A New Justification of the Unnormalized Dempster's Rule of Combination from the Least Commitment Principle

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Abstract

The conjunctive weight function is an equivalent representation of a non dogmatic belief function. Denœux recently proposed new rules of combination for belief functions based on pointwise combination of conjunctive weights. This paper characterizes the rules of combination based on the conjunctive weight function that have the vacuous belief function as neutral element. The main result is that the unnormalized Dempster's rule is the least committed rule amongst those rules, for a particular informational ordering. A counterpart to this result is also presented for the disjunctive rule.

Introduction

The Transferable Belief Model (TBM) (Smets & Kennes 1994; Smets 1998) is a model for quantifying beliefs using belief functions (Shafer 1976). An essential mechanism of the TBM is the unnormalized Dempster's rule of combination. This rule, referred to as the TBM conjunctive rule in this paper, allows the fusion of belief functions. Dempster's rule and the TBM conjunctive rule have been justified by several authors. In particular, Dubois and Prade (1986a) proved the unicity of Dempster's rule under an independence assumption. Klawonn and Smets (1992) took another path and justified the TBM conjunctive rule as being the only combination that results from an associative, commutative and least committed specialization.

A limitation, which applies to both rules, is the requirement that the items of evidence combined be distinct, or in other words, that the information sources be independent. Recently, Denœux (2008) proposed a rule, called the cautious rule of combination, which does not rely on the distinctness assumption. The term cautious is reminiscent of the derivation of the rule, which is based on the least commitment principle (LCP) (Smets 1993). The LCP stipulates that one should never give more beliefs than justified by the available information, hence it promotes a cautious attitude. The cautious rule is based on the conjunctive weight function, an equivalent representation of a non dogmatic belief function. The TBM conjunctive rule can also be expressed using the conjunctive weight function, which makes it interesting to study rules based on this function.

There are important differences between the TBM conjunctive rule and the cautious rule: the cautious rule is idempotent but does not have a neutral element, whereas the TBM conjunctive rule has a neutral element, the vacuous belief function, but is not idempotent. The lack of a neutral element for the cautious rule can be somewhat disturbing, hence the question: does there exist a rule based on the conjunctive weight function, which is more "cautious" than the TBM conjunctive rule and which admits the vacuous belief function as neutral element? This paper shows that the answer is no, which can be seen as a new justification of the unnormalized Dempster's rule as the rule thus respects a central principle of the TBM.

Denœux (2008) further showed that the cautious rule belongs to an infinite family of combination rules. Besides, the cautious rule is the least committed rule in this family. Interestingly, this paper shows that a similar property holds for the TBM conjunctive rule: it belongs to an infinite family of rules that admits the vacuous belief function as neutral element and it is the least committed rule in this family. The fundamental difference between those families is the existence of a neutral element.

The rest of this paper is organized as follows. Necessary notions, such as the canonical decomposition of a belief function and the LCP, are first recalled in Section 2. Section 3 reviews existing rules of combination based on pointwise combination of conjunctive weights. The main result of this paper is given in Section 4. Section 5 presents results corresponding to the previous ones for rules based on the disjunctive weight function. Section 6 concludes the paper.

Fundamental Concepts of the TBM

Basic Definitions and Notations

In this paper, the TBM (Smets & Kennes 1994; Smets 1998) is accepted as a model to quantify uncertainties based on belief functions (Shafer 1976). The beliefs held by an agent Ag on a finite frame of discernment $\Omega = \{\omega_1, ..., \omega_K\}$ are represented by a basic belief assignment (BBA) m defined as a mapping from 2^Ω to $[0, 1]$ verifying $\sum_{A\subseteq\Omega} m(A) = 1$. Subsets A of Ω such that $m(A) > 0$ are called focal sets (FS) of m. A BBA m is said to be:

- normal if $\emptyset$ is not a focal set;
- subnormal if $\emptyset$ is a focal set;

- vacuous if Ω is the only focal set, this BBA is noted m_Ω;
- dogmatic if Ω is not a focal set;
- categorical if it has only one focal set;
- simple if if has at most two focal sets and, if it has two, Ω is one of those.

A simple BBA (SBBA) m such that $m(A) = 1 - w$ for some $A \neq \Omega$ and $m(\Omega) = w$ can be noted A^w. The vacuous BBA can thus be noted A^1 for any $A \subset \Omega$. The advantage of this notation will appear later.

