T-norm and Uninorm-Based Combination of Belief Functions

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***Abstract*—The distinctness assumption is a limitation to the use of the unnormalized Dempster's rule. Denoeux recently proposed an alternative rule, called the cautious rule, which does not rely on this assumption. He further showed that the cautious rule and the unnormalized Dempster's rule belong to two families of combination rules having different algebraic properties. This paper revisits this latter point: the cautious and unnormalized Dempster's rules can be seen as member of families of combination rules based on triangular norms and uninorms, respectively. Furthermore, both rules have a special position in their respective family: they are the least committed elements. This paper also provides a means of obtaining an infinity of rules in the family of uninorm-based combination rules.**

I. Introduction

The Transferable Belief Model (TBM) [16], [14] is a model for quantifying beliefs using belief functions [12]. An essential mechanism of the TBM is the unnormalized Dempster's rule of combination. This rule, referred to as the TBM conjunctive rule in this paper, allows the fusion of belief functions.

The TBM conjunctive rule is justified only when it is safe to assume that the items of evidence to be combined are distinct. Recently, Denœux [1] proposed a rule, called the cautious rule of combination, which does not rely on the distinctness assumption. The cautious rule is based on the weight function, an equivalent representation of a nondogmatic belief function. The TBM conjunctive rule can also be expressed using the weight function, which makes it interesting to study rules based on this function.

There are important differences between the cautious rule and the TBM conjunctive rule: the cautious rule is idempotent but does not have a neutral element, whereas the TBM conjunctive rule has a neutral element, the vacuous belief function, but is not idempotent. These differences between the TBM conjunctive rule and the cautious rule can be cast in a more general context, as these rules belong to two families of combination rules having different algebraic properties [1].

This paper provides new results related to this latter point using the concepts of t-norms [8] and uninorms [18]. Indeed, it shows that the cautious rule can be seen as a member of an infinite family of combination rules based on t-norms on $(0, +\infty]$, whereas the TBM conjunctive rule belongs to an infinite family of combination rules based on uninorms on $(0, +\infty]$ having one as neutral element. Furthermore, both rules have a special position in their respective family: they are the least committed elements, according to a particular partial ordering.

Reference [1] provided a means of obtaining rules in the family of t-norm-based combination rules. This paper completes this work in that it gives a means of obtaining rules in the family of uninorm-based combination rules. The immediate potential use of those rules is based on the remark that rules based on parameterized families of uninorms can be defined, making it possible to tune the rules so as to optimize the performance of a fusion system.

The rest of this paper is organized as follows. Necessary notions, such as the canonical decomposition of a belief function and the LCP, are first recalled in Section 2. Section 3 reviews material on t-norms, t-conorms, uninorms and extends also the definitions of those operators beyond the usual unit interval. T-norm and uninorm-based combination rules are introduced in Section 4. This latter section also gives a means of obtaining an infinity of uninorm-based combination rules. Section 5 concludes the paper.

II. Fundamental Concepts of the TBM

A. Basic Definitions and Notations

In this paper, the TBM [16], [14] is accepted as a model to quantify uncertainties based on belief functions [12]. The beliefs held by an agent Ag on a finite frame of discernment $\Omega = \{\omega_1, ..., \omega_K\}$ are represented by a basic belief assignment (BBA) m defined as a mapping from 2^Ω to $[0, 1]$ verifying $\sum_{A \subseteq \Omega} m(A) = 1$. Subsets A of Ω such that $m(A) > 0$ are called focal sets (FS) of m. A BBA m is said to be:

- vacuous if Ω is the only focal set;
- dogmatic if Ω is not a focal set;
- simple if it has at most two focal sets and, if it has two, Ω is one of those.

A simple BBA (SBBA) m such that $m(A) = 1 - w$ and $m(\Omega) = w$ for some $A \neq \Omega$ and some $w \in [0, 1]$ can be noted A^w. The vacuous BBA can thus be noted A^1 for any $A \subset \Omega$. The advantage of this notation will appear later.

Equivalent representations of a BBA m exist. In particular the commonality function is defined as:

$$q(A) = \sum_{B \supseteq A} m(B),$$

for all $A \subseteq \Omega$.

