Nonparametric Rank-based Statistics and Significance Tests for Fuzzy Data

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January 13, 2005

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Abstract

Nonparametric rank-based statistics depending only on linear orderings of the observations are extended to fuzzy data. The approach relies on the definition of a fuzzy partial order based on the necessity index of strict dominance between fuzzy numbers, which is shown to contain, in a well defined sense, all the ordinal information present in the original data. A concept of fuzzy set of linear extensions of a fuzzy partial order is introduced, allowing the approximate computation of fuzzy statistics alpha-cutwise using a Markov Chain Monte Carlo simulation approach. The usual notions underlying significance tests are also extended, leading to the concepts of fuzzy p-value, and graded rejection of the null hypothesis (quantified by a degree of possibility and a degree of necessity) at a given significance level. This general approach is demonstrated in two special cases: Kendall's rank correlation coefficient, and Wilcoxon's two-sample rank sum statistic.

Keywords: Nonparametric Statistics, Hypothesis Testing, Fuzzy Order, Rank correlation coefficient, Fuzzy Data Analysis.

1 Introduction

The nonparametric approach to statistics provides inferential procedures which rely on weaker assumptions about the underlying distributions than do standard parametric procedures for similar problems [8]. A particular class of nonparametric, distribution-free procedures is composed of hypothesis tests based on a statistic depending only on the *rank order* of observations in one or several samples. Examples of methods in this category are Kendall's test of independence using the rank correlation coefficient τ, Kendall's coefficient of concordance for measuring the agreement among several orderings of n objects and the associated significance test [18], the Wilcoxon two-sample rank test for comparing two distributions, etc. Nonparametric procedures are typically adapted to situations in which little is known regarding the distributions of the data, such as small sample problems.

In recent years, the need to abandon the assumption of absolute precision of the observations in statistics and data analysis has gained increasing recognition. Descriptive statistics as well as inferential procedures for point estimation, confidence interval estimation and hypothesis testing have been extended to fuzzy data [17, 3, 29, 11, 4]. In particular, procedures for testing fuzzy hypotheses from crisp data [25, 2, 26], or crisp hypotheses from fuzzy data [12, 13, 20] have been proposed. However, few efforts, if any, have been devoted to applying nonparametric techniques to fuzzy data. In this paper, an attempt is made to partially fill this gap by proposing extensions of nonparametric statistics and associated significance tests to fuzzy data.

The class of nonparametric procedures investigated in this paper relies only on the ordering of observations in the sample. For instance, Kendall's rank correlation coefficient between two series of observations $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ only depends on the ranks of observations in both samples [18]. It is therefore actually a measure of comparison between two linear orders. If the observations are fuzzy, it is not possible, in general, to rank them without introducing arbitrary assumptions. It is natural, however, to derive a fuzzy partial ordering of the observations. As will be shown, fuzzy extensions of nonparameric statistical procedures can be rigorously constructed by replacing linear orders with fuzzy partial orders in the definition of test statistics. This can be achieved by viewing a fuzzy partial order as a fuzzy set of compatible linear orders, and applying Zadeh's extension principle. To avoid complex discrete optimization problems, a Monte-Carlo approach will be adopted to compute numerical approximations of the fuzzy test statistics and associated p-values.

The rest of this paper is organized as follows. Section 2 recalls some classical background material on linear orders, as well as on Kendall's tau coefficient and the associated test of independence between two samples, which will be used in the rest of the paper as an example to illustrate our approach. The relationship between our work and both censorship models in classical statistics, and previous work in fuzzy statistics, is also clarified in that section.

Section 3 then proceeds by recalling more results on crisp partial orders, and presenting their applications to the extension of the tau coefficient and the associated significance test to interval data. The concepts and results introduced in this Section are useful per se, as interval data constitute an important class of imprecise data frequently encountered in practice. They also constitute an intermediate step on the way to the introduction of methods for fuzzy data, which are essentially based on the same approach extended to the level sets of fuzzy ordering relations.

The fuzzy case is thus subsequently handled in Section 4, where basic definitions regarding fuzzy orders are recalled, new definitions and results regarding linear extensions of fuzzy partial orders are introduced, and their applications to the non-parametric statistical analysis of fuzzy data are described. To show the generality of our approach, the fuzzy extension of another nonparametric procedure: the Mann-Whitney-Wilcoxon two-sample test, is finally presented in Section 5, and Section 6 concludes the paper.

2 Background

2.1 Crisp Ordering Relations

A crisp binary relation on a set U is a crisp subset $R \subseteq U^2$. In the sequel, U will always be assumed to be finite. The notation uRv will be used as a shortcut for $(u, v) \in R$.

A relation R is a partial order if it is antisymmetric and transitive. An irreflexive partial order is said to be strict. By default, all partial orders considered in this paper will be assumed to be strict. A linear order is a complete partial order. In the sequel, $\mathcal{P}_U$ and $\mathcal{L}_U$ will denote, respectively, the sets of partial and linear orders on U.

A linear order L is a linear extension of a partial order P if and only if $P \subseteq L$. A theorem due to Szpilrajn (cited in [9]) states that every partial order has at least one linear extension. The set of linear extensions of a partial order P will be noted $\Lambda(P) \subseteq \mathcal{L}_U$.

In many situations $U = \{u_1, \ldots, u_n\}$ is a set of n objects of a population Ω, described by a continuous variable $X : \Omega \to \mathbb{R}$. Let $x_i = X(u_i)$, $i = 1, \ldots, n$. Assuming that $x_i \neq x_j$ for all $i \neq j$, the n values $x_1, \ldots, x_n$ induce a strict linear order L on U such that $u_i L u_j$ iff $x_i < x_j$. We shall denote by λ the mapping from $\mathbb{R}^n$ to $\mathcal{L}_U$ such that $L = \lambda(x_1, \ldots, x_n)$.

2.2 Kendall's tau

Let $(x_1, y_1), \ldots, (x_n, y_n)$ denote the values taken by two continuous variables (X, Y) for n members $U = \{u_1, \ldots, u_n\}$ of a population. A measure of association between variables X and Y may be obtained by counting the number of pairs of observations $\{i, j\}$ which are ordered consistently by X and Y. More precisely, let $L_X = \lambda(x_1, \ldots, x_n)$ and $L_Y = \lambda(y_1, \ldots, y_n)$ denote, respectively, the linear orders on U induced by the X and Y samples (since X and Y are assumed to be continuous, such linear orders always exist). The number of pairs ordered in the same way by L_X and L_Y is the cardinality of their intersection $|L_X \cap L_Y|$. Since there are $n(n-1)/2$ distinct pairs of observations, the minimum and maximum values of $|L_X \cap L_Y|$ are, respectively, 0 and $n(n-1)/2$. If we require a correlation coefficient to be $+1$ when there is perfect positive agreement and -1 when there is perfect negative agreement, we obtain Kendall's tau coefficient [16]:

$$\tau = \frac{4|L_X \cap L_Y|}{n(n-1)} - 1. \tag{1}$$

Note that τ depends only on the two linear orders L_X and L_Y, and can actually be seen as a measure of agreement between linear orders. In the sequel, the notation $\tau(L_X, L_Y)$ will sometimes be used to emphasize this fact.

