Introduction to belief functions

Thierry Denœux

Université de Technologie de Compiègne, France
HEUDIASYC (UMR CNRS 7253)
https://www.hds.utc.fr/~tdenoeux

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Contents of this lecture

1. Historical perspective, motivations
2. Fundamental concepts: belief, plausibility, commonality, Conditioning, basic combination rules
3. Some more advanced concepts: cautious rule, multidimensional belief functions, belief functions in infinite spaces

Uncertain reasoning

- In science and engineering we always need to reason with partial knowledge and uncertain information (from sensors, experts, models, etc.)
- Different sources of uncertainty
  - Variability of entities in populations and outcomes of random (repeatable) experiments $\rightarrow$ Aleatory uncertainty. Example: drawing a ball from an urn. Cannot be reduced
  - Lack of knowledge $\rightarrow$ Epistemic uncertainty. Example: inability to distinguish the color of a ball because of color blindness. Can be reduced
- Classical ways of representing uncertainty
  1. Using probabilities
  2. Using set (e.g., interval analysis), or propositional logic

Probability theory
Interpretations

- Probability theory can be used to represent
  - Aleatory uncertainty: probabilities are considered as objective quantities and interpreted as frequencies or limits of frequencies
  - Epistemic uncertainty: probabilities are subjective, interpreted as degrees of belief
- Main objections against the use of probability theory as a model epistemic uncertainty (Bayesian model)
  1. Inability to represent ignorance
  2. Not a plausibility model of how people make decisions based on weak information

Inability to represent ignorance

The wine/water paradox

- Principle of Indifference (PI): in the absence of information about some quantity X, we should assign equal probability to any possible value of X
- The wine/water paradox

There is a certain quantity of liquids. All that we know about the liquid is that it is composed entirely of wine and water, and the ratio of wine to water is between 1/3 and 3.
What is the probability that the ratio of wine to water is less than or equal to 2?

Inability to represent ignorance

The wine/water paradox (continued)

- Let X denote the ratio of wine to water. All we know is that $X \in [1/3, 3]$. According to the PI, $X \sim \mathcal{U}_{[1/3,3]}$. Consequently

$$P(X \leq 2) = (2 - 1/3)/(3 - 1/3) = 5/8$$

- Now, let $Y = 1/X$ denote the ratio of water to wine. All we know is that $Y \in [1/3, 3]$. According to the PI, $Y \sim \mathcal{U}_{[1/3,3]}$. Consequently

$$P(Y \geq 1/2) = (3 - 1/2)/(3 - 1/3) = 15/16$$

- However, $P(X \leq 2) = P(Y \geq 1/3)$!

Decision making

Ellsberg's paradox

- Suppose you have an urn containing 30 red balls and 60 balls, either black or yellow. You are given a choice between two gambles:
  - A: You receive 100 euros if you draw a red ball
  - B: You receive 100 euros if you draw a black ball
- Also, you are given a choice between these two gambles (about a different draw from the same urn):
  - C: You receive 100 euros if you draw a red or yellow ball
  - D: You receive 100 euros if you draw a black or yellow ball
- Most people strictly prefer A to B, hence $P(red) > P(black)$, but they strictly prefer D to C, hence $P(black) > P(red)$

Set-membership approach

- Partial knowledge about some variable X is described by a set E of possible values for X (constraint)
- Example:
  - Consider a system described by the equation

$$y = f(x_1, \ldots, x_n; \theta)$$

    where y is the output, $x_1, \ldots, x_n$ are the inputs and θ is a parameter
  - Knowing that $x_i \in [\underline{x}_i, \overline{x}_i]$, $i = 1, \ldots, n$ and $\theta \in [\underline{\theta}, \overline{\theta}]$, find a set $\mathbb{Y}$ surely containing y
- Advantage: computationally simpler than the probabilistic approach in many cases (interval analysis)
- Drawback: no way to express doubt, conservative approach

Theory of belief functions
History

- A formal framework for representing and reasoning with uncertain information
- Also known as Dempster-Shafer theory or Evidence theory
- Originates from the work of Dempster (1968) in the context of statistical inference.
- Formalized by Shafer (1976) as a theory of evidence
- Popularized and developed by Smets in the 1980's and 1990's under the name Transferable Belief Model
- Starting from the 1990's, growing number of applications in information fusion, classification, reliability and risk analysis, etc.

Theory of belief functions

Main idea

- The theory of belief functions extends both the set-membership approach and Probability Theory
  - A belief function may be viewed both as a generalized set and as a non additive measure
  - The theory includes extensions of probabilistic notions (conditioning, marginalization) and set-theoretic notions (intersection, union, inclusion, etc.)
- Dempter-Shafer reasoning produces the same results as probabilistic reasoning or interval analysis when provided with the same information
- However, the greater expressive power of the theory of belief functions allows us to represent what we know in a more faithful way

Outline

1. Basics
   - Representation of evidence
   - Combination of evidence

2. Selected advanced topics
   - Informational orderings
   - Cautious rule
   - Belief functions on product spaces
   - Belief functions on infinite spaces

