Combining Random Fuzzy Sets via Rejection Sampling

Thierry Denœux[0000−0002−0660−5436]

Université de technologie de Compiègne, CNRS, Heudiasyc, Compiègne, France
Institut universitaire de France, Paris, France
thierry.denoeux@utc.fr

Abstract. Generalised evidence theory based on random fuzzy sets (RFSs) provides a flexible framework for representing and combining uncertain information on continuous domains. While several important families of RFSs admit closed-form combination rules, this is not the case in general. This paper proposes a Monte Carlo approach for combining independent RFSs using the product-intersection rule in such cases. A rejection sampling algorithm is introduced to approximate the combined RFS and to estimate belief and plausibility measures. The method is illustrated through the combination of beta random fuzzy numbers.

Keywords: Evidence theory · Belief functions · Monte Carlo simulation.

1 Introduction

Evidence theory is a widely used framework for reasoning with uncertainty [8][5]. The two main ideas underlying evidence theory are (1) the use of belief functions for representing elementary pieces of evidence, and (2) an operation, called *Dempster's rule of combination*, for pooling independent pieces of evidence. Despite its widespread acceptance, evidence theory has, until recently, mostly been applied to problems involving unknowns with finite domains. The main reason for this limitation is that evidence theory in infinite spaces has essentially relied on the theory of random sets. The standard families of random sets, such as p-boxes or consonant random sets, are not closed under Dempster's rule, making computations practically difficult.

Recently, we have proposed a generalised approach to evidence theory based on random fuzzy sets (RFSs) [2][4]. In this approach, a piece of evidence is represented by a RFS, defined as a fuzzy-valued function from a probability space to the fuzzy subsets of another space, with some measurability property [7]. Such RFSs are said to be *epistemic*, as they represent information about unknowns, as opposed to RFSs describing a random mechanism for generating fuzzy data. This generalised evidence theory also extends possibility theory, a possibility distribution being equivalent to a constant RFS. Independent RFSs can be combined conjunctively using the *product-intersection rule*, a commutative and associative operation that extends both Dempster's rule and the normalised product intersection of possibility distributions.

In [4] and [3], we have defined several families of RFSs generalising Gaussian random variables. For instance, a *Gaussian random fuzzy number* (GRFN) is defined as a Gaussian fuzzy number whose mode is a random quantity with a normal probability distribution. The GRFN family, as well as more general families based on transformations and mixtures as described in [3], are closed under the product-intersection rule, making it easy to combine pieces of evidence about continuous variables. However, despite the flexibility offered by these models, there are situations in which evidence is better represented by RFSs that cannot be combined in closed form. It is then necessary to resort to Monte Carlo methods, such as the rejection sampling (RS) algorithm described in this paper.

The rest of this paper is organised as follows. Essential definitions about RFSs and the product-intersection rule are first recalled in Section 2. The RS algorithm for combining RFSs is then described in Section 3, where some simulation results are also reported. Section 4 concludes the paper.

2 Epistemic Random Fuzzy Sets

The notions of RFS and associated belief and plausibility functions are first recalled in Section 2.1. The product-intersection rule is then described in Section 2.2.

2.1 Basic definitions

Let $(\Omega, \mathcal{A})$ and $(\Theta, \mathcal{A}')$ be measurable spaces, and let $\widetilde{X}$ be a fuzzy-valued function from Ω to the set $\mathcal{F}(\Theta)$ of fuzzy subsets of Θ. We say that $\widetilde{X}$ is *strongly measurable* iff for any $\alpha \in [0,1]$ and any $B \in \mathcal{A}'$, $\{\omega \in \Omega : {}^\alpha\widetilde{X}(\omega) \cap B \neq \emptyset\} \in \mathcal{A}$, where ${}^\alpha\widetilde{X}(\omega)$ denotes the α-cut of $\widetilde{X}(\omega)$. If the measurable space $(\Omega, \mathcal{A})$ is endowed with a probability measure P, the tuple $\widetilde{\mathfrak{R}} := (\Omega, \mathcal{A}, P, \Theta, \mathcal{A}', \widetilde{X})$ is called a RFS [1][4]. In generalised evidence theory, such a RFS is a model of a piece of evidence about an unknown X with value in Θ. The set Ω is interpreted as a set of *interpretations* of the evidence. If $\omega \in \Omega$ holds, the fuzzy subset $\widetilde{X}(\omega)$ (assumed to be P-almost surely normal) defines a fuzzy restriction on the value of X. The degrees of possibility and necessity of any $B \in \mathcal{A}'$ are then, respectively, $\Pi_{\widetilde{X}(\omega)}(B) = \sup_{\theta \in B} \widetilde{X}(\omega)(\theta)$ and $N_{\widetilde{X}(\omega)}(B) = 1 - \Pi_{\widetilde{X}(\omega)}(B^c)$. The expectations of these quantities are, respectively,

$$\mathrm{Pl}_{\widetilde{X}}(B) := \int_{\Omega} \Pi_{\widetilde{X}(\omega)}(B) \mathrm{d}P(\omega) \quad \text{and} \quad \mathrm{Bel}_{\widetilde{X}}(B) := \int_{\Omega} N_{\widetilde{X}(\omega)}(B) \mathrm{d}P(\omega).$$

