A new singular property of the unnormalized Dempster's rule among uninorm-based combination rules

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Abstract

A new family of rules of combination for belief functions was recently introduced. It is based on uninorms and an equivalent representation of a belief function known as the weight function. The unnormalized Dempster's rule was notably shown to be the least committed rule in this family. This paper shows another interesting property of this rule in this family: it is the only one for which marginalization is distributive over the combination.

Keywords: Dempster-Shafer Theory, Transferable Belief Model, Valuation Algebra, Uninorms.

1 Introduction

The Transferable Belief Model (TBM) [11, 9] is a model for quantifying beliefs using belief functions [5]. An essential mechanism of the TBM is the unnormalized Dempster's rule of combination. This rule, referred to as the TBM conjunctive rule in this paper, allows the fusion of belief functions. This rule is justified only when it is safe to assume that the items to be combined are distinct. Denœux [1] recently proposed a rule, called the cautious rule of combination, which does not rely on the distinctness assumption. This rule is based on the weight function, an equivalent representation of a belief function. Denœux [1] further showed that the cautious rule belongs to an infinite family of combination rules. Besides, the cautious rule is the least committed rule in this family. Interestingly, it can be shown [4] that a similar property holds for the TBM conjunctive rule: it belongs to an infinite family of rules based on uninorms on $(0, +\infty)$ and it is the least committed rule in this family. The fundamental difference between those two families is the existence of a neutral element.

A valuation algebra [3] is an abstract, yet useful, framework for many different AI formalisms. In particular it can be used to manage uncertainty represented by belief functions if the belief functions are combined using the TBM conjunctive rule. This paper shows that, despite the numerous properties shared by the TBM conjunctive rule and the uninorm-based combination rules, the TBM conjunctive rule is the only rule that satisfies an axiom of the valuation algebra framework. This property further singles out the TBM conjunctive rule in this family of rules.

The rest of this paper is organized as follows. Basic notions on valuation algebras are first recalled in Section 2. Section 3 summarizes the necessary concepts, such as the weight function, of the TBM. The uninorm-based combination rules are reviewed in Section 4. The main result of this paper is given in Section 5. Section 6 concludes the paper.

2 Valuation Algebras

Many formalisms dealing with information share an underlying algebraic structure with

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the essential algebraic operations of combination and marginalization [3]. Combination corresponds to aggregation of knowledge and marginalization corresponds to focusing of knowledge to a narrower domain. In many cases, the computation of the combination of the available information is computationally intractable. Provided that combination and marginalization satisfy some axioms, one can however benefit from a technique called local computation [6, 7]. In essence, this technique makes it possible to compute marginals of a combination without explicitly computing the combination. The algebraic structures with the operations of combination and marginalization satisfying these axioms are called valuation algebras [3]. In the remainder of this section, we recall necessary concepts on valuation algebras; the presentation adopted here is in line with the one of [3].

In the valuation algebra framework, it is considered that reasoning is concerned with a finite set of variables. Each variable is associated with a finite set of possible values called its frame; a variable is noted using an upper-case letter, e.g. X, and the frame of the variable is noted Ω_X. Sets of variables are noted using a lower-case letter, e.g. s. Let s be a non empty set of variables. We note Ω_s the Cartesian product of the frames Ω_X of the variables $X \in s$, and we call configurations the elements of Ω_s. Knowledge about the possible values of a set s of variables is represented by a valuation. Valuations are noted using lower-case greek letters such as φ and ψ. If φ is valuation for s, then we call s the domain of φ and we write $d(\varphi) = s$. Given a set s of variables, we may consider that there is a set Φ_s of valuations. Let r denote the set of all variables, and let $\Phi = \cup_{s \subseteq r} \Phi_s$ denote the set of all valuations. Finally, let D denote the power set of r.