Equivalent representations of a BBA m exist. In particular the implicability and commonality functions are defined, respectively, as:

$$b(A) = \sum_{B \subseteq A} m(B),$$

and

$$q(A) = \sum_{B \supseteq A} m(B),$$

for all $A \subseteq \Omega$. The BBA m can be recovered from any of these functions, for instance:

$$m(A) = \sum_{B \supseteq A} (-1)^{|B|-|A|} q(B),$$

for all $A \subseteq \Omega$ and where $|A|$ denotes the cardinality of A.

The negation (or complement) $\overline{m}$ of a BBA m is defined as the BBA verifying $\overline{m}(A) = m(\overline{A}), \forall A \subseteq \Omega$, where $\overline{A}$ denotes the complement of A (Dubois & Prade 1986b). It can be shown that the implicability function $\bar{b}$ associated to $\overline{m}$ and the commonality function q associated to m are linked by the following relation:

$$\bar{b}(A) = q(\overline{A}), \quad \forall A \subseteq \Omega.$$

The TBM conjunctive rule is noted $\textcircled{\cap}$. It is defined as follows. Let m_1 and m_2 be two BBAs, and let $m_{1\textcircled{\cap}2}$ be the result of their combination by $\textcircled{\cap}$. We have:

$$m_{1\textcircled{\cap}2}(A) = \sum_{B \cap C = A} m_1(B)\, m_2(C), \forall A \subseteq \Omega.$$

This rule is commutative, associative and admits a unique neutral element: the vacuous BBA. Let A^{w_1} and A^{w_2} be two SBBAs. Their combination by $\textcircled{\cap}$ is the SBBA $A^{w_1 w_2}$. The TBM conjunctive rule also has a simple expression in terms of commonality functions. We have:

$$q_{1\textcircled{\cap}2}(A) = q_1(A) \cdot q_2(A), \forall A \subseteq \Omega.$$

A disjunctive rule $\textcircled{\cup}$ also exists (Dubois & Prade 1986b; Smets 1993). It is defined as:

$$m_{1\textcircled{\cup}2}(A) = \sum_{B \cup C = A} m_1(B)\, m_2(C), \forall A \subseteq \Omega.$$

This rule, called the TBM disjunctive rule, has a simple expression in terms of implicability functions:

$$b_{1\textcircled{\cup}2}(A) = b_1(A) \cdot b_2(A), \forall A \subseteq \Omega.$$

The TBM disjunctive rule is commutative, associative and admits a unique neutral element: the BBA which assigns the total mass of belief to the empty set, i.e. $m(\emptyset) = 1$. This BBA, which we note $m_\emptyset$, is the negation of the neutral BBA m_Ω of the TBM conjunctive rule.

The dual nature of $\textcircled{\cap}$ and $\textcircled{\cup}$ becomes apparent when one notices that these operators are linked by De Morgan's laws:

$$\begin{aligned} \overline{m_1 \textcircled{\cup} m_2} &= \overline{m_1} \textcircled{\cap} \overline{m_2} \\ \overline{m_1 \textcircled{\cap} m_2} &= \overline{m_1} \textcircled{\cup} \overline{m_2} \end{aligned}$$

As remarked by Smets (1993), the TBM conjunctive rule is based on the assumption that the belief functions to be combined are induced by reliable sources of information, whereas the TBM disjunctive rule only assumes that at least one source of information is reliable, but we do not know which one. Both rules assume the sources of information to be independent (i.e., they are assumed to provide distinct, non overlapping pieces of evidence).

Let us now assume that $m_{1\textcircled{\cap}2}$ has been obtained by combining two BBAs m_1 and m_2, and then we learn that m_2 is in fact not supported by evidence and should be "removed" from $m_{1\textcircled{\cap}2}$. This operation is called decombination in (Smets 1995). It is well defined if m_2 is non dogmatic. Let $\textcircled{\oslash}$ denote this operator. We can write $m_{1\textcircled{\cap}2} \textcircled{\oslash} m_2 = m_1$. Let q_1 and q_2 be the commonality functions of two non dogmatic BBAs m_1 and m_2, the decombination is defined by:

$$q_{1\textcircled{\oslash}2}(A) = \frac{q_1(A)}{q_2(A)}, \ \forall A \subseteq \Omega.$$

As the quotient of two commonality functions is not always a commonality function, one should be aware that $m_1 \textcircled{\oslash} m_2$ is not necessarily a BBA.