The TBM conjunctive rule allows the fusion of belief functions. It assumes the sources of information to be reliable and distinct. It is noted $\textcircled{\cap}$ and it is defined as follows. Let m_1 and m_2 be two BBAs, and let $m_{1\textcircled{\cap}2}$ be the result of their combination by the TBM conjunctive rule. We have:

$$m_{1\textcircled{\cap}2}(A) = \sum_{B\cap C=A} m_1(B)\, m_2(C), \forall A \subseteq \Omega.$$

This rule is commutative, associative and admits a unique neutral element: the vacuous BBA. Let $\mathcal{M}$ be the set of BBAs, $(\mathcal{M}, \textcircled{\cap})$ is thus a commutative monoid. Note that reference [6] studies further properties of this monoid, called Dempster semigroup in [6], when $\mathcal{M}$ is the set of BBAs defined on binary frames of discernment.

B. Canonical Decomposition of a Belief Function

In [15], Smets proposed a solution to canonically decompose any nondogmatic BBA. This decomposition uses the concept of a generalized SBBA (GSBBA), which is defined as a function μ from 2^Ω to $\mathbb{R}$ by:

$$\begin{aligned}
\mu(A) &= 1-w,\\
\mu(\Omega) &= w,\\
\mu(B) &= 0 \quad \forall B \in 2^\Omega \setminus \{A,\Omega\},
\end{aligned}$$

for some $A \neq \Omega$ and some $w \in [0,+\infty)$. Extending the SBBA notation, any GSBBA can be noted A^w. When $w \leq 1$, μ is a SBBA. When $w > 1$, μ is no longer a BBA; Smets [15] called such function an inverse SBBA.

Smets showed that any nondogmatic BBA m can be uniquely represented as the conjunctive combination of generalized SBBAs:

$$m = \textcircled{\cap}_{A\subset\Omega} A^{w(A)},$$

with $w(A) \in (0,+\infty)$ for all $A \subset \Omega$. The weights $w(A)$ for each $A \subset \Omega$ are obtained as follows:

$$w(A) = \prod_{B\supseteq A} q(B)^{(-1)^{|B|-|A|+1}}.$$

The function $w : 2^\Omega \setminus \{\Omega\} \rightarrow (0,+\infty)$ is called the weight function. It is yet another equivalent representation of a nondogmatic BBA m.

The TBM conjunctive rule has a simple expression using the weight function. Let m_1 and m_2 be two nondogmatic BBAs with weight functions w_1 and w_2. We have:

$$m_{1\textcircled{\cap}2} = \textcircled{\cap}_{A\subset\Omega} A^{w_1(A)\cdot w_2(A)},$$

and, equivalently, $w_{1\textcircled{\cap}2} = w_1 w_2$. The properties of the TBM conjunctive rule become evident when one notices that the product on $(0,+\infty]$ has one as neutral element and is commutative and associative. Besides, the product is increasing on $(0,+\infty]$, hence it happens to be an operation having similar properties as uninorms [18] on $(0,+\infty]$. Note that this fact is the starting point of the ideas developed in this paper.

C. Informational Comparison of Belief Functions

The least commitment principle (LCP) plays a similar role in the TBM as the principle of maximum entropy does in Bayesian Probability Theory. As explained in [13], it postulates that, given a set of BBAs compatible with a set of constraints, the most appropriate BBA is the least informative. This principle becomes operational through the definition of partial orderings allowing the informational comparison of BBAs. Such orderings, generalizing set inclusion, were proposed by Yager [17], and Dubois and Prade [3]. In particular, the q-ordering is defined as follows: $m_1 \sqsubseteq_q m_2$ iff $q_1(A) \leq q_2(A)$ for all $A \subseteq \Omega$.

Recently, Denœux [1] proposed a new partial ordering. It is defined as follows: given two nondogmatic BBAs m_1 and m_2, $m_1 \sqsubseteq_w m_2$ iff $w_1(A) \leq w_2(A)$ for all $A \subset \Omega$. Let us also give the following lemma which will be needed later.

Lemma 1 (Lemma 1 of [1]): Let m be a nondogmatic BBA with weight function w, and let w' be a mapping from $2^\Omega \setminus \{\Omega\}$ to $(0,+\infty)$ such that $w'(A) \leq w(A)$ for all $A \subset \Omega$. Then w' is the weight function of some BBA m'.

In the remainder of this paper, a BBA m_1 is said to be x-more committed than a BBA m_2, with $x \in \{w,q\}$, if we have $m_1 \sqsubseteq_x m_2$. Let $\mathcal{M}_{nd}$ be the set of nondogmatic BBAs, we can remark that $(\mathcal{M}_{nd}, \textcircled{\cap}, \sqsubseteq_x)$ is a partially ordered commutative monoid, i.e., for all m_1, m_2 and m_3, $m_1 \sqsubseteq_x m_2$ implies $m_1 \textcircled{\cap} m_3 \sqsubseteq_x m_2 \textcircled{\cap} m_3$.