Outline

Mass function
Definition

- Let X be a variable taking values in a finite set Ω (frame of discernment)
- Evidence about X may be represented by a mass function $m : 2^{\Omega} \to [0, 1]$ such that

$$\sum_{A \subseteq \Omega} m(A) = 1$$

- Every A of Ω such that $m(A) > 0$ is a focal set of m
- m is said to be normalized if $m(\emptyset) = 0$. This property will be assumed hereafter, unless otherwise specified

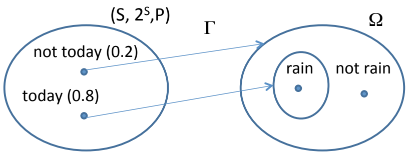
Example

- When traveling by train, you find a page of a used newspaper, with an article announcing rain for tomorrow
- The date of the newspaper is missing. If is today's newspaper, you know that it will rain tomorrow (assuming the forecast is perfectly reliable). If not, you know nothing
- Assume your subjective probability that this is today's paper is 0.8
- The frame of discernment is $\Omega = \{\text{rain}, \neg\text{rain}\}$
- The evidence can be represented by the following mass function

$$m(\{\text{rain}\}) = 0.8, \quad m(\{\text{rain}, \neg\text{rain}\}) = 0.2$$

- The mass 0.2 is not committed to $\{\neg\text{rain}\}$, because there is no evidence that it will not rain

Mass function
Source



- A mass function m on Ω may be viewed as arising from
  - A set $S = \{s_1, \ldots, s_r\}$ of states (interpretations)
  - A probability measure P on S
  - A multi-valued mapping $\Gamma : S \rightarrow 2^\Omega$
- The four-tuple $(S, 2^S, P, \Gamma)$ is called a source for m
- Meaning: under interpretation s_i, the evidence tells us that $X \in \Gamma(s_i)$, and nothing more. The probability $P(\{s_i\})$ is transferred to $A_i = \Gamma(s_i)$
- $m(A)$ is the probability of knowing only that $X \in A$, given the available evidence

Mass functions
Special cases

- If the evidence tells us that $X \in A$ for sure and nothing more, for some $A \subseteq \Omega$, then we have a logical mass function m_A such that $m_A(A) = 1$
  - m_A is equivalent to A
  - Special case: $m_?$, the vacuous mass function, represents total ignorance
- If each interpretation s_i of the evidence points to a single value of X, then all focal sets are singletons and m is said to be Bayesian. It is equivalent to a probability distribution
- A Dempster-Shafer mass function can thus be seen as
  - a generalized set
  - a generalized probability distribution

Belief function

- The total degree of support for A can be defined as the probability that the evidence implies A

$$Bel(A) = P(\{s \in S | \Gamma(s) \subseteq A\}) = \sum_{B \subseteq A} m(B)$$

- Function $Bel : 2^\Omega \to [0, 1]$ is called a belief function
- It is a completely monotone capacity: it verifies $Bel(\emptyset) = 0$, $Bel(\Omega) = 1$ and

$$Bel\left(\bigcup_{i=1}^{k} A_i\right) \geq \sum_{\emptyset \neq I \subseteq \{1,\ldots,k\}} (-1)^{|I|+1} Bel\left(\bigcap_{i \in I} A_i\right)$$

for any $k \geq 2$ and for any family $A_1, \ldots, A_k$ in 2^Ω
- Conversely, to any completely monotone capacity Bel corresponds a unique mass function m such that

$$m(A) = \sum_{\emptyset \neq B \subseteq A} (-1)^{|A|-|B|} Bel(B), \quad \forall A \subseteq \Omega$$

Plausibility function

- The plausibility of A is the probability that the evidence is consistent with A. It is defined as

$$Pl(A) = P(\{s \in S | \Gamma(s) \cap A \neq \emptyset\}) = \sum_{B \cap A \neq \emptyset} m(B)$$

- Properties:
  - $Pl(\emptyset) = 0$, $Pl(\Omega) = 1$
  - $Bel(A) \leq Pl(A), \forall A \subseteq \Omega$
  - $Pl(A) = 1 - Bel(\overline{A}), \forall A \subseteq \Omega$
- If m is Bayesian, $Bel = Pl$ (probability measure)

Example

A	$\emptyset$	$\{\text{rain}\}$	$\{\neg\text{rain}\}$	$\{\text{rain}, \neg\text{rain}\}$
$m(A)$	0	0.8	0	0.2
$Bel(A)$	0	0.8	0	1
$pl(A)$	0	1	0.2	1

- We observe that

$$Bel(A \cup B) \geq Bel(A) + Bel(B) - Bel(A \cap B)$$

$$Pl(A \cup B) \leq Pl(A) + Pl(B) - Pl(A \cap B)$$

Wine/water paradox revisited

- Let X denote the ratio of wine to water. All we know is that $X \in [1/3, 3]$. This is modeled by the logical mass function m_X such that $m_X([1/3, 3]) = 1$. Consequently:

$$Bel_X([2, 3]) = 0, \quad Pl_X([2, 3]) = 1$$

- Now, let $Y = 1/X$ denote the ratio of water to wine. All we know is that $Y \in [1/3, 3]$. This is modeled by the logical mass function m_Y such that $m_Y([1/3, 3]) = 1$. Consequently:

$$Bel_Y([1/3, 1/2]) = 0, \quad Pl_Y([1/3, 1/2]) = 1$$

Relations between *m*, *Bel* et *Pl*

- Let m be a mass function, Bel and Pl the corresponding belief and plausibility functions
- Relations:

$$Bel(A) = 1 - Pl(\overline{A}), \quad \forall A \subseteq \Omega$$

$$m(A) = \sum_{\emptyset \neq B \subseteq A} (-1)^{|A|-|B|} Bel(B), \forall A \subseteq \Omega$$