Example 1 Let us consider the data in Table 1. We have

$$L_X = \{(u_1,u_3),(u_1,u_4),(u_2,u_3),(u_2,u_1),(u_2,u_4),(u_3,u_4)\},$$

$$L_Y = \{(u_1,u_2),(u_1,u_4),(u_3,u_1),(u_3,u_2),(u_3,u_4),(u_4,u_2)\},$$

$$L_X \cap L_Y = \{(u_1,u_4),(u_3,u_4)\}.$$

Hence, $\tau = (4 \times 2)/(4 \times 3) - 1 = -1/3$.

In addition to being used as correlation coefficient, τ may also be used as a test statistic [16]. Consider the problem of testing hypothesis H_0 of independence between X and Y in the population, against the alternative hypothesis H_1 of non-independence. Under H_0, τ may be shown to have expected value zero, variance equal to

$$Var(\tau) = \frac{2(2n+5)}{9n(n-1)},$$

and to tend to the normal form very quickly as $n \to \infty$. The normal approximation is usually considered to be valid as long as $n \geq 8$. Independence may thus be rejected at the α level if the following condition holds:

$$|\tau| > \Phi^{-1}\left(1 - \frac{\alpha}{2}\right)\sqrt{\frac{2(2n+5)}{9n(n-1)}}, \tag{2}$$

where Φ denotes the distribution function of the standard normal distribution. Such a test may be formally represented by a function $\varphi_\alpha : \mathbb{R}^{2n} \to \{0,1\}$ such that

$\varphi_\alpha[(x_1, y_1), \ldots, (x_n, y_n)]$ equals 1 if H_0 is rejected at the α level, and 0 otherwise. Here, φ_α is thus defined as:

$$\varphi_\alpha[(x_1, y_1), \ldots, (x_n, y_n)] = \begin{cases} 1 & \text{if } |\tau| > \Phi^{-1}\left(1 - \frac{\alpha}{2}\right)\sqrt{\frac{2(2n+5)}{9n(n-1)}} \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

Since the significance level is often somewhat arbitrary [19, page 70], it is often interesting to determine the smallest significant level at which the null hypothesis would be rejected for the given observation, which is called the significance probability, or p-value. By definition, H_0 is thus rejected at a given significance level α if and only if the p-value is smaller than α. For fixed τ, the condition for H_0 to be rejected in our case can be found from Eq. (2) to be $\alpha > p(\tau)$ with

$$p(\tau) = 2\left[1 - \Phi\left(|\tau|\sqrt{\frac{9n(n-1)}{2(2n+5)}}\right)\right], \tag{4}$$

which is thus the expression of the significance probability. Equation (3) can be written as:

$$\varphi_\alpha[(x_1, y_1), \ldots, (x_n, y_n)] = \begin{cases} 1 & \text{if } p(\tau) < \alpha \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

EXAMPLE 2 A simple data set of $n = 10$ observations (x_i, y_i) is shown in Figure 1. We have for this data $\tau = 0.69$ and $p = 0.0056$. Hence, the null hypothesis of independence between X and Y is rejected at any significance level $\alpha > 0.0056$.

2.3 Censored data analysis and fuzzy statistics

The problem addressed in this paper is, to some extent, related to the issue of censored data as encountered in classical statistics, in the context of survival analysis. Such data typically arise when an event of interest, such as a disease or a failure, is only partially observed, because information is gathered at certain examination times. Two usual models are random right-censorship and random interval-censorship. In the first case, the observations are assumed to be of the form $Y_i = \min(X_i, W_i)$, $i = 1, \ldots, n$, where the X_i are the (partially observed) survival times, and the W_i are the censoring times. In this model, both survival and censoring times are assumed to be random, and mutually independent. Rank tests for such data have been proposed by several authors (see, e.g. [1][21]). In the case of so-called random interval censored data, the event is only known to happen between two random examination times. The observations are thus of the form (U_i, V_i), $i = 1, \ldots, n$, and it is only known that $U_i \leq X_i \leq V_i$ for all i. Here again it is customary to assume independence between survival times X_i and censoring interval endpoints. Nonparametric tests for interval censored data have been developped by Pan [23], [24], among others.

Although the interval data considered in this paper have the same form as interval censored data, the problem addressed here is different. We are not concerned with imprecision arising from *random* inspection times, but with the situation in which the result of a random experiment is reported from the observer to the statistician with some imprecision, arising from its *limited perception or recollection* of the precise numerical values [11, page 314]. Linguistic data such as "moderately expensive" to describe the price of an item, "around 20 degrees" or "warm" to report a temperature then arise, which cannot easily be modeled using classical subsets, but lend themselves quite naturally to a description using fuzzy sets. The relevance of a fuzzy set-theoretic approach to data analysis is particularly obvious in sensory studies, in which human subjects are asked to assess products according to their perception of attributes related to the color, smell, taste or texture of the objects [27]. The motivation for the present study actually comes from the authors' work in this application domain.

The use of fuzzy set-theoretic methods in statistics has been well studied in the past twenty years, as indicated by the existence of several monographs on this topic [17, 3, 29, 4]. As remarked by Gebhardt et al. [11, page 317], two main directions have been followed.

The first one is based on the concept of fuzzy random variable, defined as a function from the set Ω of all possible outcomes of a random experiment, to the set of fuzzy subsets of the real line, verifying certain properties making it a valid generalization of a real random variable. This approach is well founded mathematically as it allows to extend important limit theorems in the field of fuzzy statistics [17]. However, the corresponding mathematical apparatus is not always needed, or even relevant. As noted by Gebhardt et al. [11, page 317]: "Fuzzy random variables describe situations where the uncertainty and imprecision of observing a random value (...) is functionally dependent of the respective outcome ω. If observation conditions are not influenced by the random experiment (...), then theoretical considerations in fuzzy statistics become much simpler, since in this case we do not need anymore a concept of a fuzzy random variable. It suffices to generalize operations of traditional statistical inference for crisp data to operations on possibility distributions using the well-known extension principle". Our work is clearly in line with this second approach. We thus consider fuzzy data $\widetilde{x}_1, \ldots, \widetilde{x}_n$ (where each $\widetilde{x}_i$ is a fuzzy number) as an imperfect specification of a partially observed realization $x_1, \ldots, x_n$ of an i.i.d. sample $X_1, \ldots, X_n$ with parent distribution F_X. This approach has been used to extend several parametric tests such as the test on the variance of a normal distribution with known mean [11, page 321]. The goal of this paper is to extend this approach to nonparametric rank tests, a task which, to our knowledge, had not been undertaken before.

3 Analysis of interval data

3.1 Partial Order Induced by Crisp Real Intervals

As in Section 2.2, let us consider again a set $U = \{u_1, \ldots, u_n\}$ of n objects described by a variable X. As before, x_i denotes the value taken by X for object u_i, $i = 1, \ldots, n$. However, we now assume that we only have partial knowledge of the x_i's, in the form of lower and upper bounds. Each object u_i is thus described by a nonempty real interval $\overline{x}_i$ such that $x_i \in \overline{x}_i$ ($\overline{x}_i$ may be closed, semi-closed or open). Let $L_X = \lambda(x_1, \ldots, x_n)$ denote the linear order among the n objects induced by X. As a consequence of partial ignorance of the x_i's, L_X is, in general, partially unknown. It may then be wondered how to represent the partial knowledge of L_X induced by the n intervals $\overline{x}_i$, $i = 1, \ldots, n$.

To answer this question, let us consider the following partial order on U:

$$P = \{(u_i, u_j) \in U^2 \mid \sup \overline{x}_i \leq \inf \overline{x}_j\}. \tag{6}$$

Such a partial order, denoted $P = \pi(\overline{x}_1, \ldots, \overline{x}_n)$, is called an interval order [9]. The following theorem states that the set $\Lambda(P)$ is exactly the set of possible values of L_X. Hence, all the ordinal information induced by the intervals $\overline{x}_i$, $i = 1, \ldots, n$ is contained in P.