D. The Cautious Rule of Combination

The TBM conjunctive rule is justified only when it is safe to assume that the items of evidence to be combined are distinct. When this assumption does not hold, an alternative consists in adopting a cautious, or conservative, attitude to the merging of belief functions by applying the LCP [4], [1], [2].

Let us recall the building blocks of the cautious conjunctive merging of belief functions. Suppose we get two reliable sources of information which provide two BBAs m_1 and m_2. Upon receiving those two pieces of information, the agent's state of belief should be represented by a BBA m_{12} more informative than m_1 and m_2. Let $\mathcal{S}_x(m)$ be the set of BBAs m' such that $m' \sqsubseteq_x m$, for some $x \in \{w,q\}$. Hence $m_{12} \in \mathcal{S}_x(m_1)$ and $m_{12} \in \mathcal{S}_x(m_2)$ or, equivalently, $m_{12} \in \mathcal{S}_x(m_1) \cap \mathcal{S}_x(m_2)$. According to the LCP, the x-least committed BBA should be chosen in $\mathcal{S}_x(m_1) \cap \mathcal{S}_x(m_2)$. This defines a conjunctive combination rule if the x-least committed BBA exists and is unique.

Choosing the w-ordering yields an interesting solution [1, Proposition 4] which Denœux uses to define an idempotent rule called the cautious rule.

Definition 1 (Definition 1 of [1]): Let m_1 and m_2 be two nondogmatic BBAs, and let $m_{1\textcircled{\wedge}2} = m_1 \textcircled{\wedge} m_2$ denote the result of their combination by the cautious rule. The weight function of the BBA $m_{1\textcircled{\wedge}2}$ is:

$$w_{1\textcircled{\wedge}2}(A) = w_1(A) \wedge w_2(A), \forall A \subset \Omega. \tag{1}$$

We thus have:

$$m_{1\textcircled{\wedge}2} = \textcircled{\cap}_{A\subset\Omega} A^{w_1(A)\wedge w_2(A)}.$$

The cautious rule is obviously commutative, associative, and increasing with respect to the $\sqsubseteq_w$ ordering: if $m_1 \sqsubseteq_w m_2$, then $m_1 \owedge m \sqsubseteq_w m_2 \owedge m$ for all m. These properties are due to similar properties of the minimum, which in turn happens to be an operation having similar properties as triangular norms [8] on $(0, +\infty]$, with $+\infty$ as neutral element. Eventually, remark that $(\mathcal{M}_{nd}, \owedge, \sqsubseteq_w)$ is a partially ordered commutative semigroup.

III. T-norms, T-conorms and Uninorms

Triangular norms (t-norms) and triangular conorms (t-conorms) [8] are well known binary operations on $[0, 1]$. They are noted, respectively, $\top$ and $\perp$. Uninorms [18] generalize those two operations. They are noted $*$. Uninorms are defined on $[0, 1]$, such that for all $w, x, y, z \in [0, 1]$, the following properties are satisfied:

$$x * y = y * x \text{ (commutativity)}, \tag{2}$$

$$w * x \geq y * z, \text{if } w \geq y \text{ and } x \geq z \text{ (monotonicity)}, \tag{3}$$

$$x * (y * z) = (x * y) * z \text{ (associativity)}, \tag{4}$$

$$\exists\, e \in [0, 1], x * e = e * x = x \text{ (neutral element)}. \tag{5}$$

Uninorms generalize t-norms and t-conorms in that those latter two operators are uninorms such that $e = 1$ and $e = 0$, respectively. The concepts of t-norms and t-conorms are dual in some sense [8]. Indeed, let N be a mapping from $[0, 1]$ to $[0, 1]$ defined by $N(x) = 1 - x$, for all $x \in [0, 1]$. Then, the dual t-conorm $\perp$ of a t-norm $\top$ is obtained as follows:

$$x \perp y = N(N(x) \top N(y)),\ x, y \in [0, 1]. \tag{6}$$

In the rest of this paper, we will use the terms t-norms, t-conorms and uninorms to designate, more generally, operations satisfying properties (2)-(5) on other intervals than $[0, 1]$. In particular, an operation on $(0, +\infty]$, noted $\star$, satisfying properties (2)-(5) for $e = +\infty$ is called t-norm on $(0, +\infty]$. Similarly an operation on $(0, +\infty]$, noted $\circ$, satisfying properties (2)-(5) for $e = 1$ is called uninorm on $(0, +\infty]$. Eventually, we will also need to consider operations on $[1, +\infty]$, noted $\diamond$, satisfying properties (2)-(5) for $e = 1$; such operations $\diamond$ are called t-conorms on $[1, +\infty]$. The reasons for these slight generalizations of the definitions of those familiar operations will become apparent later. Let us first introduce some usual conventions [8, pp xviii] related to the use of the extended real line: we have $1/+\infty = 0$ and $1/0 = +\infty$.