- m, Bel et Pl are thus three equivalent representations of
  - a piece of evidence or, equivalently
  - a state of belief induced by this evidence

Relationship with Possibility theory

- When the focal sets of m are nested: $A_1 \subset A_2 \subset \ldots \subset A_r$, m is said to be consonant
- The following relations then hold

$$Pl(A \cup B) = \max\left(Pl(A), Pl(B)\right), \quad \forall A, B \subseteq \Omega$$

- Pl is this a possibility measure, and Bel is the dual necessity measure
- The possibility distribution is the contour function

$$pl(x) = Pl(\{x\}), \quad \forall x \in \Omega$$

- The theory of belief function can thus be considered as more expressive than possibility theory

Credal set

- A probability measure P on Ω is said to be compatible with m if

$$\forall A \subseteq \Omega, \quad Bel(A) \leq P(A) \leq Pl(A)$$

- The set $\mathcal{P}(m)$ of probability measures compatible with m is called the credal set of m

$$\mathcal{P}(m) = \{P : \forall A \subseteq \Omega, Bel(A) \leq P(A)\}$$

- Bel is the lower envelope of $\mathcal{P}(m)$

$$\forall A \subseteq \Omega, \quad Bel(A) = \min_{P \in \mathcal{P}(m)} P(A)$$

- Not all lower envelopes of sets of probability measures are belief functions!

Outline

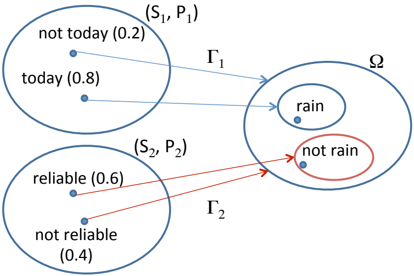
Dempster's rule

Rain example continued

- The first item of evidence gave us: $m_1(\{\text{rain}\}) = 0.8$, $m_1(\Omega) = 0.2$
- New piece of evidence: upon arriving in the train station, someone tells you that it will not rain tomorrow. Your probability that this prediction is reliable is 0.6
- This second piece if evidence can be represented by the mass funtion: $m_2(\{\neg\text{rain}\}) = 0.6$, $m_2(\Omega) = 0.4$
- How to combine these two pieces of evidence?

Dempster's rule

Justification



- If interpretations $s_1 \in S_1$ and $s_2 \in S_2$ both hold, then $X \in \Gamma_1(s_1) \cap \Gamma_2(s_2)$
- If the two pieces of evidence are independent, then the probability that s_1 and s_2 both hold is $P_1(\{s_1\})P_2(\{s_2\})$
- If $\Gamma_1(s_1) \cap \Gamma_2(s_2) = \emptyset$, we know that s_1 and s_2 cannot hold simultaneously
- The joint probability distribution on $S_1 \times S_2$ must be conditioned to eliminate such pairs

Computation

	reliable (0.6)	not reliable (0.4)
today (0.8)	$\emptyset$, 0.48	$\{\text{rain}\}$, 0.32
not today (0.2)	$\{\neg\text{rain}\}$, 0.12	Ω, 0.08

We then get the following combined mass function,

$$
\begin{aligned}
m(\{\text{rain}\}) &= 0.32/0.52 \approx 0.62 \\
m(\{\neg\text{rain}\}) &= 0.12/0.52 \approx 0.23 \\
m(\Omega) &= 0.08/0.52 \approx 0.15
\end{aligned}
$$

Dempster's rule

Definition

- Let m_1 and m_2 be two mass functions and

$$K = \sum_{B \cap C = \emptyset} m_1(B) m_2(C)$$

  their degree of conflict

- If $K < 1$, then m_1 and m_2 can be combined as

$$(m_1 \oplus m_2)(A) = \frac{1}{1-K} \sum_{B \cap C = A} m_1(B) m_2(C), \quad \forall A \neq \emptyset$$

  and $(m_1 \oplus m_2)(\emptyset) = 0$

Dempster's rule

Properties

- Commutativity, associativity. Neutral element: $m_?$
- Generalization of intersection: if m_A and m_B are categorical mass functions and $A \cap B \neq \emptyset$, then

$$m_A \oplus m_B = m_{A \cap B}$$

- Generalization of probabilistic conditioning: if m is a Bayesian mass function and m_A is a logical mass function, then $m \oplus m_A$ is a Bayesian mass function corresponding to the conditioning of m by A
- Notation for conditioning (special case):

