A New Approach to Assess Risk in Water Treatment using the Belief Function Framework*

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Abstract – *A methodology is proposed for assessing the risk to produce non-compliant potable water, taking into account the quality of the raw water, as well as characteristics of the treatment unit and different failure modes. Belief functions are used to describe expert knowledge of treatment process efficiency, failure rates, latency times and raw water quality. Evidential reasoning provides mechanisms to combine this information and assess the plausibility of non-compliant water production. This approach may be used by treatment plant designers to choose the optimal architecture, given a user-defined level of residual risk.*

Keywords: Risk Assessment, Belief Functions, Dempster-Shafer theory, Evidence Theory, Transferable Belief Model, Drinking Water Production.

1 Problem description and probabilistic solution

To manage the risk due to drinking water treatment process, it is fundamental to take into account the design and good dimensioning of the plant, during construction or rehabilitation steps. It is thus of primary importance to define the efficient but just necessary process line which will make it possible to lower the raw water quality parameters below the standards set by contractual clauses or by regulation.

The probabilistic approach described in [2] allows to define the probability of the undesirable event "production of non-compliant water", by taking into account the quality of the resource to be treated (i.e., the estimated probability to find a given concentration of an undesirable component), characteristics of the treatment unit (efficiency of the treatment steps), as well as different failure modes that can occur in the process line (failure rate and latency time of each failure mode).

Initially, the efficiency of each basic treatment step over each quality parameter is modelled by a transfer function, which gives the output concentration as a function of the input concentration for the corresponding quality parameter. In most cases, it is equivalent to a reduction factor (exceptions are undesirable components that can be introduced in treated water by the treatment process itself).

The second step of the approach consists in determining the failure modes that can affect the different treatment steps, synthesized in FMECA (Failure Mode Effects and Criticality Analyses) arrays, allowing to clearly identify, for each fault, possible causes, failure rate, detection means, corresponding latency time, qualitative effects, and finally the degraded transfer function, which is equivalent to a new reduction factor, smaller than or equal to the nominal reduction.

The third step allows to define acceptable water quality thresholds for raw water: by inversion of all global transfer functions (nominal and degraded ones), and by application of these inverse functions on all treated water thresholds (national regulation and internal recommendations), it is possible to define a series of thresholds now relative to raw water quality: if one or more raw water quality parameters exceeds the corresponding threshold, a scenario of non-compliant treated water is identified. Such scenarios are then synthesized by fault trees for each quality standard.

Finally, a global indicator of compliant water unavailability is calculated using the probability of the elementary events (unavailability of treatment steps and probability for the resource to exceed the different thresholds). This unavailability indicator is compared to the objective risk level, which allows to define an effective treatment line minimizing sanitary and financial risks.

However, one major difficulty in applying risk assessment methods in the environmental engineering domain is that basic data are not perfectly known and are often determined by expert judgment with a high level of uncertainty. We therefore propose to model the uncertainty on raw water quality, process line efficiency and state of the treatment plant (nominal or failure mode) in the belief function framework. This framework was chosen because of its flexibility for representing weak forms of knowledge, and because it generalizes Probability Theory, allowing to recover the classical results when all required data are available. Belief functions are used to describe expert knowledge of treatment steps efficiency, failure rates, latency times and raw water quality. By combining these basic belief functions, it is possible to obtain an assessment of the plausibility to produce non-compliant water.

2 Modeling uncertainties

As previously mentioned, the probabilistic solution assumes the availability of precise and complete prior knowledge of transfer function, failures rates and latency times, as well as enough historical data to estimate the distribution of water quality parameters. However, such knowledge and data are usually not available, in particular in the case of call for bids, where a proposal must be submitted based on partial information. Moreover, transfer functions, latency times and failure rates can only be obtained by laboratory tests, expert knowledge or feedback from operational sites, which generally does not allow to obtain reliable estimates for a specific site. This is why an approach integrating these various uncertainties was developed, using the belief function framework.

2.1 Notations and background

The interpretation of belief functions adopted in this paper is that of Smets' Transferable Belief Model (TBM) [8]. In this model, a belief function is understood as representing an agent's state of belief, without resorting to an underlying probability model. Only the essential definitions and specific notations will be given here. A detailed exposition of the TBM may be found in [8]. A basic belief assignment (bba) on domain (or frame of discernment) X is noted m^X (for convenience, we use the same notation X for a variable and its domain). It is defined as a function from the powerset 2^X of X to $[0,1]$ verifying $\sum_{A\subseteq X} m^X(A) = 1$. The corresponding belief and plausibility functions are defined, respectively, as: $bel^X(A) = \sum_{\emptyset\neq B\subseteq A} m^X(B)$ and $pl^X(A) = \sum_{B\cap A\neq\emptyset} m^X(B)$.