It can be shown that $\mathrm{Bel}_{\widetilde{X}}$ is a belief function, and $\mathrm{Pl}_{\widetilde{X}}$ is the dual plausibility function [1].

2.2 Product-intersection Rule

Let $\widetilde{\mathfrak{R}}_i := (\Omega_i, \mathcal{A}_i, P_i, \Theta, \mathcal{A}', \widetilde{X}_i)$, $i = 1, 2$ be two RFSs representing independent pieces of evidence about an unknown X taking values in Θ. The consistency between interpretations ω_1 and ω_2 can be measured by the height of the product intersection of their images. The *fuzzy subset of consistent pairs of interpretations* can then be defined as

$$\widetilde{\Theta}^*(\omega_1, \omega_2) := \text{hgt}\left(\widetilde{X}_1(\omega_1) \cap_p \widetilde{X}_2(\omega_2)\right),$$

where $\cap_p$ denotes the product intersection of fuzzy sets. Define the *degree of conflict* between the two RFSs as $\kappa = 1 - (P_1 \otimes P_2)(\widetilde{\Theta}^*)$, and consider the function $\widetilde{X}_1 \cap_p^* \widetilde{X}_2 : \Omega_1 \times \Omega_2 \to \mathcal{F}(\Theta)$ defined as

$$(\widetilde{X}_1 \cap_p^* \widetilde{X}_2)(\omega_1, \omega_2) := \begin{cases} \widetilde{X}_1(\omega_1) \cap_p^* \widetilde{X}_2(\omega_2) & \text{if hgt}\left(\widetilde{X}_1(\omega_1) \cap_p \widetilde{X}_2(\omega_2)\right) > 0, \\ \emptyset & \text{otherwise,} \end{cases}$$

where $\cap_p^*$ denotes the normalised product intersection. If $\kappa > 0$ and if $\widetilde{X}_1 \cap_p^* \widetilde{X}_2$ is strongly measurable, we can define the *orthogonal sum* of $\widetilde{\mathfrak{R}}_1$ and $\widetilde{\mathfrak{R}}_2$ as the tuple

$$\widetilde{\mathfrak{R}}_1 \oplus \widetilde{\mathfrak{R}}_2 := (\Omega_1 \times \Omega_2, \mathcal{A}_1 \otimes \mathcal{A}_2, P_{12}^*, \Theta, \mathcal{A}', \widetilde{X}_1 \cap_p^* \widetilde{X}_2),$$

where P_{12}^* is the product measure $P_1 \otimes P_2$ conditioned on fuzzy set $\widetilde{\Theta}^*$, i.e.,

$$P_{12}^*(B) := (P_1 \otimes P_2)(B \mid \widetilde{\Theta}^*) = \frac{(P_1 \otimes P_2)(B \cap_p \widetilde{\Theta}^*)}{(P_1 \otimes P_2)(\widetilde{\Theta}^*)}.$$

This operation, called the *product-intersection rule*, is commutative and associative [4]. It reduces to Dempster's rule when applied to random sets.

3 Rejection Sampling Algorithm

A RS algorithm for combining RFSs is first described in Section 3.1. An example of application is then presented in Section 3.2.

3.1 Rejection Sampling

Let $\widetilde{\mathfrak{R}}_i = (\Omega_i, \mathcal{A}_i, P_i, \Theta, \mathcal{A}', \widetilde{X}_i)$, $i = 1, \ldots, n$ be n RFSs and assume that we want to compute their product-intersection $\widetilde{\mathfrak{R}} = \widetilde{\mathfrak{R}}_1 \oplus \ldots \oplus \widetilde{\mathfrak{R}}_n$. For simplicity, suppose that the images $\widetilde{X}_i(\omega_i)$ belong to a parametrised family of subsets of Θ closed under the normalised product intersection $\cap_p^*$. We can approximate $\widetilde{\mathfrak{R}}$ by drawing randomly N tuples $(\omega_{1k}, \ldots, \omega_{nk})$, $k = 1, \ldots, N$ from the product measure $P_1 \otimes \ldots \otimes P_n$ conditioned on fuzzy set $\widetilde{\Theta}^*(\omega_1, \ldots, \omega_n) = \text{hgt}\left(\widetilde{X}_1(\omega_1) \cap_p \ldots \cap_p \widetilde{X}_n(\omega_n)\right)$, and computing the focal sets $\widetilde{X}_1(\omega_{1k}) \cap_p^* \ldots \cap_p^* \widetilde{X}_n(\omega_{nk})$. Here, the main difficulty is to sample from $(P_1 \otimes \ldots \otimes P_n)(\cdot \mid \widetilde{\Theta}^*)$, which may not belong to a standard parameterised family of probability measures, even if $P_1 \otimes \ldots \otimes P_n$ does. The RS algorithm [6, page 155] is a classical approach to this problem.