$$m \oplus m_A = m(\cdot|A)$$

Dempster's rule

Expression using commonalities

- Commonality function: let $Q : 2^{\Omega} \to [0,1]$ be defined as

$$Q(A) = \sum_{B \supseteq A} m(B), \quad \forall A \subseteq \Omega$$

- Conversely,

$$m(A) = \sum_{B \supseteq A} (-1)^{|B \setminus A|} Q(B)$$

- Expression of $\oplus$ using commonalities:

$$(Q_1 \oplus Q_2)(A) = \frac{1}{1-K} Q_1(A) \cdot Q_2(A), \quad \forall A \subseteq \Omega, A \neq \emptyset$$

$$(Q_1 \oplus Q_2)(\emptyset) = 1$$

Remarks on normalization

- Mass functions expressing pieces of evidence are always normalized
- Smets introduced the unnormalized Dempster's rule (TBM conjunctive rule $\textcircled{\cap}$), which may yield an unnormalized mass function
- He proposed to interpret $m(\emptyset)$ as the mass committed to the hypothesis that X might not take its value in Ω (open-world assumption)
- I now think that this interpretation is problematic, as $m(\emptyset)$ increases mechanically when combining more and more items of evidence
- Claim: unnormalized mass functions (and $\textcircled{\cap}$) are convenient mathematically, but only normalized mass functions make sense
- In particular, *Bel* and *Pl* should always be computed from normalized mass functions

TBM disjunctive rule

- Let (S_1, P_1, Γ_1) and (S_2, P_2, Γ_2) be sources associated to two pieces of evidence
- If interpretation $s_k \in S_k$ holds and piece of evidence k is reliable, then we can conclude that $X \in \Gamma_k(s_k)$
- If interpretation $s \in S_1$ and $s_2 \in S_2$ both hold and we assume that at least one of the two pieces of evidence is reliable, then we can conclude that $X \in \Gamma_1(s_1) \cup \Gamma_2(s_2)$
- This leads to the TBM disjunctive rule:

$$(m_1 \textcircled{\cup} m_2)(A) = \sum_{B \cup C = A} m_1(B) m_2(C), \quad \forall A \subseteq \Omega$$

- $Bel_1 \textcircled{\cup} Bel_2 = Bel_1 \cdot Bel_2$

Outline

Informational comparison of belief functions

- Let m_1 et m_2 be two mass functions on Ω
- In what sense can we say that m_1 is more informative (committed) than m_2?
- Special case:
  - Let m_A and m_B be two logical mass functions
  - m_A is more committed than m_B iff $A \subseteq B$
- Extension to arbitrary mass functions?

Plausibility ordering

- m_1 is pl-more committed than m_2 (noted $m_1 \sqsubseteq_{pl} m_2$) if

$$Pl_1(A) \leq Pl_2(A), \quad \forall A \subseteq \Omega$$

  or, equivalently,

$$Bel_1(A) \geq Bel_2(A), \quad \forall A \subseteq \Omega$$

- Imprecise probability interpretation:

$$m_1 \sqsubseteq_{pl} m_2 \Leftrightarrow \mathcal{P}(m_1) \subseteq \mathcal{P}(m_2)$$

- Properties:
  - Extension of set inclusion:

$$m_A \sqsubseteq_{pl} m_B \Leftrightarrow A \subseteq B$$

  - Greatest element: vacuous mass function $m_?$

Commonality ordering

- If $m_1 = m \oplus m_2$ for some m, and if there is no conflict between m and m_2, then $Q_1(A) = Q(A)Q_2(A) \leq Q_2(A)$ for all $A \subseteq \Omega$
- This property suggests that smaller values of the commonality function are associated with richer information content of the mass function
- m_1 is q-more committed than m_2 (noted $m_1 \sqsubseteq_q m_2$) if

$$Q_1(A) \leq Q_2(A), \quad \forall A \subseteq \Omega$$

- Properties:
  - Extension of set inclusion:

$$m_A \sqsubseteq_q m_B \Leftrightarrow A \subseteq B$$

  - Greatest element: vacuous mass function $m_?$

Strong (specialization) ordering

- m_1 is a specialization of m_2 (noted $m_1 \sqsubseteq_s m_2$) if m_1 can be obtained from m_2 by distributing each mass $m_2(B)$ to subsets of B:

$$m_1(A) = \sum_{B \subseteq \Omega} S(A, B) m_2(B), \quad \forall A \subseteq \Omega,$$

  where $S(A, B) =$ proportion of $m_2(B)$ transferred to $A \subseteq B$

- S: specialization matrix
- Properties:
  - Extension of set inclusion
  - Greatest element: $m_?$
  - $m_1 \sqsubseteq_s m_2 \Rightarrow \begin{cases} m_1 \sqsubseteq_{pl} m_2 \\ m_1 \sqsubseteq_q m_2 \end{cases}$

Least Commitment Principle

Definition

Definition (Least Commitment Principle)

When several belief functions are compatible with a set of constraints, the least informative according to some informational ordering (if it exists) should be selected

A very powerful method for constructing belief functions!