Two operations are defined for valuations in the valuation algebra framework. The combination of valuations is a binary operation $\otimes : \Phi \times \Phi \to \Phi$, which is assumed to be commutative and associative, hence Φ is a commutative semigroup under combination. Furthermore, if φ is a valuation for s and ψ is a valuation for t, then $\varphi \otimes \psi$ is a valuation for $s \cup t$. For any set s, we also require the existence of a neutral element e_s such that $\varphi \otimes e_s = e_s \otimes \varphi = \varphi$, for all $\varphi \in \Phi_s$. Hence, Φ_s is a commutative monoid under combination $\otimes$. The marginalization of a valuation is a binary operation $\downarrow : \Phi \times D \to \Phi$. For any valuation φ and domain s, a valuation $\varphi^{\downarrow s}$ with domain $s \cap d(\varphi)$ is associated. $\varphi^{\downarrow s}$ is called the marginal of φ for s. Marginalization corresponds to focusing of the knowledge represented by φ for $d(\varphi)$ to the smaller domain $s \cap d(\varphi)$.

Now, suppose a knowledge base consisting of a finite set of valuations $\varphi_1, ..., \varphi_m$, and let $\varphi_1 \otimes ... \otimes \varphi_m$ represent the combined knowledge, which we call the joint valuation. The problem of inference is to marginalize the joint valuation to a domain s of interest. The straightforward way to perform inference is to compute $(\varphi_1 \otimes ... \otimes \varphi_m)^{\downarrow s}$. Using this approach, the number of variables increases with each combination. This is a problem for most instantiations (such as belief function theory) of this abstract framework as complexity grows with the domain size. However, if combination and marginalization satisfy some axioms, then the marginal $(\varphi_1 \otimes ... \otimes \varphi_m)^{\downarrow s}$ can be computed without explicitly computing the joint valuation. The axioms are the following [3] (the fifth axiom is the crucial one for local computation):

1. *Semigroup*: Φ is a commutative semigroup under combination. For each $s \in D$, there is an element e_s with $d(e_s) = s$ such that for all $\varphi \in \Phi$ with $d(\varphi) = s, e_s \otimes \varphi = \varphi \otimes e_s = \varphi$.

2. *Domain of combination*: for $\varphi, \psi \in \Phi$,
$$d(\varphi \otimes \psi) = d(\varphi) \otimes d(\psi).$$

3. *Marginalization*: for $\varphi \in \Phi$ and $s \in D$,
$$\begin{aligned} \varphi^{\downarrow s} &= \varphi^{\downarrow s \cap d(\varphi)}, \\ d(\varphi^{\downarrow s}) &= s \cap d(\varphi), \\ \varphi^{\downarrow d(\varphi)} &= \varphi. \end{aligned}$$

4. *Transitivity of marginalization*: for $\varphi \in \Phi$, $(\varphi^{\downarrow s})^{\downarrow t} = \varphi^{\downarrow s \cap t}$.

5. *Distributivity of marginalization over combination*: for $\varphi, \psi \in \Phi$ with $d(\varphi) = s$

$$(\varphi \otimes \psi)^{\downarrow s} = \varphi \otimes \psi^{\downarrow s}.$$

6. *Neutrality*: for $s, t \in D$, $e_s \otimes e_t = e_{s \cup t}$.

Definition 1. *A set Φ of valuations with set D of domains, combination $\otimes$, and marginalization $\downarrow$, which satisfies these six axioms is called a valuation algebra. It is denoted by $(\Phi, D, \otimes, \downarrow)$.*

3 The Transferable Belief Model

3.1 Basic Definitions and Notations

In this paper, the TBM [11, 9] is accepted as a model to quantify uncertainties based on belief functions [5]. The beliefs held by an agent *Ag* on a finite frame of discernment $\Omega = \{\omega_1, ..., \omega_K\}$ are represented by a basic belief assignment (BBA) m defined as a mapping from 2^Ω to $[0,1]$ verifying $\sum_{A \subseteq \Omega} m(A) = 1$. Subsets A of Ω such that $m(A) > 0$ are called focal sets (FS) of m. A BBA m is said to be: *vacuous* if Ω is the only focal set; *dogmatic* if Ω is not a focal set; *simple* if if has at most two focal sets and, if it has two, Ω is one of those.

A simple BBA (SBBA) m such that $m(A) = 1 - w$ and $m(\Omega) = w$ for some $A \neq \Omega$ and some $w \in [0,1]$ can be noted A^w. The advantage of this notation will appear later.