Canonical Decomposition of a Belief Function

In (Smets 1995), Smets proposed a solution to canonically decompose any non dogmatic BBA. This decomposition uses the concept of a generalized SBBA (GSBBA) which is defined as a function μ from 2^Ω to $\mathbb{R}$ by:

$$\begin{aligned} \mu(A) &= 1 - w, \\ \mu(\Omega) &= w, \\ \mu(B) &= 0 \ \forall B \in 2^\Omega \setminus \{A, \Omega\}, \end{aligned}$$

for some $A \neq \Omega$ and some $w \in [0, +\infty)$. Extending the SBBA notation, any GSBBA can be noted A^w. When $w \leq 1$, μ is a SBBA. When $w > 1$, μ is no longer a BBA; Smets (1995) called such function an inverse SBBA.

Smets showed that any non dogmatic BBA m can be uniquely represented as the conjunctive combination of generalized SBBAs:

$$m = \textcircled{\cap}_{A \subset \Omega} A^{w(A)},$$

with $w(A) \in (0, +\infty)$ for all $A \subset \Omega$. The weights $w(A)$ for each $A \subset \Omega$ are obtained as follows:

$$w(A) = \prod_{B \supseteq A} q(B)^{(-1)^{|B|-|A|+1}}.$$

If the weights are such that $w(A) \leq 1$ for each $A \subset \Omega$, then m is said to be separable. The function $w : 2^\Omega \setminus \{\Omega\} \rightarrow$

$(0, +\infty)$ is called the conjunctive weight function. It is another equivalent representation of a non dogmatic BBA m.

Using a similar reasoning, Denœux (2008) showed that any subnormal BBA m can be uniquely represented as the disjunctive combination of *negative* GSBBAs. A negative GSBBA is a function from 2^Ω to $\mathbb{R}$ assigning a mass $v \geq 0$ to $\emptyset$, a mass $1 - v$ to $A \neq \emptyset$, and a zero mass for all $B \in 2^\Omega \setminus \{A, \emptyset\}$; this function is noted A_v. We have thus, for any subnormal BBA m:

$$m = \bigcirc\!\!\!\!\cup_{A \neq \emptyset} A_{v(A)},$$

with $v(A) \in (0, +\infty)$ for all $A \neq \emptyset$. The weights $v(A)$ for each $A \neq \emptyset$ are obtained as follows:

$$v(A) = \prod_{B \subseteq A} b(B)^{(-1)^{|A|-|B|+1}}.$$

The function $v : 2^\Omega \setminus \{\emptyset\} \to (0, +\infty)$ is called the disjunctive weight function. This function is related to the conjunctive weight function $\overline{w}$ associated to the negation $\overline{m}$ of m by the equation

$$v(A) = \overline{w}(\overline{A}), \ \forall A \neq \emptyset.$$

The TBM conjunctive and disjunctive rules have simple expressions in terms of weight functions (Denœux 2008). Let m_1 and m_2 be two non dogmatic BBAs with conjunctive weight functions w_1 and w_2. We have:

$$m_{1 \bigcirc\!\!\!\!\cap 2} = \bigcirc\!\!\!\!\cap_{A \subset \Omega} A^{w_1(A) \cdot w_2(A)}.$$

Now, let m_1 and m_2 be two subnormal BBAs with disjunctive weight functions v_1 and v_2. We have:

$$m_{1 \bigcirc\!\!\!\!\cup 2} = \bigcirc\!\!\!\!\cup_{A \neq \emptyset} A_{v_1(A) \cdot v_2(A)}.$$

Informational Comparison of Belief Functions

The least commitment principle (LCP) of the TBM postulates that, given a set of BBAs compatible with a set of constraints, the most appropriate BBA is the least informative (Smets 1993). It is thus somewhat similar to the principle of minimal specificity in possibility theory. The LCP becomes operational through the definition of partial orderings allowing the informational comparison of BBAs. Such orders, generalizing set inclusion, were proposed by Yager (1986), and Dubois and Prade (1986b). Recently, Denœux (2008) proposed two new partial orderings. They are defined as follows:

- w-ordering: given two non dogmatic BBAs m_1 and m_2, $m_1 \sqsubseteq_w m_2$ iff $w_1(A) \leq w_2(A)$ for all $A \subset \Omega$;
- v-ordering: given two subnormal BBAs m_1 and m_2, $m_1 \sqsubseteq_v m_2$ iff $v_1(A) \geq v_2(A)$ for all $A \neq \emptyset$;

A BBA m_1 is said to be x-more committed than a BBA m_2, with $x \in \{w, v\}$, if we have $m_1 \sqsubseteq_x m_2$.