THEOREM 1 *Let $P = \pi(\overline{x}_1, \ldots, \overline{x}_n)$ be the partial order on a set U of n objects, induced by n intervals $\overline{x}_i, i = 1, \ldots, n$, such that $\exists\ (x_1, \ldots, x_n) \in \overline{x}_1 \times \ldots \times \overline{x}_n$ with $x_i \neq x_j$, $\forall i \neq j$. Let $\Lambda(P)$ denote the set of linear extensions of P. We have:*

$$\begin{aligned}\Lambda(P) &= \{L \in \mathcal{L}_U,\ \mid \exists (x_1, \ldots, x_n) \in \overline{x}_1 \times \ldots \times \overline{x}_n, L = \lambda(x_1, \ldots, x_n)\} \\ &= \lambda(\overline{x}_1 \times \ldots \times \overline{x}_n).\end{aligned}$$

Proof. See Appendix A.

As a consequence of the above theorem, P may be seen as a representation of the available knowledge concerning the unknown true linear order L_X. The relationship between mappings λ, π and Λ is illustrated in Figure 2.

3.2 Rank correlation coefficient for interval data

Definition

In this section, we present the definition of the Kendall's tau coefficient between interval-valued data samples, as recently introduced by the authors [15]. Let us assume that the available data takes the form of n pairs of intervals $(\overline{x}_i, \overline{y}_i)$, $i = 1, \ldots, n$, where each interval $\overline{x}_i$ (respectively, $\overline{y}_i$) is known to contain the value taken by variable X (respectively, Y) for object u_i. As shown above, all the ordinal information regarding both samples is contained in the induced partial orders $P_X = \pi(\overline{x}_1, \ldots, \overline{x}_n)$ and $P_Y = \pi(\overline{y}_1, \ldots, \overline{y}_n)$. Consequently, the set $T(P_X, P_Y)$ of possible values for the Kendall's correlation coefficient between the two samples must depend only on P_X and P_Y. Indeed, this set contains all possible values for the tau coefficient $\tau(L_X, L_Y)$ for all linear extensions L_X of P_X and all linear extensions L_Y of P_Y. This set, however, does not admit a simple representation. For that reason, we prefer to define the Kendall's tau coefficient between interval-valued samples $\overline{x}_1, \ldots, \overline{x}_n$ and $\overline{y}_1, \ldots, \overline{y}_n$ as the convex hull of $T(P_X, P_Y)$, i.e., the smallest closed interval $\overline{\tau}(P_X, P_Y)$ containing $T(P_X, P_Y)$:

$$\overline{\tau}(P_X, P_Y) = \left[\min_{L_X \in \Lambda(P_X), L_Y \in \Lambda(P_Y)} \tau(L_X, L_Y), \max_{L_X \in \Lambda(P_X), L_Y \in \Lambda(P_Y)} \tau(L_X, L_Y) \right] \quad (7)$$

EXAMPLE 3 Let us consider the data set in Table 2, which is an imprecise version of the data used in Example 1.

The induced partial orders are:

$$P_X = \{(u_1, u_3), (u_1, u_4), (u_2, u_3), (u_2, u_4)\}$$

$$P_Y = \{(u_1, u_2), (u_1, u_4), (u_3, u_2), (u_3, u_4), (u_4, u_2)\}.$$

To simplify the notations, let us denote a linear order by the corresponding permutation of indices. For instance, the linear order $<$ such that $u_2 < u_1 < u_3 < u_4$ will be noted $(2, 1, 3, 4)$. We can see that P_X has four linear extensions:

$$P_X \cup \{(1,2), (3,4)\} = (1,2,3,4) \qquad P_X \cup \{(2,1), (3,4)\} = (2,1,3,4)$$

$$P_X \cup \{(1,2), (4,3)\} = (1,2,4,3) \qquad P_X \cup \{(2,1), (4,3)\} = (2,1,4,3),$$

and P_Y has two linear extensions:

$$P_Y \cup \{(1,3)\} = (1,3,4,2) \qquad P_Y \cup \{(3,1)\} = (3,1,4,2).$$

The tau coefficients for the $4 \times 2 = 8$ possible combinations of a linear extension of P_X and a linear extension of P_Y are given in Table 3.

We can see that there are four possible values of τ:

$$T = \{-2/3, -1/3, 0, 1/3\}.$$

Hence, $\overline{\tau} = [-2/3, 1/3]$.

Significance test

As a consequence of the tau statistics being imprecise, the result of a significance test (with the same hypotheses as in Section 2.2) may become indeterminate. To see this, let us consider the p-value for the significance test given by (4). Denoting by τ^- and τ^+ the lower and upper bounds of $\overline{\tau}$, we have

$$\min_{\tau^- \leq \tau \leq \tau^+} |\tau| = \max(0, \tau^-, -\tau^+) \tag{8}$$

and

$$\max_{\tau^- \leq \tau \leq \tau^+} |\tau| = \max(\tau^+, -\tau^-). \tag{9}$$

Consequently, the range of $p(\tau)$ is the interval $\overline{p}(\overline{\tau}) = [p^-(\overline{\tau}), p^+(\overline{\tau})]$ with

$$p^-(\overline{\tau}) = 2\left[1 - \Phi\left(\max(\tau^+, -\tau^-)\sqrt{\frac{9n(n-1)}{2(2n+5)}}\right)\right] \tag{10}$$

$$p^+(\overline{\tau}) = 2\left[1 - \Phi\left(\max(0, \tau^-, -\tau^+)\sqrt{\frac{9n(n-1)}{2(2n+5)}}\right)\right]. \tag{11}$$

As shown in Section 2.2, the result of the test at a given significance level α may be deduced by comparing the p-value to α. Since the p-value is now only known to lie in the interval $\overline{p}(\overline{\tau})$, three cases arise:

1. if $p^+(\overline{\tau}) < \alpha$, then $p(\tau)$ is surely less than α, and H_0 is rejected;
2. if $p^-(\overline{\tau}) > \alpha$, then $p(\tau)$ is surely greater than α, and H_0 is not rejected;
3. if $p^-(\overline{\tau}) < \alpha < p^+(\overline{\tau})$, then the position of $p(\tau)$ relative to α is unknown, and the result of the test is indeterminate.

This may be expressed formally using the φ_α function:

$$\varphi_\alpha[(\overline{x}_1, \overline{y}_1)\ldots,(\overline{x}_n, \overline{y}_n)] = \begin{cases} 1 & \text{if } p^+(\overline{\tau}) < \alpha \\ 0 & \text{if } p^-(\overline{\tau}) > \alpha \\ \{0,1\} & \text{otherwise.} \end{cases} \tag{12}$$

The fact that the test result may be indeterminate may be found troublesome by readers accustomed to standard statistical procedures. However, this indeterminacy is merely a consequence of the ambiguity of the observation (or perception) process, which is propagated in the calculations. This ambiguity cannot be resolved without making additional assumptions, and no such assumptions are made in the conservative approach described in this paper. When the final imprecision is too high, no conclusion can be drawn, which in practice may be resolved by acquiring more information, either as additional observations, or in the form of more accurate information regarding the cases already collected.