Equivalent representations of a BBA m exist. In particular the commonality function is defined as:

$$q(A) = \sum_{B \supseteq A} m(B), \; \forall A \subseteq \Omega.$$

The TBM conjunctive rule is noted $\bigcirc\!\!\!\!\cap$. It is defined as follows. Let m_1 and m_2 be two BBAs, and let $m_{1 \bigcirc\!\!\!\!\cap 2}$ be the result of their combination by $\bigcirc\!\!\!\!\cap$. We have, for all $A \subseteq \Omega$:

$$m_{1 \bigcirc\!\!\!\!\cap 2}(A) \;\; = \sum_{B \cap C = A} m_1(B)\, m_2(C).$$

This rule is commutative, associative and admits a unique neutral element: the vacuous BBA. Let A^{w_1} and A^{w_2} be two SBBAs. Their combination by $\bigcirc\!\!\!\!\cap$ is the SBBA $A^{w_1 w_2}$.

3.2 Canonical Decomposition of a Belief Function

In [10], Smets proposed a solution to canonically decompose any nondogmatic BBA. This decomposition uses the concept of a generalized SBBA (GSBBA) which is defined as a function μ from 2^Ω to $\mathbb{R}$ by:

$$\begin{aligned} \mu(A) &= 1 - w, \\ \mu(\Omega) &= w, \\ \mu(B) &= 0 \;\; \forall B \in 2^\Omega \backslash \{A, \Omega\}, \end{aligned}$$

for some $A \neq \Omega$ and some $w \in [0, +\infty)$. Extending the SBBA notation, any GSBBA can be noted A^w. When $w \leq 1$, μ is a SBBA. When $w > 1$, μ is no longer a BBA; Smets [10] called such function an inverse SBBA.

Smets showed that any nondogmatic BBA m can be uniquely represented as the conjunctive combination of generalized SBBAs:

$$m = \bigcirc\!\!\!\!\cap_{A \subset \Omega} A^{w(A)},$$

with $w(A) \in (0, +\infty)$ for all $A \subset \Omega$. The weights $w(A)$ for each $A \subset \Omega$ are obtained as follows:

$$w(A) = \prod_{B \supseteq A} q(B)^{(-1)^{|B| - |A| + 1}}.$$

The function $w : 2^\Omega \backslash \{\Omega\} \to (0, +\infty)$ is called the weight function. It is another equivalent representation of a nondogmatic BBA m.

The TBM conjunctive rule has a simple expression using the weight function. Let m_1 and m_2 be two nondogmatic BBAs with weight functions w_1 and w_2. We have:

$$m_{1 \bigcirc\!\!\!\!\cap 2} \;=\; \bigcirc\!\!\!\!\cap_{A \subset \Omega} A^{w_1(A) \cdot w_2(A)}.$$

3.3 Informational Comparison of Belief Functions

Another important concept of the TBM is the least commitment principle [8]. This principle plays a similar role as the principle of maximum entropy does in Bayesian probability theory. It stipulates that one should never give more beliefs than justified by the available information. It becomes operational through the definition of partial orderings allowing the informational comparison of BBAs. Such orders, generalizing set inclusion, were proposed by Yager [12], and Dubois and Prade [2]. Recently, Denœux [1] proposed a new partial ordering, called the w-ordering. It is defined as follows: given two nondogmatic BBAs m_1 and m_2, $m_1 \sqsubseteq_w m_2$, i.e. m_1 is w-more committed than m_2, iff $w_1(A) \leq w_2(A)$ for all $A \subset \Omega$. Let us also recall the following lemma related to the w-ordering, which will be needed later.