Given a bba $m^{X\times Y}$ defined on the Cartesian product of two domains X and Y, the marginal bba $m^{X\times Y\downarrow X}$ on X is defined, for all $A \subseteq X$, as

$$m^{X\times Y\downarrow X}(A) = \sum_{\{B\subseteq X\times Y \mid Proj(B\downarrow X)=A\}} m^{X\times Y}(B), \quad (1)$$

where $Proj(B \downarrow X)$ denotes the projection of B onto X, defined as

$$Proj(B \downarrow X) = \{x \in X \mid \exists y \in Y, (x,y) \in B\}\ . \quad (2)$$

Conversely, let m^X be a bba on X. Its *vacuous extension* on $X \times Y$ is defined as:

$$m^{X\uparrow X\times Y}(B) = \begin{cases} m^X(A) & \text{if } B = A\times Y, \\ & \text{for some } A \subseteq X, \\ 0 & \text{otherwise.} \end{cases} \quad (3)$$

Another useful notion is that of *ballooning extension* [7]. Let $m^X[y]$ denote the conditional bba on X, given that $Y = y$. The ballooning extension of $m^X[y]$ on $X \times Y$ is the least committed bba, whose conditioning on y yields $m^X[y]$ (see [7] for detailed justification). It is obtained for all $B \subseteq X \times Y$ as:

$$m^X[y]^{\Uparrow X\times Y}(B) = \begin{cases} m^X[y](A) & \text{if } B = C \\ 0 & \text{otherwise} \end{cases} \quad (4)$$

where $C = (A\times\{y\})\cup(X\times(Y\setminus\{y\}))$ for some $A \subseteq X$.

Let us now consider two bba's m_1^X and m_2^X induced by two distinct sources of information. If both sources are known to be reliable, they can be combined using the (unnormalized) Dempster's rule of combination, leading to a new bba $m^X_{1\textcircled{\cap}2} = m_1^X \textcircled{\cap} m_2^X$, defined as:

$$m^X_{1\textcircled{\cap}2}(A) = \sum_{B\cap C=A} m_1^X(B)m_2^X(C). \quad (5)$$

Finally, the TBM is based on a two level mental model: the *credal level* where beliefs are entertained and represented by belief functions, and the *pignistic level* where decisions are made. The *pignistic transformation* maps a bba m^X to a probability measure $BetP^X$ on X, defined as:

$$BetP^X(A) = \sum_{B\subseteq X} \frac{m^X(B)}{1-m^X(\emptyset)}\frac{|A\cap B|}{|B|}, \quad \forall A \subseteq X. \quad (6)$$

2.2 Application of the TBM

Available data. As previously explained, the principal difficulty when modeling such a problem is to obtain enough good quality data. Information necessary to evaluate the non-compliant water production risk are abatement rates (modeling the efficiency of the various treatment steps), occurrence probabilities of different failure modes (requiring the knowledge of failure rates and latency times), and finally, the distribution of raw water quality parameters. To take into account uncertainties on these data, we suppose that, for each operating modes x_i, $(i = 0..., n)$, the abatement rate for a given parameter is modeled by a triangular fuzzy number, noted $\widetilde{\alpha}_i = (\alpha_i^-, \alpha_i^0, \alpha_i^+)$. We also suppose that $[\lambda_i^-, \lambda_i^+]$ and $[T_i^-, T_i^+]$ denote interval-valued assessments of failure rate λ_i and latency time T_i. The precise knowledge of one or more of these data would result in $\alpha_i^- = \alpha_i^0 = \alpha_i^+$, $\lambda_i^- = \lambda_i^+$ or $T_i^- = T_i^+$. Finally, the probability distribution of the raw water quality parameter of interest is no longer assumed to be exactly known. Instead, we now suppose, more realistically, that a finished sample of measurement $C_{in,1}, ..., C_{in,K}$ has been observed.

Discretization. In the probabilistic approach, the output concentration was discretized in only two categories, depending on whether water met a fixed quality limit or not. However, this approach is too restrictive and results in a loss of information. To refine this discretization, we now define $\ell + 1$ thresholds σ_k, $k = 0, \dots, \ell$, which induce $\ell + 2$ possible states s_k, $k = 0, \dots, \ell + 1$ for the output water. This approach is more realistic, since risk studies are generally made on several sanitary gravity level. This new discretization will thus allow a criticality classification of non-compliant production scenarios.