Outline

Cautious rule
Motivations

- The basic rules $\oplus$ and $\textcircled{\cup}$ assume the sources of information to be independent, e.g.
  - experts with non overlapping experience/knowledge
  - non overlapping datasets
- What to do in case of non independent evidence?
  - Describe the nature of the interaction between sources (difficult, requires a lot of information)
  - Use a combination rule that tolerates redundancy in the combined information
- Such rules can be derived from the LCP using suitable informational orderings

Cautious rule
Principle

- Two sources provide mass functions m_1 and m_2, and the sources are both considered to be reliable
- After receiving these m_1 and m_2, the agent's state of belief should be represented by a mass function m_{12} more committed than m_1, and more committed than m_2
- Let $\mathcal{S}_x(m)$ be the set of mass functions m' such that $m' \sqsubseteq_x m$, for some $x \in \{pl, q, s, \cdots\}$. We thus impose that

$$m_{12} \in \mathcal{S}_x(m_1) \cap \mathcal{S}_x(m_2)$$

- According to the LCP, we should select the x-least committed element in $\mathcal{S}_x(m_1) \cap \mathcal{S}_x(m_2)$, if it exists

Cautious rule
Problem

- The above approach works for special cases
- Example (Dubois, Prade, Smets 2001): if m_1 and m_2 are consonant, then the q-least committed element in $\mathcal{S}_q(m_1) \cap \mathcal{S}_q(m_2)$ exists and it is unique: it is the consonant mass function with commonality function $Q_{12} = \min(Q_1, Q_2)$
- In general, neither existence nor uniqueness of a solution can be guaranteed with any of the x-orderings, $x \in \{pl, q, s\}$
- We need to define a new ordering relation

Simple and separable mass functions

- Definition: m is simple mass function if it has the following form

$$\begin{aligned} m(A) &= 1 - w(A) \\ m(\Omega) &= w(A) \end{aligned}$$

for some $A \subset \Omega$, $A \neq \emptyset$ and $w(A) \in [0, 1]$. It is denoted by $A^{w(A)}$.

- Property: $A^{w_1(A)} \oplus A^{w_2(A)} = A^{w_1(A) w_2(A)}$
- A (normalized) mass function is separable if it can be written as the $\oplus$ combination of simple mass functions

$$m = \bigoplus_{\emptyset \neq A \subset \Omega} A^{w(A)}$$

with $0 \leq w(A) \leq 1$ for all $A \subset \Omega$, $A \neq \emptyset$

The *w*-ordering

- Let m_1 and m_2 be two mass functions
- We say that m_1 is *w*-less committed than m_2 (denoted by $m_1 \sqsubseteq_w m_2$) if

$$m_1 = m_2 \oplus m$$

  for some separable mass function m
- How to check this condition?

Weight function
Definition

- Let m be a non dogmatic mass function, i.e., $m(\Omega) > 0$
- The weight function $w : 2^\Omega \to (0, +\infty)$ is defined by $w(\Omega) = 1$ and
$$\ln w(A) = -\sum_{B \supseteq A} (-1)^{|B|-|A|} \ln Q(B), \quad \forall A \subset \Omega$$
- It can be shown that Q can be recovered from w as follows
$$\ln Q(A) = -\sum_{\Omega \supset B \not\supseteq A} \ln w(B), \quad \forall A \subseteq \Omega$$
- m can also be recovered from w by
$$m = \bigoplus_{\emptyset \neq A \subset \Omega} A^{w(A)}$$
although $A^{w(A)}$ is not a proper mass function when $w(A) > 1$

Weight function
Properties

- m is separable iff

$$w(A) \leq 1, \quad \forall A \subset \Omega, A \neq \emptyset$$

- Dempster's rule can be computed using the w-function by

$$m_1 \oplus m_2 = \bigoplus_{\emptyset \neq A \subset \Omega} A^{w_1(A) w_2(A)}$$

- Characterization of the w-ordering

$$m_1 \sqsubseteq_w m_2 \Leftrightarrow w_1(A) \leq w_2(A), \quad \forall A \subset \Omega, A \neq \emptyset$$

Cautious rule
Definition

- Let m_1 and m_2 be two non dogmatic mass functions with weight functions w_1 and w_2
- The w-least committed element in $\mathcal{S}_w(m_1) \cap \mathcal{S}_w(m_2)$ exists and is unique. It is defined by:

$$m_1 \owedge m_2 = \bigoplus_{\emptyset \neq A \subset \Omega} A^{\min(w_1(A), w_2(A))}$$

- Operator $\owedge$ is called the (normalized) cautious rule

Cautious rule
Computation

Cautious rule computation

m-space		w-space
m_1	$\longrightarrow$	w_1
m_2	$\longrightarrow$	w_2
$m_1 \textcircled{\scriptsize$\wedge$} m_2$	$\longleftarrow$	$\min(w_1, w_2)$

Cautious rule
Properties

- Commutative, associative
- Idempotent : $\forall m,\ m \textcircled{\wedge} m = m$
- Distributivity of $\oplus$ with respect to $\textcircled{\wedge}$

$$(m_1 \oplus m_2) \textcircled{\wedge} (m_1 \oplus m_3) = m_1 \oplus (m_2 \textcircled{\wedge} m_3), \forall m_1, m_2, m_3$$

The same item of evidence m_1 is not counted twice!