Lemma 1 (Lemma 1 of [1])**.** *Let m be a nondogmatic BBA with weight function w, and let w' be a mapping from $2^\Omega \backslash \Omega$ to $(0, +\infty)$ such that $w'(A) \leq w(A)$ for all $A \subset \Omega$. Then w' is the weight function of some BBA m'.*

3.4 Operations on Product Spaces

Let $m^{X \times Y}$ denote a BBA defined on the Cartesian product $\Omega_X \times \Omega_Y$ of the frames of two variables X and Y. The marginal BBA $m^{X \times Y \downarrow X}$ is defined, for all $A \subseteq \Omega$, as

$$m^{X \times Y \downarrow X}(A) = \sum_{\{B \subseteq \Omega_X \times \Omega_Y, B \downarrow \Omega_X = A\}} m^{X \times Y}(B), \tag{1}$$

for all $A \subseteq \Omega_X$, and where $B \downarrow \Omega_X$ denotes the projection of B onto Ω_X.

Conversely, let m^X be a BBA defined on Ω_X. Its vacuous extension on $\Omega_X \times \Omega_Y$ is defined as [8]:

$$m^{X \uparrow X \times Y}(B) = \begin{cases} m^X(A) & \text{if } B = A \times \Omega_Y, \\ & \text{for some } A \subseteq \Omega_X, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

Given two BBAs m_1^X and m_2^Y, their conjunctive combination on $X \times Y$ can be obtained by combining their vacuous extensions on $X \times Y$ using (2). Formally:

$$m_1^X \circledcap m_2^Y = m_1^{X \uparrow X \times Y} \circledcap m_2^{Y \uparrow X \times Y}. \tag{3}$$

4 Uninorm-based Combination Rules

We have seen that the TBM conjunctive rule is based on pointwise multiplication of weights. The product on $(0, +\infty)$ is commutative, associative and increasing. It also has 1 as neutral element, which makes it a uninorm [13] on $(0, +\infty)$. Hence, the TBM conjunctive rule belongs to the family of rules based on pointwise combination of weights using uninorms on $(0, +\infty)$ having 1 as neutral element. Let us now recall a result presented in [4] related to this family of rules.

Theorem 1 (Theorem 1 of [4])**.** *Let $\circ$ be a binary operator on $(0, +\infty)$ with 1 as two-sided neutral element (i.e. $1 \circ x = x \circ 1 = x$) such that $\exists x, y, x \circ y > xy$. There exist two nondogmatic BBAs m_1 and m_2 on a frame Ω such that the function obtained by pointwise combination using $\circ$ of the weight functions associated to m_1 and m_2 is not a weight function of some nondogmatic BBA.*

Proposition 1. *Let w_1 and w_2 be the weight functions associated to two nondogmatic BBAs m_1 and m_2. Let $\circ$ be an operator on $(0, +\infty)$ with 1 as neutral element and such that $x \circ y \leq xy$ for all $x, y \in (0, +\infty)$. Then the function $w_{1 \circledcirc 2}$ defined by :*

$$w_{1 \circledcirc 2}(A) = w_1(A) \circ w_2(A), \forall A \subset \Omega, \tag{4}$$

is a weight function associated to some nondogmatic BBA.

Proof. Direct from Lemma 1. □

Theorem 1 and Proposition 1 define new combination rules which can be formally defined as follows.

Definition 2 (Uninorm-based combination rule)**.** *Let $\circ$ be a uninorm on $(0, +\infty)$ having 1 as neutral element and such that $x \circ y \leq xy$ for all $x, y \in (0, +\infty)$. Let m_1 and m_2 be two nondogmatic BBAs. Their combination using the uninorm-based rule, or u-rule for short, is noted $m_{1 \circledcirc 2} = m_1 \circledcirc m_2$. It is defined as a BBA with the following weight function:*

$$w_{1 \circledcirc 2}(A) = w_1(A) \circ w_2(A), \forall A \subset \Omega.$$

We thus have:

$$m_1 \circledcirc m_2 = \circledcap_{A \subset \Omega} A^{w_1(A) \circ w_2(A)}.$$

Proposition 2. *Let $\mathcal{M}_{nd}$ be the set of nondogmatic BBAs, and let $\circledcirc$ be a u-rule. Then $(\mathcal{M}_{nd}, \circledcirc)$ is a commutative monoid, with the vacuous BBA as neutral element.*

Proof. This proposition results directly from the properties of the uninorm $\circ$. $\square$

It can be shown that there exists an infinity of uninorms $\circ$ on $(0, +\infty)$ having 1 as neutral element and such that $x \circ y \leq xy$ for all $x, y \in (0, +\infty)$. The TBM conjunctive rule is thus the w-least committed u-rule. Note that this fact can be seen as a new justification of the TBM conjunctive rule since the rule thus respects a central principle of the TBM.