For a given mode x_i of the treatment plant, the output threshold σ_k defines an input threshold $\theta_{k,i}$ (the input concentration must be less than $\theta_{k,i}$ for the output concentration to be less than σ_k when the treatment plant is in mode x_i). In order to recover the classical limit, one of the output thresholds must be the norm ($N = \sigma_k$ for some k) and the input thresholds must at least contain the $n+1$ values $\theta_{k,i}$ obtained with that k and all operating modes i of the treatment plant. However, the discretization can take into account more values than this minimal set. We note η_j $(0 \le j \le m)$ the input thresholds arranged in decreasing order and e_j $(0 \le j \le m + 1)$ the corresponding input states. Note that we define $\ell + 1$ thresholds for each of the $n + 1$ modes, so that $m+1 \le (n+1)(l+1)$ (the upper bound may not be strict because some of the thresholds may be equal). This discretization scheme is represented in Figure 1.

We thus have three underlying variables: the discretized input concentration taking values in $E = \{e_0, \dots, e_{m+1}\}$, the discretized output concentration taking values in $S = \{s_0, \dots, s_{\ell+1}\}$, and the plant state in $X = \{x_0, \dots, x_n\}$. The available pieces of information will now be modeled using the belief function framework: uncertainties on raw water quality as well as imprecise failure probabilities previously defined will induce belief functions bel^E and bel^X on E and X, respectively. For each operating mode x_i, the fuzzy transfer function will also be represented by a conditional belief function $bel^{E\times S}[x_i]$ on the joint space $E \times S$; this conditional belief function may be converted to an unconditional belief function bel_i^Y on $Y = X \times E \times S$ using the ballooning extension (4). The three variables and associated belief functions can be represented by a valuation network [6] as shown in Figure 2.

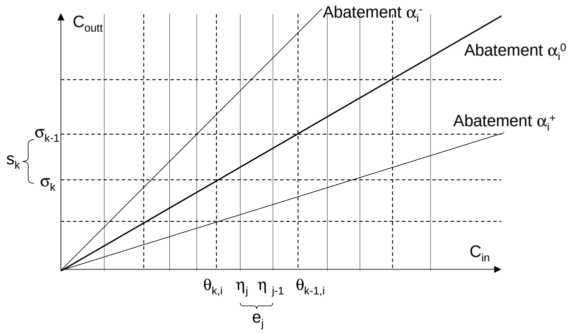

Figure 1: Discretization of input and output concentration, and fuzzy transfer functions.

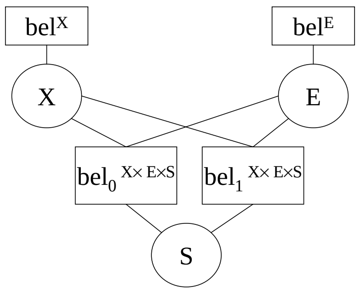

Figure 2: Representation of the 3 frames of discernment and associated belief functions (case of one failure mode).

Belief on X. In the probabilistic case, knowledge of failure rates λ_i and latency times T_i induce a probability function on X. Since λ_i and T_i are now only known to lie in given intervals, it is possible to define an imprecise probability $[p_i^-; p_i^+]$ for each operating mode of the treatment plant as

$$p_i^- = \lambda_i^- T_i^-, \quad p_i^+ = \lambda_i^+ T_i^+, \quad i = 1, \dots, n,$$

$$p_0^- = 1 - \sum_{i=1}^{n} \lambda_i^+ T_i^+, \quad p_0^+ = 1 - \sum_{i=1}^{n} \lambda_i^- T_i^-.$$

A family $\mathcal{P}^X$ of probability distributions on X is defined by the constraints $p_i^- \le p_i \le p_i^+ (i = 0, ..., n)$, and the lower and upper probability of an event $A \subseteq X$ are given by:

$$P^-(A) = \max\left(\sum_{x_i \in A} p_i^-, 1 - \sum_{x_i \notin A} p_i^+\right) \quad (7)$$

$$P^+(A) = \min\left(\sum_{x_i \in A} p_i^+, 1 - \sum_{x_i \notin A} p_i^-\right) \quad (8)$$