Rules Based on Weight Functions

We have seen that the TBM conjunctive and disjunctive rules are based on pointwise multiplication of weights. Those rules are justified only when it is safe to assume that the items of evidence combined are distinct. When this assumption does not hold, an alternative consists in adopting a cautious, or conservative, attitude to the merging of belief functions by applying the LCP (Dubois, Prade, & Smets 2001; Denœux 2008; Destercke, Dubois, & Chojnacki 2007).

Let us recall the building blocks of the cautious conjunctive merging of belief functions. Suppose we get two reliable sources of information which provide two BBAs m_1 and m_2. Upon receiving those two pieces of information, the agent's state of belief should be represented by a BBA m_{12} more informative than m_1 and m_2. Let $\mathcal{S}_x(m)$ be the set of BBAs m' such that $m' \sqsubseteq_x m$, for some $x \in \{v, w\}$. Hence $m_{12} \in \mathcal{S}_x(m_1)$ and $m_{12} \in \mathcal{S}_x(m_2)$ or, equivalently, $m_{12} \in \mathcal{S}_x(m_1) \cap \mathcal{S}_x(m_2)$. According to the LCP, the x-least committed BBA should be chosen in $\mathcal{S}_x(m_1) \cap \mathcal{S}_x(m_2)$. This defines a conjunctive combination rule if the x-least committed BBA exists and is unique.

Choosing the w-ordering yields an interesting solution (Denœux 2008, Proposition 4) which Denœux uses to define the so-called cautious rule.

Definition 1 (Definition 1 of (Denœux 2008))**.** *Let m_1 and m_2 be two non dogmatic BBAs, and let $m_{1 \bigcirc\!\!\!\!\wedge 2} = m_1 \bigcirc\!\!\!\!\wedge m_2$ denote the result of their combination by the cautious rule. The conjunctive weight function of the BBA $m_{1 \bigcirc\!\!\!\!\wedge 2}$ is:*

$$w_{1 \bigcirc\!\!\!\!\wedge 2}(A) = w_1(A) \wedge w_2(A), \forall A \subset \Omega,$$

where $\wedge$ denotes the minimum operator. We thus have:

$$m_{1 \bigcirc\!\!\!\!\wedge 2} = \bigcirc\!\!\!\!\cap_{A \subset \Omega} A^{w_1(A) \wedge w_2(A)}.$$

The cautious rule is idempotent, which makes it suitable to combine belief functions induced by non distinct pieces of evidence. The cautious rule is also commutative, associative, and increasing with respect to the $\sqsubseteq_w$ ordering: if $m_1 \sqsubseteq_w m_2$, then $m_1 \bigcirc\!\!\!\!\wedge m \sqsubseteq_w m_2 \bigcirc\!\!\!\!\wedge m$ for all m. These properties are due to similar properties of the minimum.

A similar approach can also be applied for disjunctive mergings in which case the resulting BBA m_{12} should be the x-most committed BBA amongst the BBAs which are x-less committed than m_1 and m_2, with $x \in \{v, w\}$ (Denœux 2008). Denœux showed that using the v-ordering yields an interesting solution, from which he defined an idempotent rule called the bold rule.

Definition 2 (Definition 2 of (Denœux 2008))**.** *Let m_1 and m_2 be two subnormal BBAs, and let $m_{1 \bigcirc\!\!\!\!\vee 2} = m_1 \bigcirc\!\!\!\!\vee m_2$ denote the result of their combination by the bold rule. The disjunctive weight function of the BBA $m_{1 \bigcirc\!\!\!\!\vee 2}$ is:*

$$v_{1 \bigcirc\!\!\!\!\vee 2}(A) = v_1(A) \wedge v_2(A), \forall A \neq \emptyset.$$

We thus have:

$$m_{1 \bigcirc\!\!\!\!\vee 2} = \bigcirc\!\!\!\!\cup_{A \neq \emptyset} A_{v_1(A) \wedge v_2(A)}.$$

The bold rule has similar properties as the cautious rule since they are both based on the minimum.