Finally, we can define the combination on product spaces of two BBAs by a u-rule as done for the conjunctive rule in (3). Formally, given two BBAs m_1^X and m_2^Y, their combination by a u-rule $\circledcirc$ on $X \times Y$ is defined as:

$$m_1^X \circledcirc m_2^Y = m_1^{X \uparrow X \times Y} \circledcirc m_2^{Y \uparrow X \times Y}. \quad (5)$$

5 Main Result

Let $\mathcal{M}$ be the set of BBAs, and let $\downarrow$ be the projection operation as defined by (1). It has been shown in [7] that the structure $(\mathcal{M}, 2^\Omega, \textcircled{\cap}, \downarrow)$ satisfies all the axioms of valuation algebras. Indeed, it is clear that axioms 3 and 4 are satisfied by belief functions (and thus nondogmatic belief functions). It can also easily be shown that axioms 1, 2 and 6 are respected by belief functions combined using the TBM conjunctive rule. The proof for axiom 5 is however much more complex.

We have seen that the TBM conjunctive rule belongs to the family of uninorm-based combination rules, hence the rule $\textcircled{\cap}$ has common properties with the u-rules. It is thus interesting to know if more properties are shared by those rules. In the remainder of this section, we investigate if the algebraic structure $(\mathcal{M}_{nd}, 2^\Omega, \circledcirc, \downarrow)$ is a valuation algebra for at least one u-rule based on a uninorm $\circ$ different from the product.

As for the TBM conjunctive rule, determining if this structure is a valuation algebra depends mainly on the fifth axiom. Indeed, we can first remark that axioms 3 and 4 do not need to be studied as they are independent from the combination rule used. Furthermore, axiom 2 is satisfied by the definition given by (5) of the combination by a u-rule on product spaces. Axioms 1 and 6 are direct consequences of $(\mathcal{M}_{nd}, \circledcirc)$ being a commutative monoid having the vacuous BBA as neutral element and of (5).

Proposition 3 below shows that axiom 5 is not satisfied by u-rules different from the TBM conjunctive rule.

Proposition 3. *Let $\circledcirc$ be a u-rule. Let s and t be two sets of variables. If the binary operator $\circ$ underlying the u-rule $\circledcirc$ is different from the product, i.e. if $\exists x, y \in (0, +\infty)$ such that $x \circ y \neq xy$, then there exist two nondogmatic BBAs m_1 defined on a frame of discernment Ω_s and m_2 defined on a frame of discernment Ω_t such that*

$$(m_1 \circledcirc m_2)^{\downarrow t} \neq m_1^{\downarrow t} \circledcirc m_2. \quad (6)$$

Proof. (Sketch) Let x and y be two arbitrary numbers in $(0, +\infty)$ such that $x \circ y \neq xy$. Let s and t be two sets of variables. Further, let m_1 be a BBA defined on the frame of discernment Ω_s and m_2 be a BBA defined on the frame of discernment Ω_t. From the fact that the $\textcircled{\cap}$ rule satisfies axiom 5, we have:

$$(m_1 \textcircled{\cap} m_2)^{\downarrow t} = m_1^{\downarrow t} \textcircled{\cap} m_2. \quad (7)$$

The proof consists in choosing the BBAs m_1 and m_2 such that we have $(m_1 \circledcirc m_2)^{\downarrow t} = (m_1 \textcircled{\cap} m_2)^{\downarrow t}$ and $m_1^{\downarrow t} \circledcirc m_2 \neq m_1^{\downarrow t} \textcircled{\cap} m_2$. Hence, from (7) we have

$$\begin{aligned} (m_1 \circledcirc m_2)^{\downarrow t} &= (m_1 \textcircled{\cap} m_2)^{\downarrow t} \\ &= m_1^{\downarrow t} \textcircled{\cap} m_2 \\ &\neq m_1^{\downarrow t} \circledcirc m_2 \end{aligned}$$