The following simple example will serve to further illustrate this important point. Consider the problem of testing hypotheses about the mean of a Gaussian distribution with unknown variance, based on a sample of size n, using the well known Student-t test. Assume that, instead of observing the sample data or the exact value of the t statistic, you are only given an interval in which this statistic surely lies, *and you have no idea how this interval was constructed*, i.e., you have no model for the degradation of the data in the observation process. All you know is that the given interval contains the true value of the test statistic. Obviously, there are cases where you will be able to give the result of the test, while in other cases the result will be indeterminate. Refusing to consider the interval data on the ground that it will not allow to make a decision in all cases would surely be a very radical view. On the other hand, assuming a model of the observation process without any evidence to support this model would not be good statistical practice. The approach adopted in this paper is both pragmatic and cautious, in that it merely propagates imprecision in the calculations, and accepts that the decision may, in some cases, be indeterminate.

Example 4 Figures 3 and 4 show two data sets generated from the data of Example 2 by the transformation $\overline{x}_i = [x_i - r, x_i + r]$ and $\overline{y}_i = [y_i - r, y_i + r]$ $(i = 1, \dots, 10)$, with $r = 0.1$ and $r = 0.2$, respectively. In the first case we have $\overline{\tau} = [0.51, 0.87]$ and $\overline{p} = [0.0005, 0.0397]$. The null hypothesis is thus still rejected at the 5% significance level. In the second case, we have: $\overline{\tau} = [0.29, 0.96]$ and $\overline{p} = [0.0001, 0.2449]$: the result of the test is indeterminate.

Computational procedure

Until now, we have left aside the important practical issue of computing τ^- and τ^+. A simplistic approach for solving the optimization problems in (7) might be to generate all the linear extensions of P_X and P_Y. However, this approach is impractical due to the potentially exponential number of linear extensions of a partial order, which can reach $n!$ in the limit case of an empty partial order. An algorithm for computing $\overline{\tau}$ directly without generating linear extensions has been proposed by Hébert et al. [15]. However, this algorithm (the complexity analysis of which remains to be done) is only applicable to very small problems.

A more general approach is to compute approximations of τ^- and τ^+ using a Monte Carlo simulation method. This is made possible by the availability of efficient Markov chain Monte Carlo techniques for generating uniformly randomly linear extensions of a partial order [5, 14]. The most efficient algorithm to date seems to be that proposed by Bubley and Dyer [5] which has a running time of $O(n^3 \log n\varepsilon^{-1})$, where ε is the desired accuracy. This algorithm is described in Appendix B. It may be used to repeatedly generate $L_X \in \Lambda(P_X)$, $L_Y \in \Lambda(P_Y)$, compute $\tau(L_X, L_Y)$, and store this value if it is smaller than the current minimum, or larger than the current maximum. The algorithm may be stopped when the minimum and maximum have not changed in the last η iterations. It may be noticed that the resulting approximations $\widehat{\tau}^-$ and $\widehat{\tau}^+$ are biased estimates of τ^- and τ^+, since we can only have $\widehat{\tau}^- \geq \tau^-$ and $\widehat{\tau}^+ \leq \tau^+$. However, this bias can be made arbitrarily small by increasing the number of iterations. In practice, very good approximations have been obtained for medium size problems ($n \approx 30$) with only a few thousand iterations.

4 Analysis of fuzzy data

4.1 Fuzzy Orderings

Fuzzy counterparts to some of the relational concepts recalled in Section 2.1 were introduced by Zadeh [30]. A recent overview of fuzzy relations is presented in [22]. A thorough treatment with emphasis on preference modeling is provided in [10]. A fuzzy relation $\widetilde{R}$ on a set U is a fuzzy subset of U^2. The degree of membership of a pair (u, v) in $\widetilde{R}$ will be noted $\mu_{\widetilde{R}}(u, v) \in [0, 1]$. The (max-min) composition of two fuzzy relations $\widetilde{R}$ and $\widetilde{Q}$ is noted $\widetilde{R} \circ \widetilde{Q}$ and is defined by:

$$(\widetilde{R} \circ \widetilde{Q})(u, w) = \bigvee_v \mu_{\widetilde{R}}(u, v) \wedge \mu_{\widetilde{Q}}(v, w), \quad \forall (u, w) \in U^2, \tag{13}$$

where $\vee$ and $\wedge$ denote the maximum and minimum operators, respectively. The inverse of a fuzzy relation $\widetilde{R}$ is defined by $\mu_{\widetilde{R}^{-1}}(u, v) = \mu_{\widetilde{R}}(v, u)$, for all $(u, v) \in U^2$.

The above definitions regarding properties of crisp relations can easily be extended to fuzzy relations using the classical definition of the inclusion of fuzzy sets:

$$\widetilde{R} \subseteq \widetilde{Q} \Leftrightarrow \forall (u, v) \in U^2, \mu_{\widetilde{R}}(u, v) \leq \mu_{\widetilde{Q}}(u, v),$$

and the natural fuzzy set connectives:

$$\mu_{\widetilde{R} \cap \widetilde{Q}}(u, v) = \mu_{\widetilde{R}}(u, v) \wedge \mu_{\widetilde{Q}}(u, v)$$

and

$$\mu_{\widetilde{R} \cup \widetilde{Q}}(u, v) = \mu_{\widetilde{R}}(u, v) \vee \mu_{\widetilde{Q}}(u, v).$$

More precisely, a fuzzy relation $\widetilde{R}$ on U is:

- reflexive if $I \subseteq \widetilde{R}$ $(\forall u \in U, \mu_{\widetilde{R}}(u, u) = 1)$;
- irreflexive if $I \cap \widetilde{R} = \emptyset$ $(\forall u \in U, \mu_{\widetilde{R}}(u, u) = 0)$;
- symmetric if $\widetilde{R}^{-1} = \widetilde{R}$ $(\forall (u, v) \in U^2, \mu_{\widetilde{R}}(u, v) = \mu_{\widetilde{R}}(v, u))$;
- antisymmetric if $\widetilde{R} \cap \widetilde{R}^{-1} \subseteq I$ $(\forall (u, v) \in U^2, \mu_{\widetilde{R}}(u, v) > 0$ and $\mu_{\widetilde{R}}(v, u) > 0 \Rightarrow u = v)$;
- (max-min) transitive if $\widetilde{R}^2 \subseteq \widetilde{R}$ $(\forall (u, w) \in U^2, \mu_{\widetilde{R}}(u, w) \geq \bigvee_v \mu_{\widetilde{R}}(u, v) \wedge \mu_{\widetilde{R}}(v, w))$.

Zadeh [30] defines a fuzzy partial order as an antisymmetric and transitive fuzzy relation. The following theorem [30] expresses a relationship between crisp and fuzzy orders.

THEOREM 2 *Let $\widetilde{R}$ be a fuzzy partial order on U. For all $\beta \in (0, 1]$, the β-cut of $\widetilde{R}$, defined as $\widetilde{R}^\beta = \{(u, v) \in U^2 \mid \mu_{\widetilde{R}}(u, v) \geq \beta\}$ is a crisp partial order.*

Note that we use the notation β as an index for the level sets of a fuzzy set instead of the more usual notation α, to avoid confusion with the significance level of a hypotheses test, a concept that will be used later in this paper.