These lower and upper probabilities do not, in general, verify the axioms of belief and plausibility measures. However, we may represent this information in the belief function framework by the most specific bba m^X (according, e.g., to the nonspecificity uncertainty measure [5]), whose set of compatible probability functions includes $\mathcal{P}^X$. This may be obtained by solving the following linear program:

$$\min_{m^X} \sum_{\emptyset \neq A \subseteq X} m^X(A) \log |A|$$

under the constraints:

$$bel^X(A) \leq P^-(A) \leq P^+(A) \leq pl^X(A), \quad \forall A \subseteq X.$$

Belief on E. The available information on E consists of a finite sample $C_{in,1}, \ldots, C_{in,K}$ of observed values of the raw water quality parameter under study. A simple approach to build a belief function on E might be to consider the histogram, i.e., to define $m^E(e_j)$ as the relative frequency of observations falling in class e_j. This approach, however, is not satisfactory in the small sample case because it does not take into account the sample size. Ideally, the inferred belief function should reflect the amount of available information, and hence the sample size.

One way to achieve this goal is to generate B bootstrap replicates of the data [4]. Let us denote by $p_b(e_j)$ the relative frequency of class e_j in bootstrap sample b, and let us define $p^-(e_j)$ and $p^+(e_j)$ as, say, the 1st and 9th deciles of the distribution $p_b(e_j), b = 1, \ldots, B$. We then obtain lower and upper probabilities on E, which define a family $\mathcal{P}^E$ of probability distributions. As before, we may translate this information in the belief function format by considering the most specific bba m^E, whose set of compatible probability distributions includes $\mathcal{P}^E$. However, as E may be much larger than X, the complexity of this solution might become too high. A simpler approach is to restrict the number of focal elements of m^E. For instance, if m^E is constrained to be quasi-Bayesian (i.e., to have only singletons and E as focal elements), the solution can be shown to be:

$$m^E(\{e_j\}) = p^{*-}(e_j) \quad j = 0, \ldots, m+1 \quad (9)$$

$$m^E(E) = 1 - \sum_{j=0}^{m+1} m^E(\{e_j\}) \quad (10)$$

$$m^E(A) = 0 \quad \forall A \subseteq E, A \neq E, |A| \neq 1 \quad (11)$$

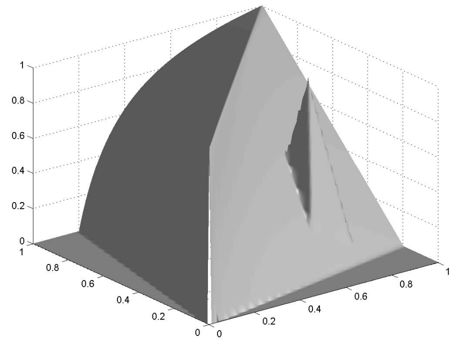

Figure 3: Possibility distribution on $E \times S$ and discretization procedure. The possibility distribution defined by (13) is represented by two pieces of hyperbolic paraboloids.

with

$$p^{*-}(e_j) = \max\left(p^-(e_j), 1 - \sum_{j' \neq j}^{m+1} p^+(e_{j'})\right) \quad (12)$$

for $j = 0, \ldots, m+1$.

Belief on $X \times E \times S$. The fuzzy abatement rate $\widetilde{\alpha}_i$ for operating mode i may be seen as defining a fuzzy relation between input and output concentrations. This fuzzy relation may be expressed as a possibility distribution π_i on variables C_{in} and C_{out} defined as a function of the ratio $\rho = C_{out}/C_{in}$ as:

$$\pi_i(C_{in}, C_{out}) = \begin{cases} 0 & \text{if } \rho \leq \beta_i^- \text{ or } \rho \geq \beta_i^+ \\ \dfrac{\rho - \beta_i^-}{\beta_i^0 - \beta_i^-} & \text{if } \beta_i^- < \rho \leq \beta_i^0 \\ \dfrac{\beta_i^+ - \rho}{\beta_i^+ - \beta_i^0} & \text{if } \beta_i^0 < \rho < \beta_i^+ \end{cases} \quad (13)$$

with $\beta_i^- = 1 - \alpha_i^+, \beta_i^0 = 1 - \alpha_i^0$ and $\beta_i^+ = 1 - \alpha_i^-$.

In order to avoid discontinuity problems linked to a division of space $E \times S$ into rectangles, a "smooth" discretization procedure was adopted. The value of the discretized possibility distribution in the rectangle $R_{jk} = [\eta_j, \eta_{j+1}) \times [\sigma_k, \sigma_{k+1})$ was computed as the height of the intersection between the quadratic possibility distribution defined by (13) and a pyramidal membership function with support R_{jk} and with kernel equal to the center of R_{jk} (Figure 3).