We get $m_1^{\downarrow t} \circledcirc m_2 \neq m_1^{\downarrow t} \textcircled{\cap} m_2$ by choosing the BBAs m_1^s and m_2^t such that:

- $\exists B \in 2^{\Omega_t} \setminus \{\Omega_t\}$ such that $w_1^{s \downarrow t}(B) = x$ and $w_2^t(B) = y$, with $w_1^{s \downarrow t}$ the weight function associated to $m_1^{s \downarrow t}$,

- $\forall A \in 2^{\Omega_t} \setminus \{\Omega_t, B\}$, $w_1^{s\downarrow t}(A) = 1$ or $w_2^t(A) = 1$.

The weight functions $w^t_{1 \circledcirc 2}$ and $w^t_{1 \Cap 2}$ associated respectively to $m_1^{\downarrow t} \circledcirc m_2$ and $m_1^{\downarrow t} \Cap m_2$ are thus as follows:

$$\begin{aligned} w^t_{1 \circledcirc 2}(B) &\neq w^t_{1 \Cap 2}(B), \\ w^t_{1 \circledcirc 2}(A) &= w^t_{1 \Cap 2}(A), \text{ for } A \neq B. \end{aligned}$$

Consequently

$$m_1^{\downarrow t} \circledcirc m_2 \neq m_1^{\downarrow t} \Cap m_2.$$

We get $(m_1 \circledcirc m_2)^{\downarrow t} = (m_1 \Cap m_2)^{\downarrow t}$ by choosing the BBAs m_1^s and m_2^t such that:

- $\forall A \in 2^{\Omega_{s \cup t}} \setminus \{\Omega_{s \cup t}\}$, $w_1^{s \uparrow s \cup t}(A) = 1$ or $w_2^{t \uparrow s \cup t}(A) = 1$, with $w_1^{s \uparrow s \cup t}$ the weight function associated to $m_1^{s \uparrow s \cup t}$, and $w_2^{t \uparrow s \cup t}$ the weight function associated to $m_2^{t \uparrow s \cup t}$.

The weight functions $w^{s \cup t}_{1 \circledcirc 2}$ and $w^{s \cup t}_{1 \Cap 2}$ associated respectively to $m_1 \circledcirc m_2$ and $m_1 \Cap m_2$ are thus as follows:

$$w^{s \cup t}_{1 \circledcirc 2}(A) = w^{s \cup t}_{1 \Cap 2}(A), \forall A \in 2^{\Omega_{s \cup t}} \setminus \Omega_{s \cup t}.$$

Consequently we have $m_1 \circledcirc m_2 = m_1 \Cap m_2$ and thus $(m_1 \circledcirc m_2)^{\downarrow t} = (m_1 \Cap m_2)^{\downarrow t}$.

Let us now provide the BBAs m_1 and m_2 which verify the above scheme.

The operator $\circ$ is such that $\exists x, y$, $x \circ y \neq xy$, this implies that $x, y \in (0, +\infty) \setminus \{1\}$ as 1 is the neutral element of $\circ$. In the remainder of this proof, we consider the cases where:

- Case 1: $x \vee y < 1$,
- Case 2: $x \wedge y > 1$,
- Case 3: $x \vee y > 1$ and $x \wedge y < 1$.

We must thus provide a pair of BBAs m_1 and m_2 verifying the above scheme for each of those three cases. Let us first provide two frames of discernment Ω_s and Ω_t on which we are going to define our three pairs of BBA. Let X and Z be two binary variables whose frames are $\Omega_X = \{x_1, x_2\}$ and $\Omega_Z = \{z_1, z_2\}$, and let Y be a ternary variable whose frame is $\Omega_Y = \{y_1, y_2, y_3\}$. Let t denote the set composed of the variables Y and Z and let s denote the set composed of the variables X, Y and Z. Tables 1 and 2 give explicit names to the configurations of the frames Ω_t and Ω_s.