To draw from a target distribution f, the RS algorithm uses another distribution g from which we know how to sample, and an envelope $e(x) = g(x)/\alpha \geq q(x)$ for some constant $\alpha \leq 1$, where $q(x) = f(x)/C$ is some function proportional to f. One iteration of the algorithm consists in generating a candidate value X^* from g and U from the standard uniform distribution $\text{Unif}(0,1)$. If $U \leq q(X^*)/e(X^*)$, X^* is recorded as an element of the target random sample; otherwise, it is discarded. The proportion of accepted draws is α/C.

This algorithm can be applied to our problem a follows. Let us denote by f_i the density functions associated with P_i, $i = 1, \ldots, n$. The target density is

$$f(\omega_1, \ldots, \omega_n) = \frac{f_1(\omega_1) \ldots f_n(\omega_n) \widetilde{\Theta}^*(\omega_1, \ldots, \omega_n)}{1 - \kappa} \tag{1}$$

where κ is the degree of conflict. Let $q(\omega_1, \ldots, \omega_n)$ denote the numerator on the right-hand side of (1), $C = 1/(1-\kappa)$, and $g(\omega_1, \ldots, \omega_n) = f_1(\omega_1) \ldots f_n(\omega_n)$. Let us define the envelope as $e(\omega_1, \ldots, \omega_n) = g(\omega_1, \ldots, \omega_n)$. The algorithm thus draws U from $\text{Unif}(0,1)$ and $(\omega_1^*, \ldots, \omega_n^*)$ from g. If $U \leq \widetilde{\Theta}^*(\omega_1^*, \ldots, \omega_n^*)$, the normalised product intersection $\widetilde{X}_1(\omega_1^*) \cap_p^* \ldots \cap_p^* \widetilde{X}_n(\omega_n^*)$ is computed and recorded. The algorithm stops when a predefined number N of focal sets $\widetilde{F}_i$, $i = 1, \ldots, N$ have been obtained. The procedure is described in Algorithm 1. The rejection probability is equal to the κ. It can be estimated by one minus the average of $\widetilde{\Theta}^*(\omega_1^*, \ldots, \omega_n^*)$ over all the draws from the product distribution g (and not only the accepted ones). The degrees of plausibility and belief of a subset $A \subseteq \mathbb{R}$ for the combined RFS $\widetilde{X} = \widetilde{X}_1 \oplus \ldots \oplus \widetilde{X}_n$ can be estimated, respectively, by

$$\widehat{\text{Pl}}_{\widetilde{X}}(A) = \frac{1}{N} \sum_{i=1}^{N} \Pi_{\widetilde{F}_i}(A) \quad \text{and} \quad \widehat{\text{Bel}}_{\widetilde{X}}(A) = \frac{1}{N} \sum_{i=1}^{N} N_{\widetilde{F}_i}(A).$$

The number N of focal sets can be chosen so as to achieve a desired standard error for these estimates.

3.2 Application Example

To illustrate the application of the algorithm described in Section 3.1, let us consider the combination of beta random fuzzy numbers (BRFN) defined as follows. We define a *beta fuzzy number* with mode $m \in [0,1]$ and concentration $c > 0$ as a fuzzy subset of $\mathbb{R}$ with membership function

$$\widetilde{x}(x) = \left(\frac{x}{m}\right)^{mc} \left(\frac{1-x}{1-m}\right)^{(1-m)c} \mathbb{1}_{[0,1]}(x).$$

We write $\widetilde{x} = \text{BFN}(m,c)$. The family of beta fuzzy numbers is closed under the normalised product intersection. Given n beta fuzzy numbers $\widetilde{x}_i = \text{BFN}(m_i, c_i)$,

Algorithm 1 RS algorithm for combining RFSs.