After discretization, we obtain for each state x_i a conditional possibility distribution $\pi^{E \times S}[x_i]$, which can be seen as defining a consonant bba $m^{E \times S}[x_i]$ (see [3]). This conditional bba can be converted into a joint bba m_i^Y using the ballooning extension (4).

Combination and marginalization. The final step is to combine all the available evidence, and marginalize on S. For that purpose, bel^X and bel^E must be

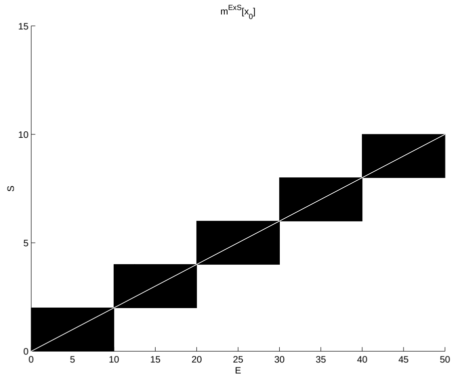

Figure 4: Possibility distribution on $E \times S$ conditional on the nominal state, when the transfer function is precisely known, and graphical representation of this transfer function.

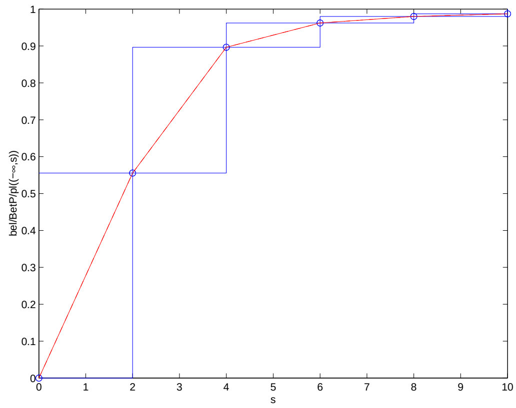

Figure 5: Belief distribution, pignistic probability and plausibility distribution for the output concentration to be smaller than an output thresholds s.

extended to the product space $Y = X \times E \times S$ using the vacuous extension (3). The resulting belief functions are combined using Dempster's rule, and the result is marginalized on S. Formally, the final bba m^S on S is thus defined as:

$$m^S = \left((\circledcap_{i=0}^{n} m_i^Y) \circledcap m^{X\uparrow Y} \circledcap m^{E\uparrow Y} \right)^{\downarrow S}. \quad (14)$$

Note that these operations can be performed very efficiently using local computation algorithms such as the one described in [6].

This modeling approach was shown in [1] to provide results identical to those of the probabilistic method, in the case where all input data are known precisely.

3 Simulation results

As an example, let us consider a simple problem with one normal mode x_0, one failure mode x_1 and six output concentration states. The cases of precise and imprecise data will be considered successively.

In the first case, the abatement rates as well as the failure rate and latency time of the failure mode are assumed to be known: $\alpha_0 = 0.8$, $\alpha_1 = 0.2$, $\lambda_1 = 1e^{-3}$ and $T_1 = 2 \times 24$. Figure 4 shows to the conditional possibility distribution on space $E \times S$ related to the nominal state x_0. The corresponding figure for state x_1 would be similar with a larger slope. Only "diagonal" rectangles (i.e. those rectangles whose diagonal is crossed by the line $\rho = 1 - \alpha_0$) receive a possibility value equal to 1. The resulting mass function has a unique focal element composed of these rectangles.

Figure 5 shows the belief, the pignistic probability and the plausibility for the output concentration to be lower than an output level s. We can see that beliefs, plausibilities and pignistic plausibilies are equal at discretization thesholds σ_k. This confirms that, in the case of precise data, the solution given by the belief function approach is identical to that given by the probabilistic method. In that case, we also have $bel^S((s, +\infty)) = 1 - bel^S(-\infty, s))$ for $s \in \{\sigma_0, \ldots, \sigma_\ell\}$, and similar relations for pl^S and $BetP^S$.

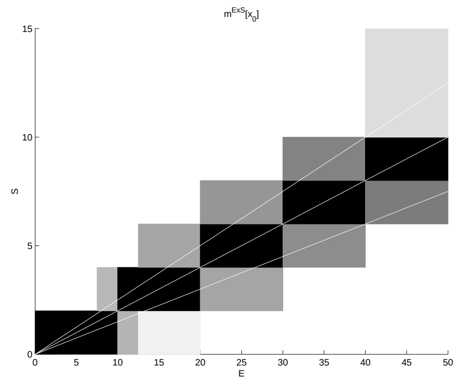

Figure 6: Possibility distribution on $E \times S$ conditional on the nominal state, and fuzzy transfer function.