- No neutral element, but $m_? \textcircled{\wedge} m = m$ iff m is separable

Basic rules

Sources	independent	dependent
All reliable	$\oplus$	$\owedge$
At least one reliable	$\textcircled{\cup}$	$\ovee$

$\ovee$ is the bold disjunctive rule

Outline

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   - Representation of evidence
   - Combination of evidence
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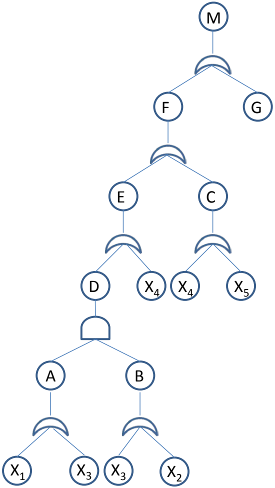
   - Cautious rule
   - Belief functions on product spaces
   - Belief functions on infinite spaces

Belief functions on product spaces

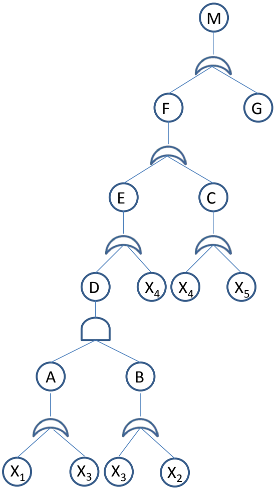
Motivation

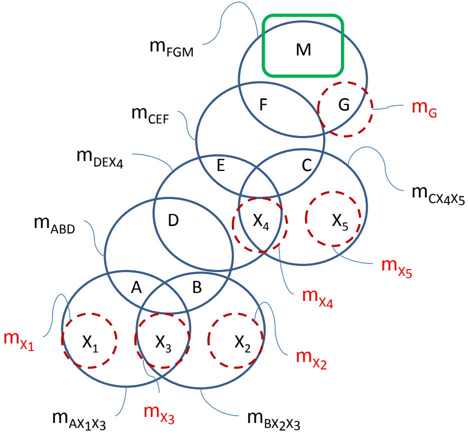


- In many applications, we need to express uncertain information about several variables taking values in different domains
- Example: fault tree (logical relations between Boolean variables and probabilistic or evidential information about elementary events)

Fault tree example

(Dempster & Kong, 1988)





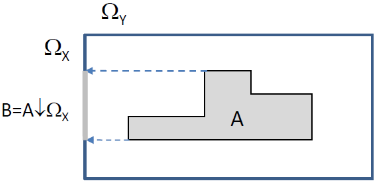
Hypergraph

Multidimensional belief functions

Marginalization, vacuous extension

- Let X and Y be two variables defined on frames Ω_X and Ω_Y
- Let $\Omega_{XY} = \Omega_X \times \Omega_Y$ be the product frame
- A mass function m^{XY} on Ω_{XY} can be seen as an generalized relation between variables X and Y
- Two basic operations on product frames
  1. Express a joint mass function m^{XY} in the coarser frame Ω_X or Ω_Y (marginalization)
  2. Express a marginal mass function m^X on Ω_X in the finer frame Ω_{XY} (vacuous extension)

Marginalization



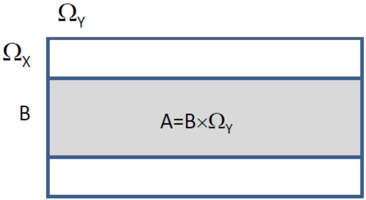
- Problem: express m^{XY} in Ω_X
- Solution: transfer each mass $m^{XY}(A)$ to the projection of A on Ω_X

- Marginal mass function

$$m^{XY\downarrow X}(B) = \sum_{\{A\subseteq\Omega_{XY}, A\downarrow\Omega_X = B\}} m^{XY}(A) \quad \forall B \subseteq \Omega_X$$

- Generalizes both set projection and probabilistic marginalization

Vacuous extension



- Problem: express m^X in Ω_{XY}
- Solution: transfer each mass $m^X(B)$ to the cylindrical extension of B: $B \times \Omega_Y$

- Vacuous extension:

$$m^{X\uparrow XY}(A) = \begin{cases} m^X(B) & \text{if } A = B \times \Omega_Y \\ 0 & \text{otherwise} \end{cases}$$

Operations in product frames

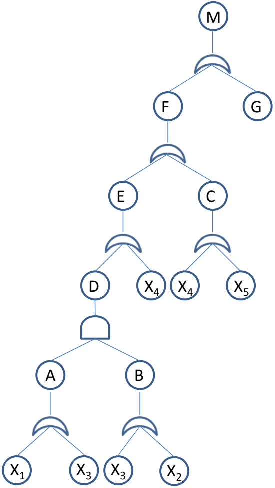
Application to approximate reasoning

- Assume that we have:
  - Partial knowledge of X formalized as a mass function m^X
  - A joint mass function m^{XY} representing an uncertain relation between X and Y
- What can we say about Y?
- Solution:

$$m^Y = \left(m^{X\uparrow XY} \oplus m^{XY}\right)^{\downarrow Y}$$

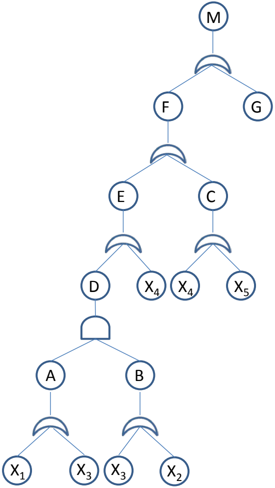
- Infeasible with many variables and large frames of discernment, but efficient algorithms exist to carry out the operations in frames of minimal dimensions