Lemma 2: The minimum is the largest t-norm on $(0, +\infty]$.

Proof: Any t-norm on $(0, +\infty]$ is noted $\star$. It has by definition $+\infty$ as neutral element and is increasing, hence we have $x \star y \leq x \star +\infty = x$ and $x \star y \leq +\infty \star y = y$, so $x \star y \leq x \wedge y$, for all $x, y \in (0, +\infty]$. ∎

It is possible to obtain an infinity of t-norms on $(0, +\infty]$ using usual t-norms on $[0, 1]$ and usual t-conorms on $[0, 1]$ as shown by the following proposition.

Proposition 1: Let $\top$ be a positive t-norm on $[0, 1]$, i.e., $x, y \in (0, 1]$ implies $x \top y > 0$, and let $\perp$ be a t-conorm on $[0, 1]$. Then the operator $\star_{\top,\perp}$ defined by

$$x \star_{\top,\perp} y = \begin{cases} x \top y & \text{if } x \vee y \leq 1, \\ \left(\frac{1}{x} \perp \frac{1}{y}\right)^{-1} & \text{if } x \wedge y > 1, \\ x \wedge y & \text{otherwise,} \end{cases} \tag{7}$$

for all $x, y \in (0, +\infty]$ is a t-norm on $(0, +\infty]$.

Proof: (Sketch) Commutativity, associativity, and monotonicity of $\star_{\top,\perp}$ can be proved using Lemma 2 of [1]. Let us now show that $+\infty$ is a neutral element for $\star_{\top,\perp}$: for $x \leq 1$, $+\infty \star_{\top,\perp} x = +\infty \wedge x = x$, and for $x > 1$, $+\infty \star_{\top,\perp} x = \left(0 \perp \frac{1}{x}\right)^{-1} = x$. ∎

Another remark of interest for a latter part of this paper is that the concepts of t-norms on $[0, 1]$ and t-conorms on $[1, +\infty]$ may be seen as dual using a mechanism similar to (6) as shown by the following proposition.

Proposition 2: Let I be a mapping from $[1, +\infty]$ to $[0, 1]$ defined by $I(x) = 1/x$, for all $x \in [1, +\infty]$. Let $\top$ be a t-norm on $[0, 1]$. Then, the operator $\diamond_\top$ defined by:

$$x \diamond_\top y = I(I(x) \top I(y)), \tag{8}$$

for all $x, y \in [1, +\infty]$ is a t-conorm on $[1, +\infty]$.

Proof:

- The commutativity of $\diamond_\top$ results from the commutativity of $\top$.
- Monotonicity: let us show that $w \diamond_\top x \geq y \diamond_\top z$ if $w \geq y$ and $x \geq z$, for $w, x, y, z \in [1, +\infty]$.
  By the monotonicity of $\top$, we have:
$$\begin{aligned} (1/w) \top (1/x) &\leq (1/y) \top (1/z) \\ ((1/w) \top (1/x))^{-1} &\geq ((1/y) \top (1/z))^{-1} \\ w \diamond_\top x &\geq y \diamond_\top z \end{aligned}$$
- Associativity: let $x, y, z \in [1, +\infty]$.
$$\begin{aligned} x \diamond_\top (y \diamond_\top z) &= x \diamond_\top ((1/y) \top (1/z))^{-1} \\ &= \left(\frac{1}{x} \top \frac{1}{((1/y) \top (1/z))^{-1}}\right)^{-1} \\ &= ((1/x) \top (1/y) \top (1/z))^{-1} \\ &= \left(\frac{1}{((1/x) \top (1/y))^{-1}} \top \frac{1}{z}\right)^{-1} \\ &= ((1/x) \top (1/y))^{-1} \diamond_\top z \\ &= (x \diamond_\top y) \diamond_\top z \end{aligned}$$
- Neutral element: $x \diamond_\top 1 = \left(\frac{1}{x} \top \frac{1}{1}\right)^{-1} = x$

∎

The product is commutative, associative and increasing. It also has 1 as neutral element, which makes it a uninorm on $(0, +\infty]$. Proposition 1 has provided a means to obtain t-norms on $(0, +\infty]$ out of usual t-norms and t-conorms. The following proposition shows that uninorms on $(0, +\infty]$ having 1 as neutral element can be obtained out of usual t-norms.