4.2 Linear Extensions of a Fuzzy Partial Order

The concept of linear extension can be extended to fuzzy partial orders. Let $\widetilde{P}$ be a fuzzy partial order on U. A crisp linear order L may be considered to be a linear extension of $\widetilde{P}$ to the extent that it includes $\widetilde{P}$ (we recall that L and $\widetilde{P}$ are, respectively, crisp and fuzzy subsets of U^2). The fuzzy set $\Lambda(\widetilde{P}) \in [0,1]^{\mathcal{L}_U}$ of linear extensions of $\widetilde{P}$ may thus be defined by:

$$\mu_{\Lambda(\widetilde{P})}(L) = I(\widetilde{P}, L) \quad \forall L \in \mathcal{L}_U$$

where I is an inclusion index. A general family of inclusion measures is:

$$I(F,G) = \inf_u i(\mu_F(u), \mu_G(u))$$

where F and G are two fuzzy subsets of a common domain, and i is a fuzzy implication operator [6, page 59]. A common choice of implication operator is $i(a,b) = \max(1-a, b)$. This leads to

$$\begin{aligned} I(F,G) &= \inf_u (1-\mu_F(u)) \vee \mu_G(u) \\ &= 1 - \sup_u \mu_F(u) \wedge (1-\mu_G(u)) \\ &= 1 - h(F \cap \overline{G}) \end{aligned}$$

where h denotes the height of a fuzzy set. We thus arrive at

$$\begin{aligned} \mu_{\Lambda(\widetilde{P})}(L) &= 1 - h(\widetilde{P} \cap \overline{L}) \\ &= 1 - \bigvee_{u,v} \mu_{\widetilde{P}}(u,v) \wedge (1-\mu_L(u,v)), \end{aligned}$$

where the supremum has been replaced by the maximum, since we are dealing with a finite domain U^2.

THEOREM 3 *Let $\widetilde{P}$ be a fuzzy partial order on U, and $\Lambda(\widetilde{P}) \in [0,1]^{\mathcal{L}_U}$ the fuzzy set of linear extensions of $\widetilde{P}$. We have*

1. $\Lambda(\widetilde{P})^{\beta} = \Lambda(\widetilde{P}^{(1-\beta)+})$, $\forall \beta \in (0,1]$;

2. $\Lambda(\widetilde{P})^{\beta+} = \Lambda(\widetilde{P}^{(1-\beta)})$, $\forall \beta \in [0,1)$,

where the superscript $\beta+$ denotes the strong β-cut.

Proof. Let $\beta \in (0,1]$. For all $L \in \mathcal{L}_U$,

$$\begin{aligned}
L \in \Lambda(\widetilde{P})^{\beta} &\Leftrightarrow 1 - \sup_{u,v} \mu_{\widetilde{P}}(u,v) \wedge (1 - \mu_L(u,v)) \geq \beta \\
&\Leftrightarrow \sup_{u,v} \mu_{\widetilde{P}}(u,v) \wedge (1 - \mu_L(u,v)) \leq 1 - \beta \\
&\Leftrightarrow \forall (u,v) \in U^2, \mu_{\widetilde{P}}(u,v) \wedge (1 - \mu_L(u,v)) \leq 1 - \beta \\
&\Leftrightarrow \forall (u,v) \in U^2, \mu_{\widetilde{P}}(u,v) > 1 - \beta \Rightarrow 1 - \mu_L(u,v) = 0 \\
&\Leftrightarrow \forall (u,v) \in \widetilde{P}^{(1-\beta)+}, (u,v) \in L \\
&\Leftrightarrow L \in \Lambda(\widetilde{P}^{(1-\beta)+}).
\end{aligned}$$

A similar line of reasoning may be applied to prove that $\Lambda(\widetilde{P})^{\beta+} = \Lambda(\widetilde{P}^{1-\beta})$ for all $\beta \in [0,1)$. $\square$

EXAMPLE 5 Let us consider the fuzzy partial order $\widetilde{P}$ described by Table 4. It has three distinct strong β-cuts:

$$\widetilde{P}^{0+} = \{(u_1,u_3),(u_1,u_4),(u_2,u_3),(u_2,u_1),(u_2,u_4),(u_3,u_4)\}$$

$$\widetilde{P}^{1/3+} = \{(u_1,u_3),(u_1,u_4),(u_2,u_3),(u_2,u_1),(u_2,u_4)\}$$

$$\widetilde{P}^{2/3+} = \{(u_1,u_3),(u_1,u_4),(u_2,u_3),(u_2,u_4)\}$$

We can see that $\widetilde{P}^{0+}$ is a linear order. Hence

$$\Lambda(\widetilde{P}^{0+}) = \Lambda(\widetilde{P})^{1} = \{(2,1,3,4)\}.$$

We then have:

$$\Lambda(\widetilde{P}^{1/3+}) = \Lambda(\widetilde{P})^{2/3} = \{(2,1,3,4),(2,1,4,3)\}$$

and

$$\Lambda(\widetilde{P}^{2/3+}) = \Lambda(\widetilde{P})^{1/3} = \{(2,1,3,4),(2,1,4,3),(1,2,3,4),(1,2,4,3)\}.$$

The fuzzy set of linear extensions is, consequently:

$$\Lambda(\widetilde{P}) = \left\{ \frac{(2,1,3,4)}{1}, \frac{(2,1,4,3)}{2/3}, \frac{(1,2,3,4)}{1/3}, \frac{(1,2,4,3)}{1/3} \right\}, \tag{14}$$

where the classical notation $\{u/\mu_F(u)\}$ is used to denote a fuzzy set F.

4.3 Fuzzy Partial Order Induced by Fuzzy Intervals

Let us now assume that the unknown values x_i, $i = 1, \ldots, n$ are constrained by fuzzy intervals $\widetilde{x}_i$, $i = 1, \ldots, n$. The membership function $\mu_{\widetilde{x}_i}$ is a possibility distribution related to the unknown value of x_i. Let $\widetilde{x}_i$ and $\widetilde{x}_j$ be two fuzzy intervals related to two values x_i and x_j. Assuming $\widetilde{x}_i$ and $\widetilde{x}_j$ to be noninteractive, the possibility distribution concerning the joint values (x_i, x_j) is given by:

$$\mu_{(\widetilde{x}_i,\widetilde{x}_j)}(x_i, x_j) = \mu_{\widetilde{x}_i}(x_i) \wedge \mu_{\widetilde{x}_j}(x_j).$$

According to this possibility distribution, the necessity of the event $x_i \leq x_j$ is given by:

$$\begin{aligned} N(x_i \leq x_j) &= 1 - \Pi(x_i > x_j) && (15) \\ &= 1 - \sup_{x_i > x_j} \mu_{\widetilde{x}_i}(x_i) \wedge \mu_{\widetilde{x}_j}(x_j). && (16) \end{aligned}$$

This quantity, sometimes referred to as the necessity index of strict dominance (NSD), is classically used to compare fuzzy numbers [7]. Let $\mu_{\widetilde{P}}(u_i, u_j) = N(x_i \leq x_j)$. This fuzzy relation is antisymmetric and transitive: it is a fuzzy partial order on U [22]. Note that this relation is dual to the fuzzy relation $\mu_{\widetilde{R}}(u_i, u_j) = \Pi(x_i < x_j)$, i.e., we have $\mu_{\widetilde{P}}(u_i, u_j) = 1 - \mu_{\widetilde{R}}(u_j, u_i)$ for all u_i and u_j. However, $\widetilde{R}$ is not transitive. Hence, $\widetilde{P}$, and not $\widetilde{R}$, is the fuzzy counterpart of P in (6). The previous notation will be extended as follows: $\widetilde{P} = \pi(\widetilde{x}_1, \ldots, \widetilde{x}_n) \in \widetilde{\mathcal{P}}_U$, where $\widetilde{\mathcal{P}}_U$ denotes the set of fuzzy partial orders on U. The relationship between $\widetilde{P}$ and P is given by the following theorem.