This latter fact leads us to two related observations using the following reasoning. It is well known that the minimum is a triangular norm, i.e., it is a commutative, associative and monotonic operator on $[0, 1]$, which satisfies the "boundary condition" (Klement, Mesiar, & Pap 2000, p.4), meaning that the upper bound of $[0,1]$ serves as a neutral element. Interestingly, the minimum satisfies the same properties on $(0,+\infty]$, with $+\infty$ serving as neutral element. This leads us to introduce the following definition.

Definition 3. *A t-norm on* $(0,+\infty]$ *is a binary operation on* $(0,+\infty]$*, which is commutative, associative, monotonic, and which admits* $+\infty$ *as neutral element.*

We can further remark that the minimum is the largest t-norm on $(0,+\infty]$, much as it is the largest t-norm on $[0,1]$.

Lemma 1. *The minimum is the largest t-norm on* $(0,+\infty]$.

Proof. Let $\star$ be any t-norm on $(0,+\infty]$. For all $x,y \in (0,+\infty]$, we have $x \star y \leq x \star +\infty = x$ and $x \star y \leq +\infty \star y = y$, so $x \star y \leq x \wedge y$, . □

From this later lemma and Lemma 2 immediately below, it may easily be shown that there exists an infinite family of combination rules based on pointwise combination of conjunctive weights using t-norms on $(0+\infty]$. Besides, the cautious rule is the least committed element of this family.

Lemma 2 (Lemma 1 of (Denœux 2008))**.** *Let* m *be a non dogmatic BBA with conjunctive weight function* w*, and let* w' *be a mapping from* $2^{\Omega}\backslash\Omega$ *to* $(0,+\infty)$ *such that* $w'(A) \leq w(A)$ *for all* $A \subset \Omega$*. Then* w' *is the conjunctive weight function of some BBA* m'.

Similarly, from Lemma 3 below and Lemma 1, it may easily be shown that the bold rule is the most committed element in the family of rules based on pointwise combination of disjunctive weights using t-norms on $(0,+\infty]$.

Lemma 3. *Let* m *be a subnormal BBA with disjunctive weight function* v*, and let* v' *be a mapping from* $2^{\Omega}\backslash\emptyset$ *to* $(0,+\infty)$ *such that* $v'(A) \leq v(A)$ *for all* $A \neq \emptyset$*. Then* v' *is the disjunctive weight function of some BBA* m'.

Proof. The proof is similar to the proof of Lemma 1 of (Denœux 2008). □

Main Result

The TBM conjunctive rule has the vacuous BBA as neutral element. This property is interesting in the context of conjunctive merging, as the vacuous BBA represents total ignorance. Rules that are based on t-norms on $(0,+\infty]$ do not possess this property. Hence, the TBM conjunctive rule does not belong to this latter family; it belongs to the family of rules based on pointwise combination of conjunctive weights and which admit the vacuous BBA as neutral element. This section studies this family.

It is clear that a combination rule for belief functions based on pointwise combination of conjunctive weights using a binary operator on $(0,+\infty)$, has the vacuous BBA as neutral element iff 1 is a neutral element of the binary operator. We can further make the following remark. Let w_1 and w_2 be the conjunctive weight functions associated to two non dogmatic BBAs m_1 and m_2. For any binary operator $*$ on $(0,+\infty)$ with 1 as neutral element, such that $x * y \leq xy$ for all $x,y \in (0,+\infty)$, it may easily be shown from Lemma 2 that $w_1 * w_2$ is the conjunctive weight function of some non dogmatic BBA.

The remainder of this section aims at proving that the TBM conjunctive rule is the w-least committed rule in the family of rules based on pointwise combination of conjunctive weights and having the vacuous BBA as neutral element. This is achieved by showing that, for any binary operator $*$ on $(0,+\infty)$ with 1 as neutral element, such that $x * y > xy$ for some $x,y \in (0,+\infty)$, we can find two conjunctive weight functions w_1 and w_2 such that $w_1 * w_2$ is not a conjunctive weight function. Lemma 5 below is essential for this study. Let us first give the following technical lemma (due to the restricted space of this paper, the proof of Lemma 4 is omitted) in order to simplify the proof of Lemma 5.