Proposition 3: Let $\top$ be a positive t-norm on $[0,1]$ and let $\top'$ be a t-norm on $[0,1]$. Then the operator $\circ_{\top,\top'}$ defined by

$$x \circ_{\top,\top'} y \quad = \quad \begin{cases} x \top y & \text{if } x \vee y \leq 1, \\ \left(\frac{1}{x} \top' \frac{1}{y}\right)^{-1} & \text{if } x \wedge y \geq 1, \\ x \wedge y & \text{otherwise,} \end{cases} \tag{9}$$

for all $x, y \in (0, +\infty]$ is a uninorm on $(0, +\infty]$ having 1 as neutral element.

Proof: The notation is simplified in this proof: $\circ_{\top,\top'}$ is simply noted $\circ$. To further simplify the proof, we can also remark that, for all $x, y \in [1, +\infty]$, the operator $\circ$ is equivalent using Proposition 2 to a t-conorm noted $\diamond_{\top'}$ on $[1, +\infty]$, hence we can equivalently write:

$$x \circ_{\top,\top'} y \quad = \quad \begin{cases} x \top y & \text{if } x \vee y \leq 1, \\ x \diamond_{\top'} y & \text{if } x \wedge y \geq 1, \\ x \wedge y & \text{otherwise.} \end{cases} \tag{10}$$

- Commutativity of $\circ$: it results from the commutativity of $\top, \diamond_{\top'}$ and $\wedge$.
- Monotonicity: if $w \geq y$ and $x \geq z$, we must show that $w \circ x \geq y \circ z$, for $w, x, y, z \in (0, +\infty]$. The monotonicity of $\circ$ can be proved by considering three cases:
  - If $w \circ x = w \top x$, then, by the monotonicity of $\top$, $w \top x \geq y \top z$.
  - Suppose $w \circ x = w \diamond_{\top'} x$. Then:
    * if $y \circ z = y \diamond_{\top'} z$, then, by the monotonicity of $\diamond_{\top'}$, $w \diamond_{\top'} x \geq y \diamond_{\top'} z$;
    * if $y \circ z = y \wedge z$, then $w \diamond_{\top'} x \geq y \wedge z$;
    * if $y \circ z = y \top z$, then $w \diamond_{\top'} x \geq y \top z$.
  - Suppose $w \circ x = w \wedge x$. Then:
    * if $y \circ z = y \wedge z$, then, by the monotonicity of $\wedge$, $w \wedge x \geq y \wedge z$;
    * if $y \circ z = y \top z$, then $w \wedge x \geq y \top z$;
    * $y \circ z = y \diamond_{\top'} z$ is not possible since when $w \circ x = w \wedge x$, either w or x is strictly smaller than 1, hence y or z is also strictly smaller than 1.
- Neutral element:
  - if $w \leq 1$, then $w \circ 1 = w \top 1 = w$;
  - if $w \geq 1$, then $w \circ 1 = w \diamond_{\top'} 1 = w$.
- Associativity: without loss of generality, suppose $x \geq y \geq z$, and $x, y, z \in (0, \infty]$.
  - if $z \geq 1$, then

$$\begin{aligned} x \circ (y \circ z) &= x \circ (y \diamond_{\top'} z) \\ &= x \diamond_{\top'} (y \diamond_{\top'} z) \\ &= (x \diamond_{\top'} y) \diamond_{\top'} z \\ &= (x \circ y) \circ z; \end{aligned}$$

  - if $x \leq 1$, then

$$\begin{aligned} x \circ (y \circ z) &= x \circ (y \top z) \\ &= x \top (y \top z) \\ &= (x \top y) \top z \\ &= (x \circ y) \circ z; \end{aligned}$$

  - $x \geq 1 \geq y \geq z$: this situation can be broken up into three distinct situations:
    * if $x > 1 > y \geq z$, then

$$x \circ (y \circ z) = x \circ (y \top z) = x \wedge (y \top z) = y \top z,$$

and $(x \circ y) \circ z = (x \wedge y) \circ z = y \circ z = y \top z$;