THEOREM 4 *Let $\widetilde{P} = \pi(\widetilde{x}_1, \ldots, \widetilde{x}_n)$ be the fuzzy partial order on U induced by the fuzzy numbers $\widetilde{x}_1, \ldots, \widetilde{x}_n$. We have*

1. $\widetilde{P}^{\beta} = \pi(\widetilde{x}_1^{(1-\beta)+}, \ldots, \widetilde{x}_n^{(1-\beta)+})$, *for all* $\beta \in (0,1]$;

2. $\widetilde{P}^{\beta+} = \pi(\widetilde{x}_1^{(1-\beta)}, \ldots, \widetilde{x}_n^{(1-\beta)})$, *for all* $\beta \in [0,1)$.

Proof. For all $(i,j) \in \{1, \ldots, n\}$,

$$
\begin{aligned}
(u_i, u_j) \in \widetilde{P}^{\beta} \quad &\Leftrightarrow \quad 1 - \sup_{x_i > x_j} \mu_{\widetilde{x}_i}(x_i) \wedge \mu_{\widetilde{x}_j}(x_j) \geq \beta \\
&\Leftrightarrow \quad \sup_{x_i > x_j} \mu_{\widetilde{x}_i}(x_i) \wedge \mu_{\widetilde{x}_j}(x_j) \leq 1 - \beta \\
&\Leftrightarrow \quad \forall (x_i, x_j) \in \mathbb{R}^2, x_i > x_j \Rightarrow (\mu_{\widetilde{x}_i}(x_i) \wedge \mu_{\widetilde{x}_j}(x_j) \leq 1 - \beta) \\
&\Leftrightarrow \quad \forall (x_i, x_j) \in \mathbb{R}^2, x_i > x_j \Rightarrow (\mu_{\widetilde{x}_i}(x_i) \leq 1 - \beta \text{ or } \mu_{\widetilde{x}_j}(x_j) \leq 1 - \beta) \\
&\Leftrightarrow \quad \forall (x_i, x_j) \in \mathbb{R}^2, (\mu_{\widetilde{x}_i}(x_i) > 1 - \beta \text{ and } \mu_{\widetilde{x}_j}(x_j) > 1 - \beta) \Rightarrow x_i \leq x_j \\
&\Leftrightarrow \quad \widetilde{x}_i^{(1-\beta)+} < \widetilde{x}_j^{(1-\beta)+}.
\end{aligned}
$$

The second part of the theorem can be proved in the same way, starting from a strict inequality in the first line. □

Finally, the following theorem shows that the fuzzy set $\Lambda(\widetilde{P})$ of linear extensions of the fuzzy partial order $\widetilde{P} = \pi(\widetilde{x}_1, \ldots, \widetilde{x}_n)$ can be obtained by applying the extension principle to the mapping λ from $\mathbb{R}^n$ to $\mathcal{L}_U$.

THEOREM 5 *Let* $\widetilde{P} = \pi(\widetilde{x}_1, \ldots, \widetilde{x}_n)$ *be the fuzzy partial order induced by* n *fuzzy intervals* $\widetilde{x}_1, \ldots, \widetilde{x}_n$. *Let* $\Lambda(\widetilde{P}) \in [0,1]^{\mathcal{L}_U}$ *be the fuzzy set of linear extensions of* $\widetilde{P}$. *We have, for all* $L \in \mathcal{L}_U$:

$$\mu_{\Lambda(\widetilde{P})}(L) = \sup_{L = \lambda(x_1, \ldots, x_n)} \mu_{\widetilde{x}_1}(x_1) \wedge \ldots \wedge \mu_{\widetilde{x}_n}(x_n)$$

Proof. From Theorem 4 we have, for all $\beta \in (0,1]$, $\pi(\widetilde{x}_1^{\beta}, \ldots, \widetilde{x}_n^{\beta}) = \widetilde{P}^{(1-\beta)+}$ and, from Theorem 3, $\Lambda(\widetilde{P}^{(1-\beta)+}) = \lambda(\widetilde{P})^{\beta}$. Hence, from Theorem 1,

$$\lambda(\widetilde{P})^{\beta} = \Lambda \circ \pi(\widetilde{x}_1^{\beta}, \ldots, \widetilde{x}_n^{\beta}) = \lambda(\widetilde{x}_1^{\beta} \times \ldots \times \widetilde{x}_n^{\beta}).$$

Let F the fuzzy subset of $\mathcal{L}_U$ defined by

$$\mu_F(L) = \sup_{L = \lambda(x_1, \ldots, x_n)} \mu_{\widetilde{x}_1}(x_1) \wedge \ldots \wedge \mu_{\widetilde{x}_n}(x_n).$$

Since $\mathcal{L}_U$ is finite, we have $F^\beta = \lambda(\widetilde{x}_1^\beta \times \ldots \times \widetilde{x}_n^\beta)$ (see, e.g., [6, page 50]). Since $\lambda(\widetilde{P})$ and F have the same β-cuts for all $\beta \in (0, 1]$, they are equal, which completes the proof. □

4.4 Fuzzy rank correlation between two fuzzy partial orders

Definition

As before, let us denote by $(x_1, y_1), \ldots, (x_n, y_n)$ the values taken by two continuous variables (X, Y) for n members of a population. We now assume that, because of the imperfectness of the observation process, these values are only partially known, and are constrained by possibility distributions. The available data thus consists of n pairs of fuzzy numbers $(\widetilde{x}_1, \widetilde{y}_1), \ldots, (\widetilde{x}_n, \widetilde{y}_n)$. Let $\widetilde{T}$ denote the fuzzy set of possible values for the Kendall's tau coefficient, defined by applying the extension principle to the function:

$$(x_1, y_1), \ldots, (x_n, y_n) \rightarrow \tau(\lambda(x_1, \ldots, x_n), \lambda(y_1, \ldots, y_n)).$$

Its membership function is thus defined as:

$$\begin{aligned} \mu_{\widetilde{T}}(t) &= \sup_{t=\tau(\lambda(x_1,\ldots,x_n),\lambda(y_1,\ldots,y_n))} \left(\bigwedge_{i=1}^n \mu_{\widetilde{x}_i}(x_i) \right) \wedge \left(\bigwedge_{i=1}^n \mu_{\widetilde{y}_i}(y_i) \right) \\ &= \sup_{t=\tau(L_X,L_Y)} \left(\sup_{L_X=\lambda(x_1,\ldots,x_n)} \bigwedge_{i=1}^n \mu_{\widetilde{x}_i}(x_i) \right) \wedge \left(\sup_{L_Y=\lambda(y_1,\ldots,y_n)} \bigwedge_{i=1}^n \mu_{\widetilde{y}_i}(y_i) \right) \end{aligned}$$

Using Theorem 5, this expression can be simplified as:

$$\mu_{\widetilde{T}}(t) = \sup_{t=\tau(L_X,L_Y)} \mu_{\Lambda(\widetilde{P}_X)}(L_X) \wedge \mu_{\Lambda(\widetilde{P}_Y)}(L_Y), \tag{17}$$

with $\widetilde{P}_X = \pi(\widetilde{x}_1, \ldots, \widetilde{x}_n)$ and $\widetilde{P}_Y = \pi(\widetilde{y}_1, \ldots, \widetilde{y}_n)$. The β-cut of $\widetilde{T}$ is therefore:

$$\widetilde{T}^\beta = \{\tau(L_X, L_Y), L_X \in \Lambda(\widetilde{P}_X)^\beta, L_Y \in \Lambda(\widetilde{P}_Y)^\beta\}.$$