Lemma 4. *Let* m *be a BBA. For* $B \subset \Omega$*, the following holds:*

$$\sum_{A \subseteq B} (-1)^{|A|}\, q(A) \quad = \quad \sum_{A \cap B = \emptyset} m(A)\,.$$

Lemma 5. *Let* m *be a normal, non dogmatic BBA and such that* $m(A) > 0$*, for a subset* $A \subset \Omega$*. If the conjunctive weight function associated to* m *is increased for any subset* $B \subseteq \bar{A}$*, then* m *is not a BBA any more.*

Proof. Let m_2 be a normal and non dogmatic BBA. Let $m_1 = m_2 \oslash B^{w_2(B)} \cap B^{w_2(B)+\epsilon}$, with $B \subset \Omega$, $\epsilon > 0$ and $\exists A \subset \Omega$ such that $A \cap B = \emptyset$ and $m_2(A) > 0$. Remark that m_1 is a function obtained by increasing the conjunctive weight function associated to a BBA m_2 for a subset $B \subseteq \bar{A}$, with m_2 verifying $m_2(A) > 0$. The proof consists in proving that m_1 is not a BBA. This is done by showing that $m_1(\emptyset) < 0$. We have:

$$m_1(\emptyset) \quad = \quad \sum_{A \subseteq \Omega} (-1)^{|A|} \frac{q_2(A)}{q_0(A)} q_0'(A),$$

where q_0 and q_0' are the commonality functions associated with $B^{w_2(B)}$ and $B^{w_2(B)+\epsilon}$, respectively. We have:

$$q_0(A) = \begin{cases} 1 & \text{if } A \subseteq B, \\ w_2(B) & \text{otherwise,} \end{cases} \tag{1}$$

$$q_0'(A) = \begin{cases} 1 & \text{if } A \subseteq B, \\ w_2(B) + \epsilon & \text{otherwise.} \end{cases} \tag{2}$$

Using (1) and (2), one can obtain:

$$\begin{aligned} m_1(\emptyset) \quad &= \quad \sum_{A \subseteq B} (-1)^{|A|} q_2(A) \\ &\quad + \sum_{A \nsubseteq B} (-1)^{|A|} q_2(A) \frac{w_2(B) + \epsilon}{w_2(B)}. \end{aligned}$$

As

$$\begin{aligned} m_2(\emptyset) \quad &= \quad \sum_{A \subseteq \Omega} (-1)^{|A|} q_2(A) \\ &= \quad \sum_{A \subseteq B} (-1)^{|A|} q_2(A) + \sum_{A \nsubseteq B} (-1)^{|A|} q_2(A), \end{aligned}$$

then

$$m_1(\emptyset) \quad = \quad m_2(\emptyset) + \frac{\epsilon}{w_2\left(B\right)} \sum_{A \not\subseteq B} (-1)^{|A|} q_2(A). \quad (3)$$

We can thus remark that $m_1(\emptyset)$ is equal to $m_2(\emptyset)$, which is itself equal to 0, plus another term. Let us prove that this term is always strictly smaller than 0.

$$\begin{aligned}(3) \quad &= \quad \frac{\epsilon}{w_2\left(B\right)} \left(m_2(\emptyset) - \sum_{A \subseteq B} (-1)^{|A|} q_2(A) \right) \\ &= \quad -\frac{\epsilon}{w_2\left(B\right)} \sum_{A \subseteq B} (-1)^{|A|} q_2(A).\end{aligned}$$

We thus have from Lemma 4:

$$m_1(\emptyset) \quad = \quad -\frac{\epsilon}{w_2\left(B\right)} \sum_{A \cap B = \emptyset} m_2\left(A\right).$$

We have $\epsilon > 0$ and $w_2\left(B\right) > 0$. Furthermore, m_2 satisfies $\exists A \subset \Omega$ such that $A \cap B = \emptyset$ and $m_2(A) > 0$. Hence $m_1(\emptyset) < 0$, thus m_1 is not a BBA. □

Theorem 1. *Let $*$ be a binary operator on $(0, +\infty)$ with 1 as two-sided neutral element (i.e. $1 * x = x * 1 = x$) such that $\exists x, y$, $x * y > xy$. There exist two non dogmatic BBAs m_1 and m_2 on a frame Ω such that the function obtained by pointwise combination using $*$ of the conjunctive weight functions associated to m_1 and m_2 is not a conjunctive weight function.*