    * if $x \geq 1 = y \geq z$, then, as 1 is a neutral element for $\circ$, we have $x \circ (y \circ z) = x \circ z$ and $(x \circ y) \circ z = x \circ z$;
    * if $x = 1 \geq y \geq z$, then $x \circ (y \circ z) = y \circ z$ and $(x \circ y) \circ z = y \circ z$.
  - $x \geq y \geq 1 \geq z$: this situation can be broken up into three distinct situations:
    * if $x \geq y > 1 > z$, then

$$x \circ (y \circ z) = x \circ (y \wedge z) = x \circ z = x \wedge z = z,$$

and $(x \circ y) \circ z = (x \diamond_{\top'} y) \circ z = (x \diamond_{\top'} y) \wedge z = z$;

    * if $x \geq y = 1 \geq z$: this situation has been already handled;
    * if $x \geq y \geq 1 = z$, then $x \circ (y \circ z) = x \circ y$ and $(x \circ y) \circ z = x \circ y$.

■

An interesting feature of the uninorms on $(0, +\infty]$ having 1 as neutral element that are obtained by Proposition 3 is that they have a similar structure as the usual uninorms on $[0,1]$. Indeed, Fodor, Yager and Rybalov [5] have shown that the structure of *proper* uninorms on $[0,1]$ (uninorms for which the neutral element e is such that $e \in (0,1)$) is related to t-norms on the square $[0,e]^2$ and to t-conorms on the square $[e,1]^2$. Now, remark that the uninorms on $(0, +\infty]$ having $e = 1$ as neutral element and obtained by Proposition 3 behave like a t-norm on $(0,e]^2$ and like a t-conorm on the square $[e, \infty]^2$.

IV. T-RULES AND U-RULES

Denœux [1] showed that the cautious and TBM conjunctive rules have different algebraic properties. This section provides new results related to this fact using the concepts of t-norms and uninorms on $(0, +\infty]$ introduced in the previous section.

As previously mentioned, the minimum on $(0, +\infty]$ is a t-norm on $(0, +\infty]$. The cautious rule belongs thus to the family of rules based on pointwise combination of weights using t-norms on $(0, +\infty]$.

Proposition 4: Let w_1 and w_2 be the weight functions associated to any two nondogmatic BBAs m_1 and m_2. The function obtained by pointwise combination of the weight functions w_1 and w_2 using any t-norm on $(0, +\infty]$ is the weight function of some nondogmatic BBA.

Proof: Direct from Lemmas 1 and 2. ■

Proposition 4 allows us to define combination rules for belief functions which can be formally defined as follows.

Definition 2 (T-norm-based combination rule): . Let $\star$ be a t-norm on $(0, +\infty]$. Let m_1 and m_2 be two nondogmatic

BBAs. Their combination using the t-norm based rule, or t-rule for short, is noted $m_{1\circledast 2} = m_1 \circledast m_2$. It is defined as a BBA with the following weight function:

$$w_{1\circledast 2}(A) = w_1(A) \star w_2(A), \forall A \subset \Omega.$$

We thus have:

$$m_{1\circledast 2} = \bigcirc\!\!\!\!\cap_{A\subset\Omega} A^{w_1(A)\star w_2(A)}.$$

One can obtain t-rules using the t-norms on $(0,+\infty]$ given by Proposition 1. Remark that, from Lemma 2, the cautious rule is the w-least committed t-rule. Further, for any t-rule $\circledast$, $(\mathcal{M}_{nd}, \circledast, \sqsubseteq_w)$ is a partially ordered commutative semigroup.

We have seen that the TBM conjunctive rule is based on pointwise multiplication of weights and that the product is a uninorm on $(0,+\infty]$ with 1 as neutral element. Hence, the TBM conjunctive rule belongs to the family of rules based on pointwise combination of weights using uninorms on $(0,+\infty]$ having 1 as neutral element. Let us now recall a result presented in [10] related to this family of rules.

Theorem 1 (Theorem 1 of [10]): Let $\circ$ be a binary operator on $(0,+\infty]$ with 1 as two-sided neutral element (i.e., $1\circ x = x \circ 1 = x$) such that $\exists x, y,\ x \circ y > xy$. There exist two nondogmatic BBAs m_1 and m_2 on a frame Ω such that the function obtained by pointwise combination using $\circ$ of the weight functions associated to m_1 and m_2 is not a weight function of some nondogmatic BBA.