In the second case (imprecise data), some uncertainty was introduced on abatement rates, failure rate and latency time: $\widetilde{\alpha}_0 = (0.75, 0.8, 0.85)$, $\widetilde{\alpha}_1 = (0.15, 0.2, 0.25)$, $\lambda_1 = [0.8e^{-3}, 1.2e^{-3}]$, $T_1 = [24, 3 \times 24]$.

Figure 6 shows the conditional possibility distribution on space $E \times S$ related to the nominal state x_0; it is therefore directly comparable to Figure 4. Once again, the corresponding figure for the failure mode x_1 would be similar with larger slopes (weaker abatement rates).

Figure 7 represents the belief, the pignistic probability and the plausibility for the output concentration to be less than an output value. The dotted line represents results obtained by the classical fault tree approach. We can see that the TBM approach allows to estimate the imprecision on output data resulting from the imprecision of inputs.

Finally, we can study the evolution of belief, pignistic probability and plausibility degrees for compliant water production, as a function of the uncertainty on abate-

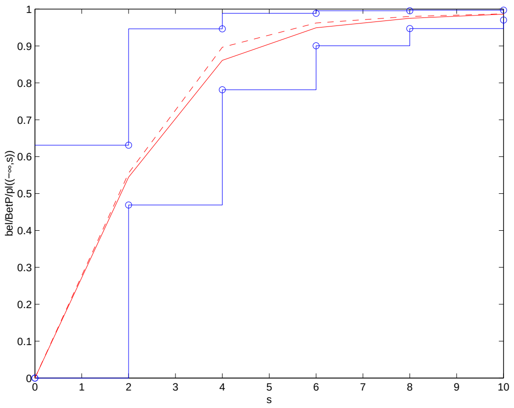

Figure 7: Belief, pignistic probability and plausibility for the output concentration to be smaller than an output value s as a function of this value.

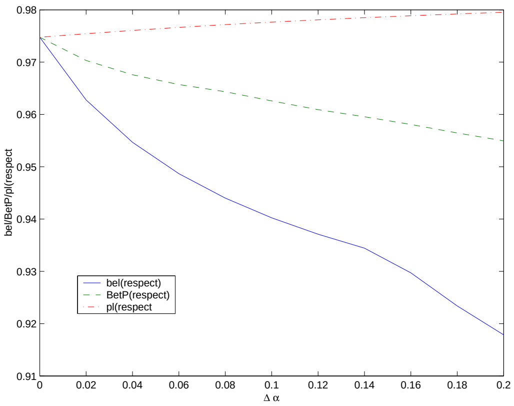

Figure 8: Variation of compliant water production belief, pignistic probability and plausibility degree as a function of uncertainty on abatement rates $\Delta\alpha$.

ment rates α. Figure 8 presents the variation of the 3 quantities in the case where $\widetilde{\alpha}_0 = (0.8 - \Delta\alpha, 0.8, 0.8 + \Delta\alpha)$ and $\widetilde{\alpha}_1 = (0.2 - \Delta\alpha, 0.2, 0.2 + \Delta\alpha)$ as a function of $\Delta\alpha$ (all other necessary data are known with perfect accuracy, see previous simulation paragraph for numeric value details). This is a good way to see the variation of output uncertainty as a function of input data incertainty. The three curves converge to the same point (equal to the probabilistic limit) when $\Delta\alpha$ tends to zero.

4 Conclusion

This paper has presented a methodology for assessing the risk with regard to unavailability of compliant water, by taking into account the quality of the resource, the characteristics of the treatment plant, and the possible failure modes of the treatment process. This approach integrates uncertainties on basic data (failure rates, latency times, raw water quality variability and efficiency of the treatment line) in order to define a degree of confidence in compliant water production. In the case where all available data are known with perfect accuracy, this method is equivalent to probabilistic method previously described and shortly recalled in this paper. Detailed simulations have shown that this approach meets operational needs. It allows the redimensioning of an existing plant or the dimensioning of a new plant, according to a risk level considered to be acceptable. It gives a level of confidence in the results as a function of uncertainty of input data. This is a very important point in ill-structured domains and poorly informed environments, where only weak forms of data and knowledge are available.

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