Computational Statistics
Chapter 1: Continuous optimization

1. The following data are assumed to be an i.i.d. sample from a Cauchy(θ,1) distribution:

   1.77, -0.23, 2.76, 3.80, 3.47, 56.75, -1.34, 4.24, -2.44, 3.29, 3.71,
   -2.40, 4.53, -0.07, -1.05, -13.87, -2.53, -1.75, 0.27, 43.21

   The density function of the Cauchy(θ,1) distribution is

   $$f(x) = \frac{1}{\pi}\left[(x-\theta)^2 + 1\right]^{-1}. \tag{1}$$

   (a) Draw a box plot and a dot plot of this dataset (use the functions `boxplot()` and `dotchart()`).

   (b) Plot the log-likelihood in the interval $[-10, 10]$. How many modes does it have?

   (c) Program the bisection method in R and apply it to these data with starting points -1 and 1. Use additional runs to explore ways in which the bisection method may fail to find the global maximum.

   (d) Program the Newton-Raphson method and/or the secant method and apply it/them to the same data. Study the behavior of the algorithm for different starting points.

   (e) Solve the same problem with the R function `optimize`.

2. The data $(1,1,1,1,1,1,2,2,2,3)$ are assumed to be an i.i.d. sample from a logarithmic distribution,

   $$f(x;\theta) = \frac{\theta^x}{x[-\log(1-\theta)]}, \quad x \in \{1,2,3,\ldots\}, \theta > 0.$$

   Estimate θ using

   (a) The Newton-Raphson method;

   (b) The Fisher scoring method.

3. The data `F5_2.txt` from Greene's book "Econometric analysis" is a macroeconomics dataset from the U.S. Department of Commerce. It contains quarterly observations from 1950I to 2000IV of some macroeconomic variables. We want to model the relation between income (variable `realdpi`) and consumption (variable `realcons`) using the following nonlinear model

$$Y = \alpha + \beta Z^{\gamma}, \tag{2}$$

where Y represents the consumption, Z is the income, and $\theta = (\alpha, \beta, \gamma)^T$ is a vector of parameters.

   (a) Assuming $\gamma = 1$, estimate α and β using linear regression (use the function `lm`). Plot the data with the least squares line, and the residuals. Does the linear model fit the data?

   (b) Considering the nonlinear model, compute the least-squares estimate of θ using the Gauss-Newton method. Display the results graphically.

4. The data `transportation.txt` from Greene's book "Econometric analysis" concern transportation equipment manufacturing in $n = 25$ states of United States. The output variable Y is `ValueAdd` and the two input variables x_1 and x_2 are `Capita` (capital input) and `Labor` (labor input). The stochastic frontier model is

$$\ln Y_i = \boldsymbol{\beta}' \ln \mathbf{x}_i + V_i - U_i \tag{3}$$

for $i \in \{1, \ldots, n\}$, where $\boldsymbol{\beta}$ is a vector of coefficients, V_i is an error term assumed to have a normal distribution $\mathcal{N}(0, \sigma_v^2)$ and U_i is a positive inefficiency term having a half-normal distribution $|\mathcal{N}(0, \sigma_u^2)|$ (i.e., the distribution of the absolute value of a normal variable). The log-likelihood function is

$$\ln L_y(\theta) = -n \ln \sigma + \frac{n}{2} \log \frac{2}{\pi} - \frac{1}{2} \sum_{i=1}^{n} \left(\frac{\epsilon_i}{\sigma} \right)^2 + \sum_{i=1}^{n} \ln \Phi \left(-\frac{\epsilon_i \lambda}{\sigma} \right), \tag{4}$$

with $\lambda = \sigma_u / \sigma_v$, $\sigma^2 = \sigma_u^2 + \sigma_v^2$ and $\theta = (\boldsymbol{\beta}, \sigma, \lambda)$.

   (a) Using function `read.table`, store the data as a data table.

   (b) Display a matrix plot of the logarithm of the data (use function `plot`).

   (c) Using function `lm`, find the least-squares estimate of $\boldsymbol{\beta}$.

   (d) Compute the maximum likelihood (ML) estimate of θ, using function `optim` with the least-squares estimates $\widehat{\boldsymbol{\beta}}_{LS}$ as a starting point.

(e) Plot contours of the log-likelihood by fixing two parameters at their ML values and letting the other two parameter vary (use function `contour`). Verify that the solution found in the previous question is a maximum.

(f) Perform again the optimization with the following starting point: $\theta_0 = (3, \widehat{\boldsymbol{\beta}}_{LS}[2:3], 0.5, 10)$. What do you observe? Check graphically that the solution found is a maximum.