Fault tree example



Cause	$m(\{1\})$	$m(\{0\})$	$m(\{0,1\})$
X_1	0.05	0.90	0.05
X_2	0.05	0.90	0.05
X_3	0.005	0.99	0.005
X_4	0.01	0.985	0.005
X_5	0.002	0.995	0.003
G	0.001	0.99	0.009
M	0.02	0.951	0.029
F	0.019	0.961	0.02

Fault tree example (continued)



Cause	$m(\{1\})$	$m(\{0\})$	$m(\{0,1\})$
M	1	0	0
G	0.197	0.796	0.007
F	0.800	0.196	0.004
⋮	⋮	⋮	⋮
X_1	0.236	0.724	0.040
X_2	0.236	0.724	0.040
X_3	0.200	0.796	0.004
X_4	0.302	0.694	0.004
X_5	0.099	0.898	0.003

Outline

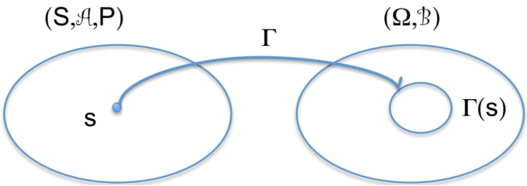
Belief function: general definition

- Let Ω be a set (finite or not) and $\mathcal{B}$ be an algebra of subsets of Ω
- A belief function (BF) on $\mathcal{B}$ is a mapping $Bel : \mathcal{B} \to [0, 1]$ verifying $Bel(\emptyset) = 0$, $Bel(\Omega) = 1$ and the complete monotonicity property: for any $k \geq 2$ and any collection $B_1, \ldots, B_k$ of elements of $\mathcal{B}$,

$$Bel\left(\bigcup_{i=1}^{k} B_i\right) \geq \sum_{\emptyset \neq I \subseteq \{1,\ldots,k\}} (-1)^{|I|+1} Bel\left(\bigcap_{i \in I} B_i\right)$$

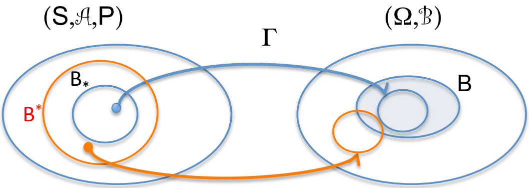
- A function $Pl : \mathcal{B} \to [0, 1]$ is a plausibility function iff $B \to 1 - Pl(\overline{B})$ is a belief function

Source



- Let S be a state space, $\mathcal{A}$ an algebra of subsets of S, $\mathbb{P}$ a finitely additive probability on $(S, \mathcal{A})$
- Let Ω be a set and $\mathcal{B}$ an algebra of subsets of Ω
- Γ a multivalued mapping from S to $2^\Omega \setminus \{\emptyset\}$
- The four-tuple $(S, \mathcal{A}, \mathbb{P}, \Gamma)$ is called a source
- Under some conditions, it induces a belief function on $(\Omega, \mathcal{B})$

Strong measurability



- Lower and upper inverses: for all $B \in \mathcal{B}$,

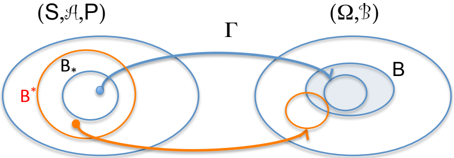
$$\Gamma_*(B) = B_* = \{s \in S | \Gamma(s) \neq \emptyset, \Gamma(s) \subseteq B\}$$

$$\Gamma^*(B) = B^* = \{s \in S | \Gamma(s) \cap B \neq \emptyset\}$$

- Γ is strongly measurable wrt $\mathcal{A}$ and $\mathcal{B}$ if, for all $B \in \mathcal{B}$, $B^* \in \mathcal{A}$
- $(\forall B \in \mathcal{B}, B^* \in \mathcal{A}) \Leftrightarrow (\forall B \in \mathcal{B}, B_* \in \mathcal{A})$

Belief function induced by a source

Lower and upper probabilities

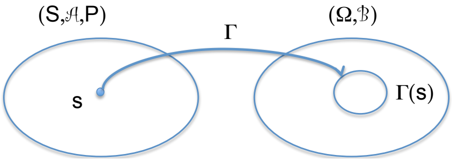


- Lower and upper probabilities:

$$\forall B \in \mathcal{B}, \quad \mathbb{P}_*(B) = \frac{\mathbb{P}(B_*)}{\mathbb{P}(\Omega^*)}, \quad \mathbb{P}^*(B) = \frac{\mathbb{P}(B^*)}{\mathbb{P}(\Omega^*)} = 1 - Bel(\overline{B})$$