Table 1: The frame Ω_s

configurations	
s_1	(x_1, y_1, z_1)
s_2	(x_1, y_1, z_2)
s_3	(x_1, y_2, z_1)
s_4	(x_1, y_2, z_2)
s_5	(x_1, y_3, z_1)
s_6	(x_1, y_3, z_2)
s_7	(x_2, y_1, z_1)
s_8	(x_2, y_1, z_2)
s_9	(x_2, y_2, z_1)
s_{10}	(x_2, y_2, z_2)
s_{11}	(x_2, y_3, z_1)
s_{12}	(x_2, y_3, z_2)

Table 2: The frame Ω_t

configurations	
t_1	(y_1, z_1)
t_2	(y_1, z_2)
t_3	(y_2, z_1)
t_4	(y_2, z_2)
t_5	(y_3, z_1)
t_6	(y_3, z_2)

Let us now provide the pairs of BBAs m_1 and m_2 satisfying the scheme described at the beginning of the proof, for each of the three possible cases. Only the case 1 is treated in details; the other cases are more tedious but nonetheless similar.

- Case 1:

  Let m_1 be a BBA defined on Ω_s as follows, for $x \in (0, 1)$:

  $$\begin{aligned} m_1^s(\{s_9, s_{10}\}) &= 1 - x, \\ m_1^s(\Omega_s) &= x. \end{aligned}$$

  Marginalizing m_1^s on Ω_t yields:

  $$\begin{aligned} m_1^{s\downarrow t}(\{t_3, t_4\}) &= 1 - x, \\ m_1^{s\downarrow t}(\Omega_t) &= x. \end{aligned}$$

The weight functions associated respectively to m_1^s and $m_1^{s\downarrow t}$ are the following:

$$\begin{aligned} w_1^s(\{s_9, s_{10}\}) &= x, \\ w_1^s(A) &= 1, \end{aligned}$$

for all $A \in 2^{\Omega_s} \setminus \{\Omega_s, \{s_9, s_{10}\}\}$, and

$$\begin{aligned} w_1^{s\downarrow t}(\{t_3, t_4\}) &= x, \\ w_1^{s\downarrow t}(A) &= 1, \end{aligned}$$

for all $A \in 2^{\Omega_t} \setminus \{\Omega_t, \{t_3, t_4\}\}$.

Now, let m_2 be a BBA defined on Ω_t as follows, for $y \in (0, 1)$:

$$\begin{aligned} m_2^t(\{t_3, t_4\}) &= 1 - y, \\ m_2^t(\Omega_t) &= y. \end{aligned}$$

Vacuously extending m_2^t on $\Omega_{t\cup s} = \Omega_s$ yields:

$$\begin{aligned} m_2^{t\uparrow s}(\{s_3, s_4, s_9, s_{10}\}) &= 1 - y, \\ m_2^{t\uparrow s}(\Omega_s) &= y. \end{aligned}$$

The weight functions associated respectively to m_2^t and $m_2^{t\uparrow s}$ are the following:

$$\begin{aligned} w_2^t(\{t_3, t_4\}) &= y, \\ w_2^t(A) &= 1, \end{aligned}$$

for all $A \in 2^{\Omega_t} \setminus \{\Omega_t, \{t_3, t_4\}\}$, and

$$\begin{aligned} w_2^{t\uparrow s}(\{s_3, s_4, s_9, s_{10}\}) &= y, \\ w_2^{t\uparrow s}(A) = 1, \end{aligned}$$

for all $A \in 2^{\Omega_s} \setminus \{\Omega_s, \{s_3, s_4, s_9, s_{10}\}\}$. For those two BBAs m_1 and m_2, we thus have:

- $\exists B = \{t_3, t_4\}$ such that $w_1^{s\downarrow t}(B) = x$, and $w_2^t(B) = y$, with $x, y \in (0, 1)$
- $\forall A \in 2^{\Omega_t} \setminus \{\Omega_t, B\}$, $w_1^{s\downarrow t}(A) = 1$ or $w_2^t(A) = 1$.
- $\forall A \in 2^{\Omega_{s\cup t}} \setminus \{\Omega_{s\cup t}\}, w_1^{s\uparrow s\cup t}(A) = 1$ or $w_2^{t\uparrow s\cup t}(A) = 1$.