Proposition 5: Let w_1 and w_2 be the weight functions associated to two nondogmatic BBAs m_1 and m_2. Let $\circ$ be an operator on $(0,+\infty]$ with 1 as neutral element and such that $x \circ y \leq xy$ for all $x, y \in (0,+\infty]$. Then the function $w_{1\circledcirc 2}$ defined by :

$$w_{1\circledcirc 2}(A) = w_1(A) \circ w_2(A), \forall A \subset \Omega, \qquad (11)$$

is a weight function associated to some nondogmatic BBA.

Proof: Direct from Lemma 1. ■

Theorem 1 and Proposition 5 allow us to define new combination rules which can be formally defined as follows.

Definition 3 (Uninorm-based combination rule): Let $\circ$ be a uninorm on $(0,+\infty]$ having 1 as neutral element and such that $x \circ y \leq xy$ for all $x, y \in (0,+\infty]$. Let m_1 and m_2 be two nondogmatic BBAs. Their combination using the uninorm-based rule, or u-rule for short, is noted $m_{1\circledcirc 2} = m_1 \circledcirc m_2$. It is defined as a BBA with the following weight function:

$$w_{1\circledcirc 2}(A) = w_1(A) \circ w_2(A), \forall A \subset \Omega.$$

We thus have:

$$m_1 \circledcirc m_2 = \bigcirc\!\!\!\!\cap_{A\subset\Omega} A^{w_1(A)\circ w_2(A)}.$$

It is clear that the TBM conjunctive rule is the w-least committed u-rule. Furthermore, for any u-rule $\circledcirc$, $(\mathcal{M}_{nd}, \circledcirc, \sqsubseteq_w)$ is a partially ordered commutative monoid, with the vacuous BBA as neutral element.

Let us now provide a means of obtaining other u-rules than the TBM conjunctive rule. For that, we merely need uninorms on $(0,+\infty]$ having 1 as neutral element and such that $x \circ y \leq xy$ for all $x, y \in (0,+\infty]$. One way of obtaining such uninorms is given in Proposition 7 below. Let us first give the following technical result needed for Proposition 7.

Proposition 6: Let $\top$ be a t-norm on $[0,1]$ verifying $x\top y \geq xy$ for all $x, y \in [0,1]$. Then the t-conorm $\diamond_\top$ on $[1,+\infty]$ obtained using (8) verifies $x \diamond_\top y \leq xy$ for all $x, y \in [1,+\infty]$.

Proof: For all $x, y \in [1,+\infty]$, we have:

$$\begin{aligned} (1/x) \top (1/y) &\geq (1/x)(1/y) \\ ((1/x) \top (1/y))^{-1} &\leq ((1/x)(1/y))^{-1} \\ x \diamond_\top y &\leq x \cdot y \end{aligned}$$

■

Proposition 7: Let $\top$ be a positive t-norm on $[0,1]$ verifying $x\top y \leq xy$ for all $x, y \in [0,1]$, and let $\top'$ be a t-norm on $[0,1]$ verifying $x\top' y \geq xy$ for all $x, y \in [0,1]$. Then the operator defined by

$$x \circ_{\top,\top'} y = \begin{cases} x\top y & \text{if } x \vee y \leq 1, \\ \left((1/x)\top'(1/y)\right)^{-1} & \text{if } x \wedge y \geq 1, \\ x \wedge y & \text{otherwise,} \end{cases}$$

for all $x, y \in (0,+\infty]$ is a uninorm on $(0,+\infty]$ having 1 as neutral element and verifying $x \circ_{\top,\top'} y \leq xy$ for all $x, y \in (0,+\infty]$.

Proof: From Proposition 3, $\circ_{\top,\top'}$ is a uninorm on $(0,+\infty]$ having 1 as neutral element. Let us now show that $x \circ_{\top,\top'} y \leq xy$ for all $x, y \in (0,+\infty]$.

- By definition, $x\top y \leq xy$ for all $x, y \in (0,1]$.
- For all $x, y \in [1,+\infty]$, $\circ_{\top,\top'}$ is equivalent using Propositions 2 and 6 to a t-conorm on $[1,+\infty]$, noted $\diamond_{\top'}$, verifying $x \diamond_{\top'} y \leq xy$ for all $x, y \in [1,+\infty]$.
- Finally, it is clear that $x \wedge y \leq x \cdot y$ if $x \vee y > 1$ and $x \wedge y < 1$. Hence, $x \circ_{\top,\top'} y \leq xy$ for all $x, y \in (0,+\infty]$.