Require: Desired number of focal sets N

$i \leftarrow 0$; $N_0 \leftarrow 0$; $S \leftarrow 0$
while $i < N$ **do**
  $N_0 \leftarrow N_0 + 1$
  Draw $(\omega_1^*, \ldots, \omega_n^*)$ from $P_1 \otimes \ldots \otimes P_n$
  $S \leftarrow S + \widetilde{\Theta}^*(\omega_1^*, \ldots, \omega_n^*)$
  Draw U from Unif$(0,1)$
  if $U \leq \widetilde{\Theta}^*(\omega_1^*, \ldots, \omega_n^*)$ **then**
    $i \leftarrow i + 1$
    $\widetilde{F}_i \leftarrow \widetilde{X}_1(\omega_1^*) \cap_p^* \ldots \cap_p^* \widetilde{X}_n(\omega_n^*)$
  end if
end while
$\widehat{\kappa} \leftarrow 1 - S/N_0$

Ensure: $\widetilde{F}_1, \ldots, \widetilde{F}_N$; $\widehat{\kappa}$

$i = 1, \ldots, n$, we have

$$\text{hgt}(\widetilde{x}_1 \cap_p \ldots \cap_p \widetilde{x}_n) = \left(\prod_{i=1}^n m_i^{m_i c_i} (1-m_i)^{(1-m_i)c_i} \right)^{-1} \times \left(\frac{\sum_{i=1}^n m_i c_i}{\sum_{i=1}^n c_i} \right)^{\sum_{i=1}^n m_i c_i} \left(\frac{\sum_{i=1}^n (1-m_i) c_i}{\sum_{i=1}^n c_i} \right)^{\sum_{i=1}^n (1-m_i) c_i}, \quad (2)$$

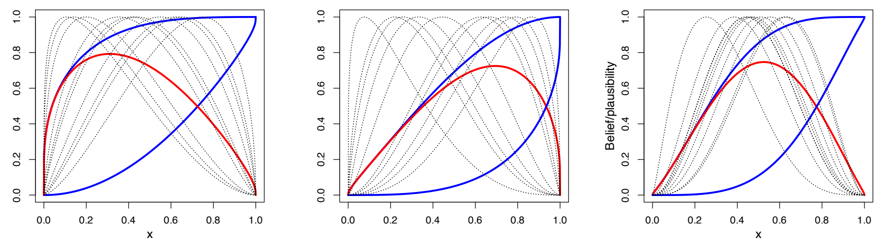
and

$$\widetilde{x}_1 \cap_p^* \ldots \cap_p^* \widetilde{x}_n = \text{BFN}\left(\frac{\sum_{i=1}^n c_i m_i}{\sum_{i=1}^n c_i}, \sum_{i=1}^n c_i \right). \quad (3)$$

We define a *beta RFN* (BRFN) as a beta fuzzy number BFN(M, c) whose mode M is a beta-distributed random variable with shape parameters $a > 0$ and $b > 0$. Such random sets thus belong to a three-parameter family denoted as $\widetilde{B}(a, b, c)$. They are suitable for representing evidence about unknowns in $[0, 1]$ such as probabilities or proportions. This family is not closed by the product-intersection rule. However, the family of beta fuzzy numbers being product-stable, the orthogonal sum of n independent BRFNs $\widetilde{X}_i$, $i = 1, \ldots, n$ with $\widetilde{X}_i \sim \widetilde{B}(a_i, b_i, c_i)$ can easily be computed using the RS algorithm. For each draw $(m_1, \ldots, m_n)$, the degree of consistency $\widetilde{\Theta}^*(m_1, \ldots, m_n)$ and the focal set $\widetilde{F}_i$ can be computed from (2) and(3), respectively. As an example, Figure 1 displays the contour function together with the lower and upper CDFs of $\widetilde{X}_1 \oplus \widetilde{X}_2$, with $\widetilde{X}_1 \sim \widetilde{B}(1, 2, 2)$ and $\widetilde{X}_2 \sim \widetilde{B}(2, 1, 3)$. In this case, the estimated degree of conflict is $\widehat{\kappa} = 0.38$.

4 Conclusion

We have described a RS algorithm for combining RFSs. This algorithm extends the applicability of generalised evidence theory by making it possible to combine



(a) (b) (c)

Fig. 1: Two BRFNs ((a) and (b)) and their orthogonal sum computed by the RS algorithm (c). For each RFS, we plot ten focal set (black dotted curves), the contour function (red curve), and the lower/upper CDFs (blue curves).

RFSs when closed-form expressions are not available. We have illustrated the application of this algorithm with the combination of BRFNs. As mentioned in Section 3.1, the rejection rate equals the degree of conflict between the combined RFSs. The RS algorithm will thus be inefficient in situations of high conflict, such as when combining a large number of RFSs. Alternative methods such as Markov Chain Monte Carlo algorithms may then be preferable. Further work in this direction will be reported in future publications.

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