- Case 2: Let m_1 be a BBA defined on Ω_s as follows:

$$\begin{aligned} m_1^s(\{s_1, s_2, s_3\}) &= m_1^s(\{s_1, s_3, s_5\}) \\ &= m_1^s(\{s_2, s_4, s_5\}) \\ &= \alpha \\ m_1^s(\Omega_s) &= 1 - 3\alpha \end{aligned}$$

for $\alpha \in (0, 1/3)$.

Let m_2 be a BBA defined on Ω_t as follows:

$$\begin{aligned} m_2^t(\{t_1, t_2\}) &= m_2^t(\{t_2, t_3\}) = \beta, \\ m_2^t(\Omega_t) &= 1 - 2\beta \end{aligned}$$

for $\beta \in (0, 0.5)$.

For those two BBAs m_1 and m_2, we have

- $\exists B = \{t_2\}$ such that $w_1^{s\downarrow t}(B) = x, x \in (1, +\infty)$ as $w_1^{s\downarrow t}(B) = f(\alpha)$ with f a surjective function from $(0, 1/3)$ to $(1, +\infty)$, and $w_2^t(B) = y, y \in (1, +\infty)$ as $w_2^t(B) = g(\beta)$ with g a surjective function from $(0, 0.5)$ to $(1, +\infty)$.
- $\forall A \in 2^{\Omega_t} \setminus \{\Omega_t, B\}$, $w_1^{s\downarrow t}(A) = 1$ or $w_2^t(A) = 1$.
- $\forall A \in 2^{\Omega_{s\cup t}} \setminus \{\Omega_{s\cup t}\}, w_1^{s\uparrow s\cup t}(A) = 1$ or $w_2^{t\uparrow s\cup t}(A) = 1$.

- Case 3:

Let m_1 be a BBA defined on Ω_s as follows:

$$\begin{aligned} m_1^s(\{s_1, s_2, s_3\}) &= m_1^s(\{s_1, s_2, s_4\}) \\ &= m_1^s(\{s_1, s_3, s_4\}) \\ &= \alpha, \\ m_1^s(\Omega_s) &= 1 - 3\alpha \end{aligned}$$

for $\alpha \in (0, 1/3)$.

Let m_2 be a BBA defined on Ω_t as follows:

$$\begin{aligned} m_2^t(\{t_1, t_2\}) &= 1 - y, \\ m_2^t(\Omega_t) &= y \end{aligned}$$

for $y \in (0, 1)$.

For those two BBAs m_1 and m_2, we have

- $\exists B = \{t_1, t_2\}$ such that $w_1^{s\downarrow t}(B) = x, x \in (1, +\infty)$ as $w_1^{s\downarrow t}(B) = f(\alpha)$ with f a surjective function from $(0, 1/3)$ to $(1, +\infty)$, and $w_2^t(B) = y, y \in (0, 1)$
- $\forall A \in 2^{\Omega_t} \setminus \{\Omega_t, B\}$, $w_1^{s\downarrow t}(A) = 1$ or $w_2^t(A) = 1$.
- $\forall A \in 2^{\Omega_{s\cup t}} \setminus \{\Omega_{s\cup t}\}, w_1^{s\uparrow s\cup t}(A) = 1$ or $w_2^{t\uparrow s\cup t}(A) = 1$.

□

6 Conclusion

New combination rules based on the weight function and uninorms have been recently proposed. A potential application of those rules is based on the remark that it seems possible to define rules based on parameterized families of uninorms, and to tune the rules so as to optimize the performance of a fusion system. Another interesting fact related to this family of uninorm-based combination rules is that the TBM conjunctive rule has been shown to be its least committed element.

This paper has shown that the TBM conjunctive rule is also the only rule in this family for which marginalization is distributive over the combination. The consequences of this result are twofold. On the one hand, it strengthens the fact that the TBM conjunctive rule has a special position in this family. On the other hand, it may also be seen as a drawback of those new rules since they do not satisfy a reasonable axiom related to operations on product spaces. This latter fact should however not overlook the performance gains for information fusion systems suggested by preliminary experiments using this new family of rules.

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