From Theorem 3, this is equal to:

$$\widetilde{T}^\beta = \{\tau(L_X, L_Y), L_X \in \Lambda(\widetilde{P}_X^{(1-\beta)+}), L_Y \in \Lambda(\widetilde{P}_Y^{(1-\beta)+})\}.$$

To facilitate the computations, we propose to approximate $\widetilde{T}$ by the "fuzzy convex hull" of $\mu_{\widetilde{T}}$, i.e., the most precise fuzzy interval $\widetilde{\tau}$ such that $\mu_{\widetilde{\tau}} \geq \mu_{\widetilde{T}}$. The β-cut of $\widetilde{\tau}$, for any $\beta \in (0,1]$ is thus the convex hull of $\widetilde{T}^{\beta}$. We have

$$\widetilde{\tau}^{\beta} = \left[\min_{L_X \in \Lambda(\widetilde{P}_X^{(1-\beta)+}), L_Y \in \Lambda(\widetilde{P}_Y^{(1-\beta)+})} \tau(L_X, L_Y), \right. \\ \left. \max_{L_X \in \Lambda(\widetilde{P}_X^{(1-\beta)+}), L_Y \in \Lambda(\widetilde{P}_Y^{(1-\beta)+})} \tau(L_X, L_Y) \right] \quad (18)$$

From (7), this can be written as:

$$\widetilde{\tau}^{\beta} = \overline{\tau} \left(\widetilde{P}_X^{(1-\beta)+}, \widetilde{P}_Y^{(1-\beta)+}) \right), \quad (19)$$

or, equivalently using Theorem 4:

$$\widetilde{\tau}^{\beta} = \overline{\tau} \left(\pi(\widetilde{x}_1^{\beta}, \ldots, \widetilde{x}_n^{\beta}), \pi(\widetilde{y}_1^{\beta}, \ldots, \widetilde{y}_n^{\beta}) \right). \quad (20)$$

Equations (19) and (20) show that $\widetilde{\tau}$ can be computed β-cutwise in two ways, either from the β-cuts of the $\widetilde{x}_i$'s and the $\widetilde{y}_i$'s, or from the strong $(1-\beta)$-cuts of the induced fuzzy partial orders. This confirms the fact that the fuzzy relations $\widetilde{P}_X$ and $\widetilde{P}_Y$ capture all the ordinal information in the data.

From a computational point of view, $\widetilde{\tau}$ can be approximated by estimating the lower and upper bounds of a few of its β-cuts in (19) using the Monte-Carlo approach described in Section 3.2, i.e., by sampling from $\Lambda(\widetilde{P}_X^{(1-\beta)+})$ and $\Lambda(\widetilde{P}_Y^{(1-\beta)+})$ according to a uniform distribution. An alternative approach might be to sample from the $\widetilde{x}_i^{\beta}$ and $\widetilde{y}_i^{\beta}$ intervals and use (20). The former approach, which consists in sampling from a finite space, is obviously much more efficient computationally than the latter, which deals with a continuous space, as will be shown in Example 7.

EXAMPLE 6 Let us consider the data in Table 5, in which each observed value is a triangular fuzzy number. The data in Tables 1 and 2 are, respectively, the cores and the supports of these fuzzy numbers.

The fuzzy partial order $\widetilde{P}_X$ induced by the $\widetilde{x}_i$'s is the one studied in Example 5 and shown in Table 4. Its fuzzy set of linear extensions is given by (14). The fuzzy partial order $\widetilde{P}_Y$ induced by the $\widetilde{y}_i$'s is shown in Table 6.

By reproducing the same line of reasoning as in Example 5, we can easily see that the fuzzy set of linear extensions of $\widetilde{P}_Y$ is:

$$\Lambda(\widetilde{P}_Y) = \left\{ \frac{(3,1,4,2)}{1}, \frac{(1,3,4,2)}{1/2} \right\}.$$

Table 7 illustrates the computation of $\widetilde{T}$ from (17). Each column in this table shows a linear extension L_X with its membership degree $\mu_{\Lambda(\widetilde{P}_X)}(L_X)$ to $\Lambda(\widetilde{P}_X)$, and each row shows a linear extension L_Y with its membership degree $\mu_{\Lambda(\widetilde{P}_Y)}(L_Y)$ to $\Lambda(\widetilde{P}_Y)$. The corresponding cell in the table contains $\tau(L_X, L_Y)$ and $\mu_{\Lambda(\widetilde{P}_X)}(L_X) \wedge \mu_{\Lambda(\widetilde{P}_Y)}(L_Y)$. We can easily deduce from this table the expression of $\widetilde{T}$:

$$\widetilde{T} = \left\{ \frac{-2/3}{2/3}, \frac{-1/3}{1}, \frac{0}{1/2}, \frac{1/3}{1/3} \right\}.$$

The β-cuts of $\widetilde{T}$ for the different values of $\beta \in (0,1]$ are:

$$\widetilde{T}^\beta = \begin{cases} \{-2/3, -1/3, 0, 1/3\} & \text{if} \quad 0 < \beta \leq 1/3 \\ \{-2/3, -1/3, 0\} & \text{if} \quad 1/3 < \beta \leq 1/2 \\ \{-2/3, -1/3\} & \text{if} \quad 1/2 < \beta \leq 2/3 \\ \{-1/3\} & \text{if} \quad 2/3 < \beta \leq 1. \end{cases}$$

We thus have:

$$\widetilde{\tau}^\beta = \begin{cases} [-2/3, 1/3] & \text{if} \quad 0 < \beta \leq 1/3 \\ [-2/3, 0] & \text{if} \quad 1/3 < \beta \leq 1/2 \\ [-2/3, -1/3] & \text{if} \quad 1/2 < \beta \leq 2/3 \\ \{-1/3\} & \text{if} \quad 2/3 < \beta \leq 1. \end{cases}$$

The membership function of $\widetilde{\tau}$ is, consequently,

$$\mu_{\widetilde{\tau}}(t) = \begin{cases} 0 & \text{if} \quad t \in (-\infty, -2/3) \cup (1/3, +\infty) \\ 2/3 & \text{if} \quad t \in [-2/3, -1/3) \\ 1 & \text{if} \quad t = -1/3 \\ 1/2 & \text{if} \quad t \in (-1/3, 0] \\ 1/3 & \text{if} \quad t \in (0, 1/3]. \end{cases}$$

Significance test

A fuzzy significance probability $\widetilde{p}(\widetilde{\tau})$ (for the test problem described in Section 2.2) can be defined by applying the extension principle to (4). Its β-cut is $\widetilde{p}(\widetilde{\tau})^\beta = [p^-(\widetilde{\tau}^\beta), p^+(\widetilde{\tau}^\beta)]$, where p^- and p^+ are defined by (10) and (11).