Proof. (Sketch) Let x, y be two numbers in $(0, +\infty)$ such that $x * y = xy + \epsilon$ for some $\epsilon > 0$. It is always possible to find two logically consistent BBAs m_1 and m_2, i.e. $m_{1 \circledcap 2}(\emptyset) = 0$, such that:

- $\exists B \in 2^\Omega \setminus \{\Omega\}$ such that $w_1\left(B\right) = x$ and $w_2\left(B\right) = y$,
- $\forall A \in 2^\Omega \setminus \{\Omega, B\}$, $w_1\left(A\right) = 1$ or $w_2\left(A\right) = 1$,
- $\exists C \in 2^\Omega$ such that $m_{1 \circledcap 2}(C) > 0$ and $C \cap B = \emptyset$.

For those BBAs, we thus have:

$$\begin{aligned} w_{1 \circledcap 2}(B) \quad &= \quad w_1(B) \cdot w_2(B), \\ w_{1 \circledcap 2}(A) \quad &= \quad \begin{cases} w_1(A) & \text{if } w_2(A) = 1, \\ w_2\left(A\right) & \text{otherwise,} \end{cases} \end{aligned}$$

for all $A \neq B$, and

$$\begin{aligned} w_1(B) * w_2(B) \quad &= \quad w_{1 \circledcap 2}(B) + \epsilon, \\ w_1(A) * w_2(A) \quad &= \quad w_{1 \circledcap 2}(A), \end{aligned}$$

for all $A \neq B$.

We have:

$$\circledcap_{A \subset \Omega} A^{w_1(A) * w_2(A)} = m_{1 \circledcap 2} \oslash B^{w_{1 \circledcap 2}(B)} \circledcap B^{w_{1 \circledcap 2}(B) + \epsilon}, \quad (4)$$

and $\exists C \in 2^\Omega$ such that $m_{1 \circledcap 2}(C) > 0$ and $C \cap B = \emptyset$. By Lemma 5, (4) is not a BBA, hence $w_1 * w_2$ is not a conjunctive weight function. □

Corollary 1. *The TBM conjunctive rule is the w-least committed rule in the family of rules based on pointwise combination of conjunctive weights and having the vacuous BBA as neutral element.*

Proof. From Theorem 1 and Lemma 2, it is clear that the rules that are based on the conjunctive weight function and which have the vacuous BBA as neutral element are based on pointwise combination of conjunctive weights using binary operators $*$ on $(0, +\infty)$ with 1 as neutral element and such that $x * y \leq xy$ for all $x, y \in (0, +\infty)$, hence the corollary. □

Remark 1. *Idempotence and having the vacuous BBA as neutral element are incompatible properties for rules based on pointwise combination of conjunctive weights.*

Proof. From Theorem 1 and Lemma 2, a rule that is based on pointwise combination of conjunctive weights and which has the vacuous BBA as neutral element is based on a binary operator $*$ having 1 as neutral element and satisfying $x * y \leq xy$, $\forall x, y \in (0, +\infty)$. Let $z \in (0, 1)$. We have $z * z \leq z^2 < z$, hence $*$ is not idempotent. □

The product on $(0, +\infty)$ is commutative, associative, and increasing. It also has 1 as neutral element, which makes the product a uninorm (Yager & Rybalov 1996) on $(0, +\infty)$. It can be shown that there exists an infinity of uninorms on $(0, +\infty)$. Hence, a consequence of the results presented in this paper is that the TBM conjunctive rule is the least committed element in the family of uninorm-based combination rules.

The Disjunctive Case

In this section, we present results corresponding to the previous ones for rules based on pointwise combination of disjunctive weights.

Corollary 2. *The TBM disjunctive rule is the v-most committed rule amongst the rules based on pointwise combination of disjunctive weights and having the BBA $m\left(\emptyset\right) = 1$ as neutral element.*

Proof. (Sketch) Let $*$ be a binary operator on $(0, +\infty)$ having 1 as neutral element. Let v_1 and v_2 be the disjunctive weight functions associated to two subnormal BBAs m_1 and m_2. Let $\overline{w_1}$ and $\overline{w_2}$ be the conjunctive weight functions associated to $\overline{m_1}$ and $\overline{m_2}$. We have:

$$\begin{aligned} \overline{\circledcup_{A \neq \emptyset} A_{v_1(A) * v_2(A)}} \quad &= \quad \circledcap_{A \neq \emptyset} \overline{A_{v_1(A) * v_2(A)}} \\ &= \quad \circledcap_{A \neq \emptyset} \bar{A}^{\overline{w_1}(\bar{A}) * \overline{w_2}(\bar{A})} \\ &= \quad \circledcap_{A \subset \Omega} A^{\overline{w_1}(A) * \overline{w_2}(A)} \quad (5) \end{aligned}$$