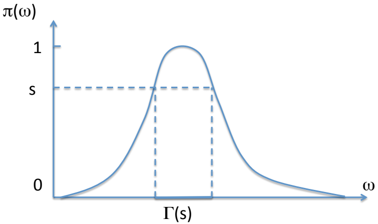
- $\mathbb{P}_*$ is a BF, and $\mathbb{P}^*$ is the dual plausibility function
- Conversely, for any belief function, there is a source that induces it (Shafer's thesis, 1973)

Interpretation



- Typically, Ω is the domain of an unknown quantity ω, and S is a set of interpretations of a given piece of evidence about ω
- If $s \in S$ holds, then the evidence tells us that $\omega \in \Gamma(s)$, and nothing more
- Then
  - $Bel(B)$ is the probability that the evidence supports B
  - $Pl(B)$ is the probability that the evidence is consistent with B

Consonant belief function



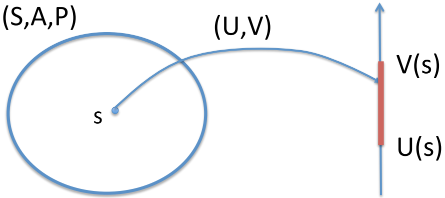
- Let π be a mapping from Ω to $S = [0, 1]$ s.t. $\sup \pi = 1$
- Let Γ be the multi-valued mapping from S to 2^Ω defined by

$$\forall s \in [0,1], \quad \Gamma(s) = \{\omega \in \Omega | \pi(\omega) \geq s\}$$

- The source $(S, \mathcal{B}(S), \lambda, \Gamma)$ defines a consonant BF on Ω, such that $pl(\omega) = \pi(\omega)$ (contour function)
- The corresponding plausibility function is a possibility measure

$$\forall B \subseteq \Omega, \quad Pl(B) = \sup_{\omega \in B} pl(\omega)$$

Random closed interval

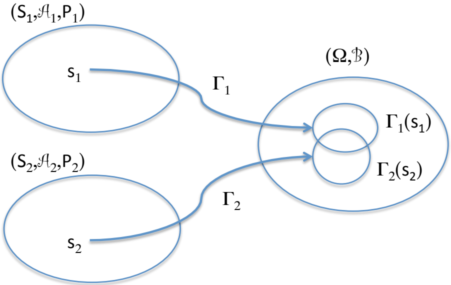


- Let (U, V) be a bi-dimensional random vector from a probability space $(S, \mathcal{A}, \mathbb{P})$ to $\mathbb{R}^2$ such that $U \leq V$ a.s.
- Multi-valued mapping:

$$\Gamma : s \rightarrow \Gamma(s) = [U(s), V(s)]$$

- The source $(S, \mathcal{A}, \mathbb{P}, \Gamma)$ is a random closed interval. It defines a BF on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$

Dempster's rule



- Let $(S_i, \mathcal{A}_i, \mathbb{P}_i, \Gamma_i)$, $i = 1, 2$ be two sources representing independent items of evidence, inducing BF Bel_1 and Bel_2
- The combined BF $Bel = Bel_1 \oplus Bel_2$ is induced by the source $(S_1 \times S_2, \mathcal{A}_1 \otimes \mathcal{A}_2, \mathbb{P}_1 \otimes \mathbb{P}_2, \Gamma_\cap)$ with

$$\Gamma_\cap(s_1, s_2) = \Gamma_1(s_1) \cap \Gamma_2(s_2)$$

Approximate computation

Monte Carlo simulation

Require: Desired number of focal sets N
  $i \leftarrow 0$
  while $i < N$ **do**
    Draw s_1 in S_1 from $\mathbb{P}_1$
    Draw s_2 in S_2 from $\mathbb{P}_2$
    $\Gamma_\cap(s_1, s_2) \leftarrow \Gamma_1(s_1) \cap \Gamma_2(s_2)$
    if $\Gamma_\cap(s_1, s_2) \neq \emptyset$ **then**
      $i \leftarrow i + 1$
      $B_i \leftarrow \Gamma_\cap(s_1, s_2)$
    end if
  end while
  $\widehat{Bel}(B) \leftarrow \frac{1}{N}\#\{i \in \{1, \ldots, N\} | B_i \subseteq B\}$
  $\widehat{Pl}(B) \leftarrow \frac{1}{N}\#\{i \in \{1, \ldots, N\} | B_i \cap B \neq \emptyset\}$

Summary

- The theory of belief functions: a very general formalism for representing imprecision and uncertainty that extends both probabilistic and set-theoretic frameworks
  - Belief functions can be seen both as generalized sets and as generalized probability measures
  - Reasoning mechanisms extend both set-theoretic notions (intersection, union, cylindrical extension, inclusion relations, etc.) and probabilistic notions (conditioning, marginalization, Bayes theorem, stochastic ordering, etc.)
- The theory of belief function can also be seen as more geneal than Possibility theory (possibility measures are particular plausibility functions)
- The mathematical theory of belief functions in infinite spaces exists. We need practical models

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cf. http://www.hds.utc.fr/~tdenoeux

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