■

Using Proposition 7, a u-rule can be associated to a pair of t-norms $\top$ and $\top'$. By choosing parameterized families of t-norms [8], it is thus possible to define parameterized families of belief function combination rules. This introduces the possibility to learn a combination rule from examples, as shown in the simple illustrative Example 1 below. This example uses the Frank family of t-norms [8], which is defined by:

$$x\top_s y = \begin{cases} x \wedge y & \text{if } s = 0, \\ xy & \text{if } s = 1, \\ \log_s\left(1 + \frac{(s^x-1)(s^y-1)}{s-1}\right) & \text{otherwise,} \end{cases}$$

for all $x, y \in [0,1]$, where s is a positive parameter. This family of t-norms is useful for the task at hand since it allows one to create t-norms $\top$ and $\top'$ which verify the conditions of Proposition 7.

Example 1: Assume that the BBAs m_1 and m_2 shown in Table I have been provided by two sensors, and expert knowledge regarding the true value of the variable of interest is represented by BBA m_e also shown in Table I. For this

TABLE I
BBAS OF EXAMPLE 1.

	$\emptyset$	$\{a\}$	$\{b\}$	$\{a,b\}$	$\{c\}$	$\{a,c\}$	$\{b,c\}$	Ω
m_1	0.1	0.1	0.1	0.1	0	0.15	0.15	0.3
m_2	0.05	0.05	0.05	0.15	0.05	0.15	0.15	0.35
m_e	0.24	0.14	0.14	0.09	0.08	0.11	0.11	0.09

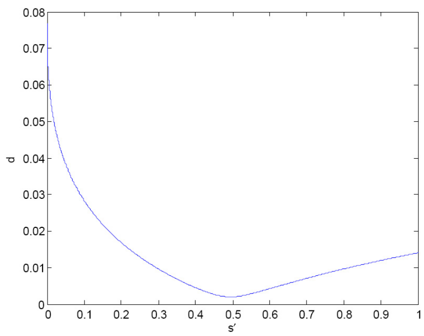

Fig. 1. Distance between the target BBA and the combined BBA, as a function of parameter s' of the Frank family of t-norms.

simple illustrative example, m_e was artificially constructed by combining m_1 and m_2 using a u-rule based on an uninorm $\circ_{\top_s,\top_{s'}}$, where $\top_{s'}$ and $\top_s$ are, respectively, two Franck t-norms with parameters $s' = 0.5$ and $s = 1/s'$, and adding a small amount of random noise.

We wish to find a u-rule $\circledcirc$ based on an uninorm $\circ_{\top_s,\top_{s'}}$, where $\top_{s'}$ and $\top_s$ are, respectively, two Franck t-norms with parameters s' and $s = 1/s'$, such that the combination of m_1 and m_2 yields a BBA as close as possible to m_e. Parameter s' was varied between 10^{-5} and 1, and for each value s', the discrepancy between $m_{12} = m_1 \circledcirc m_2$ and m_e was measured by Jousselme's distance [7], noted d. Distance d is plotted as a function of s' in Figure 1. The best fit between the combined BBA m_{12} and the target BBA m_e is obtained for $s' \approx 0.495$, which is an estimate of the true value of $s' = 0.5$.

V. CONCLUSION

This paper has brought to light two infinite families of combination rules for belief functions. Those families are different in that they yield different algebraic structures: commutative semigroups and commutative monoids. This is explained by the fact that one of the family is based on t-norms on $(0, +\infty]$ whereas the other family is based on uninorms on $(0, +\infty]$ having one as neutral element.

Of particular interest is that the well known unnormalized Dempster's rule belongs to the family of uninorm-based combination rules (u-rules) and that the more recent cautious rule belongs to the family of t-norm-based combination rules (t-rules). Moreover, those rules have a special position in their respective families: they are the least committed elements, according to a particular informational ordering.

One may object that these new combination rules, in spite of the interesting properties that they share with the unnormalized Dempster's rule or with the cautious rule, are only weakly justified. This latter fact should however not overlook the possible performance gains for information fusion systems that may be achieved using those rules. In particular, this paper has provided a means of obtaining u-rules from classical t-norms on $[0, 1]$. The existence of parameterized families of t-norms on $[0, 1]$ suggests that those u-rules can be tuned so as to optimize a performance measure, for instance an error rate in a classification problem. The interest of those rules in concrete applications remains nonetheless to be shown experimentally.

To conclude, it is worth mentioning that, despite the numerous properties shared by the unnormalized Dempster's rule and the u-rules, the unnormalized Dempster's rule is the only u-rule [11] that satisfies a particular axiom of the valuation algebra framework [9]. This property further singles out this rule among uninorm-based combination rules.

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