From Theorem 1 and Lemma 2, (5) is a BBA iff $*$ is such that $x * y \leq xy$. For any operator $*$ on $(0, +\infty)$ having 1 as neutral element and such that $x * y \leq xy$, $m_1 \circledcup m_2 \sqsubseteq_v \circledcup_{A \neq \emptyset} A_{v_1(A) * v_2(A)}$, hence the corollary. □

The TBM disjunctive rule has similar properties as the TBM conjunctive rule since they are both based on the product. It can thus be deduced from Corollary 2 that the TBM disjunctive rule is the most committed element in the family of rules based on pointwise combination of disjunctive weights using uninorms.

Proposition 1 (De Morgan's Laws). *Let $*$ be an operator on $(0, +\infty)$ with 1 as neutral element and such that $x * y \leq xy$ for all $x, y \in (0, +\infty)$. Further, let $\circledast_w$ denote a rule based on pointwise combination of conjunctive weights using operator $*$, and let $\circledast_v$ denote a rule based on pointwise combination of disjunctive weights using operator $*$. Let m_1 and m_2 be two subnormal BBAs. We have:*

$$\overline{m_1 \circledast_v m_2} = \overline{m_1} \circledast_w \overline{m_2}. \tag{6}$$

Let m_1 and m_2 be two non dogmatic BBAs. We have:

$$\overline{m_1 \circledast_w m_2} = \overline{m_1} \circledast_v \overline{m_2}. \tag{7}$$

Proof. The proof of (6) is direct using the proof of Corollary 2. The proof of (7) is similar. □

Conclusion

This paper has shown a new singular property of the unnormalized Dempster's rule of combination in the context of combination rules for belief functions based on the conjunctive weight function. It was also put forward that the unnormalized Dempster's rule of combination $\bigcirc\!\!\!\!\cap$ and the cautious rule $\bigcirc\!\!\!\!\wedge$ have fundamental different algebraic properties: the former is based on a uninorm on $(0, +\infty)$ and has a neutral element while the latter is based on a t-norm on $(0, +\infty]$ and has no neutral element. Similarly the TBM disjunctive rule $\bigcirc\!\!\!\!\cup$ is based on a uninorm on $(0, +\infty)$ and has a neutral element while the bold rule $\bigcirc\!\!\!\!\vee$ is based on a t-norm on $(0, +\infty]$ and has no neutral element. Note also that the pairs of rules $\bigcirc\!\!\!\!\cap - \bigcirc\!\!\!\!\cup$ and $\bigcirc\!\!\!\!\wedge - \bigcirc\!\!\!\!\vee$ are related by De Morgan laws.

In addition, it was revealed that to each of those four basic rules corresponds one infinite family of combination rules. Indeed, there exist two t-norm-based families that are based respectively on the conjunctive and disjunctive weight functions. There exist also two uninorm-based families that are based respectively on the conjunctive and disjunctive weight functions. Interestingly, this paper showed that the four basic rules occupy a special position in each of their respective family: the $\bigcirc\!\!\!\!\cap$ and $\bigcirc\!\!\!\!\wedge$ rules are the least committed elements, whereas the $\bigcirc\!\!\!\!\cup$ and $\bigcirc\!\!\!\!\vee$ rules are the most committed elements. This is summarized in the following table:

	conjunctive weights	disjunctive weights
uninorm	$\bigcirc\!\!\!\!\cap$	$\bigcirc\!\!\!\!\cup$
t-norm	$\bigcirc\!\!\!\!\wedge$	$\bigcirc\!\!\!\!\vee$
	least committed	most committed

Future efforts will concentrate on the performance gains for information fusion systems suggested by preliminary experiments using those new families of rules. In particular, it seems possible to define rules based on parameterized families of t-norms or uninorms, and to tune these rules so as to optimize the performance of a fusion system. Finally, properties of these rules related to their use in valuation-based systems (mainly, distributivity of marginalization over combination (Shenoy & Shafer 1990)) will be investigated.

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