Elicitation of expert opinions for constructing belief functions. In Proc. of IPMU'2006, Vol.1, (2006) 403–411.
2. B. R. Cobb and P. P. Shenoy. On the plausibility transformation method for translating belief function models to probability models. International Journal of Approximate Reasoning, **41** (2006) 314–330.
3. T. Denœux. The cautious rule of combination for belief functions and some extensions. In Proceedings of FUSION'2006, Italy, (2006) 1–8.
4. T. Denœux. Conjunctive and Disjunctive Combination of Belief Functions Induced by Non Distinct Bodies of Evidence. doi:10.1016/j.artint.2007.05.008, Artificial Intelligence, (2007).
5. D. Dubois and H. Prade. A set-theoretic view of belief functions: logical operations and approximations by fuzzy sets. Int. J. of General Systems, **12** (1986) 193–226.
6. D. Dubois and H. Prade. Representation and combination of uncertainty with belief functions and possibility measures. Computational Intelligence, **4** (1998) 244–264.
7. D. Dubois, H. Prade and Ph. Smets. "Not Impossible" vs. "Guaranteed Possible" in Fusion and Revision. In Proc. of ECSQARU'01, Springer-Verlag, (2001) 522–531.
8. P. Gärdenfors. Knowledge in Flux: Modeling the Dynamics of Epistemic States. MIT Press, Cambridge, Mass, (1988).
9. F. Klawonn and Ph. Smets. The dynamic of belief in the transferable belief model and specialization-generalization matrices. In Proc. of UAI'1992, (1992) 130–137.
10. I. Kramosil. Probabilistic Analysis of Belief Functions. Kluwer Acad. Pub., (2001).
11. Ch. Labreuche and M. Grabisch. Modeling positive and negative pieces of evidence in uncertainty. In Proc. of ECSQARU 2003, Springer-Verlag, (2003) 279–290.
12. D. Mercier, B. Quost and T. Denoeux. Refined modeling of sensor reliability in the belief function framework using contextual discounting. Information Fusion, (2006) doi:10.1016/j.inffus.2006.08.001.
13. L. J. Savage. Foundation of statistics, Wiley, New York, (1954).
14. G. Shafer. A mathematical theory of evidence. Princeton University Press, (1976).
15. P. P. Shenoy. Conditional independence in valuation-based systems. International Journal of Approximate Reasoning, **10** (1994) 203–234.
16. Ph. Smets. The canonical decomposition of a weighted belief. In Int. Joint Conf. on Artificial Intelligence, San Mateo, Ca, Morgan Kaufman, (1995) 1896–1901.
17. Ph. Smets. The Transferable Belief Model for quantified belief representation. In D. Gabbay and Ph. Smets (eds), Handbook of Defeasible reasoning and uncertainty management systems Vol.1, Kluwer Academic Pub., Dordrecht, (1998) 264–301.
18. F. Voorbraak. A computationally efficient approximation of Dempster-Shafer theory. Int. Journal of Man-Machine Studies, **30** (1989) 525–536.