Applying the extension principle to (3) or (5) results in the same possibility distribution $\mu_{\widetilde{\varphi}_\alpha}$ on $\{0,1\}$:

$$\begin{aligned} \mu_{\widetilde{\varphi}_\alpha}(1) &= \sup_{p\leq\alpha} \mu_{\widetilde{p}}(p) && (21)\\ \mu_{\widetilde{\varphi}_\alpha}(0) &= \sup_{p>\alpha} \mu_{\widetilde{p}}(p). && (22) \end{aligned}$$

The quantity $\mu_{\widetilde{\varphi}_\alpha}(1)$ may be interpreted as the possibility of the event $p \leq \alpha$, according to the possibility distribution $\widetilde{p}$. It is thus the possibility that the null hypothesis would be rejected, had the realizations of X and Y be precisely observed for the sample under study. Note that this should *not* be interpreted as the possibility that H_0 is false. The possibility distribution $\mu_{\widetilde{\varphi}_\alpha}$ is defined on the set $\{0,1\}$ of decisions, and not on the set of hypotheses. In the case of precise data, $\mu_{\widetilde{\varphi}_\alpha}(1) = 0$ whenever H_0 is rejected, although H_0 can obviously not be claimed to be impossible.

In a similiar fashion, the quantity $\mu_{\widetilde{\varphi}_\alpha}(0)$ may be interpreted as the possibility of the event $p > \alpha$, i.e., the possibility that H_0 would not be rejected, had the data been precisely oberved. Equivalently, it is equal to one minus the necessity of rejection of the null hypothesis.

The result of such a fuzzy test is actually contained in the two numbers $\mu_{\widetilde{\varphi}_\alpha}(1)$ and $\mu_{\widetilde{\varphi}_\alpha}(0)$, which can also be seen as degrees of membership in a fuzzy subset of the set $\{0,1\}$, defining a "fuzzy decision". In the case where a crisp decision is absolutely needed, this fuzzy subset may be defuzzified, which will result in rejecting H_0 whenever the possibility of rejection is greater than the possibility of non-rejection, i.e. whenever $\mu_{\widetilde{\varphi}_\alpha}(1) > \mu_{\widetilde{\varphi}_\alpha}(0)$.

Note that this approach may easily be extended to the case where the significance level α is itself fuzzy. In that case, the possibility and the necessity of the event $p \leq \alpha$ according to the joint possibility distribution of $\widetilde{p}$ and $\widetilde{\alpha}$ are, respectively,

$$\Pi(p \leq \alpha) = \sup_{p\leq\alpha} \mu_{\widetilde{p}}(p) \wedge \mu_{\widetilde{\alpha}}(\alpha) \tag{23}$$

and

$$N(p \leq \alpha) = 1 - \sup_{p>\alpha} \mu_{\widetilde{p}}(p) \wedge \mu_{\widetilde{\alpha}}(\alpha). \tag{24}$$

EXAMPLE 7 Figure 5 shows a fuzzy data set obtained by fuzzifying the data in Example 2. The data consist of $n = 10$ pairs of triangular fuzzy numbers $(\widetilde{x}_i, \widetilde{y}_i)$ $(i = 1, \ldots, 10)$, whose cores, 1/2-cuts and supports are, respectively, the data in Figures 1, 3 and 4. The fuzzy correlation coefficient $\widetilde{\tau}$ is shown in Figure 6, together with the 5% critical values. The membership function of $\widetilde{\tau}$ was approximated from 10 equally spaced β-cuts using a piecewise cubic Hermite interpolation method.

An alternative view is provided by Figure 7, where the membership function of the fuzzy p-value is plotted together with the 5% significance level. We have $\Pi(p \leq 0.05) = \mu_{\widetilde{\varphi}_{0.05}}(1) = 1$ and $\Pi(p > 0.05) = \mu_{\widetilde{\varphi}_{0.05}}(0) \approx 0.37$. It is thus fully possible that the independence hypothesis would be rejected at the 5% significance level if the precise data were available, but there is also a possibility of 0.37 that H_0 would not be rejected.

The convergence of our Monte-Carlo approach, in which linear orders are sampled from $\Lambda(\widetilde{P}_X^{(1-\beta)+})$ and $\Lambda(\widetilde{P}_Y^{(1-\beta)+})$ using the procedure described in Appendix B, was compared to the naive approach that consists in sampling from $\widetilde{x}_i^\beta$ and $\widetilde{y}_i^\beta$ using a uniform distribution. Results are shown in Figure 8, in which the absolute difference $\Delta\widetilde{\tau}^\beta$ between the estimated bounds of $\widetilde{\tau}^\beta$ are plotted as a function of the number of trials, for the two methods and different values of β. These results confirm that sampling from the fuzzy sets of linear extensions of fuzzy partial orders $\widetilde{P}_X$ and $\widetilde{P}_Y$ results in significantly faster convergence.

EXAMPLE 8 To demonstrate the application of our method to real data, let us consider the data set reported in Table 8. This data set consists of values of 5 perceptive attributes related to 8 objects, recorded during a sensory analysis experiment. The assessor was asked to to assess his perception of each attribute for each object on a scale between 0 and 100, in the form of lower and upper bounds, as well as a point estimate. In our approach, each triple is modeled by a triangular fuzzy number $\widetilde{x}_i$, and is interpreted as a possibility distribution related to an unknown value x_i, itself a realization of a random variable X_i. Randomness arises from the selection of objects (assumed to be sampled from a hypothetical population), as well as environmental factors which influence the perception by the assessor. In contrast, fuzziness arises from the limited ability of the assessor to describe his perception using numbers, which is not influenced by random factors.

The fuzzy tau coefficients between each pair of attributes are shown in Figure 9, together with the 5 % critical values. We observe that the attribute pairs (1,2), (1,3), (2,3), (2,5) and (4,5) are certainly independent at the 5 % significance level (the independence assumption would not be rejected at the significance level, whatever the values of the data within the bounds provided by the assessor). The other cases are more ambiguous. However, there is strong evidence of positive correlation for the pair (1,3), and negative correlation for the pair (3,5); in both case, it is completely possible (but not certain) that the independence assumption would be rejected at the 5 % significance level, if the precise attribute values were available.

5 Mann-Whitney-Wilcoxon two-sample test

The purpose of this section is to show that the concepts introduced in this paper can easily be applied to any nonparametric rank-based statistical procedure. The Mann-Whitney-Wilcoxon two-sample test, a well known procedure for comparing two distributions, will be used as an illustration.

5.1 Principle

The Mann-Whitney-Wilcoxon two-sample test is based on the Wilcoxon two-sample rank sum statistic W defined as follows [18]. Let $x_1, \ldots, x_n$ and $y_1, \ldots, y_m$ be two independent samples. Combine the two samples, and order the resulting sample of size $N = n+m$. Let r_i be the rank of x_i in the combined sample. Then $W = \sum_{i=1}^{n} r_i$, the sum of ranks of the x_i's. Under the hypothesis H_0 that the two samples have the same distribution, the mean and variance of W are:

$$E(W) = \frac{n(N+1)}{2}$$

$$Var(W) = \frac{nm(N+1)}{12},$$

and the distribution of W can be considered to be approximately normal whenever n and m are greater than 10. The null hypothesis can then be rejected at the α level (against the alternative hypothesis H_1 that the two samples do not have the same distribution) when

$$\left| W - \frac{n(N+1)}{2} \right| > \Phi^{-1}\left(1 - \frac{\alpha}{2}\right) \sqrt{\frac{nm(N+1)}{12}}. \tag{25}$$

The corresponding p-value $p(W)$ is:

$$p(W) = 2\left[1 - \Phi\left(\left| W - \frac{n(N+1)}{2} \right| \sqrt{\frac{12}{nm(N+1)}}\right)\right]. \tag{26}$$

5.2 Extension to fuzzy data

Let $z_1, \ldots, z_N$ denote the combined sample, and $L = \lambda(z_1, \ldots, z_N)$ the induced linear order. The Wilcoxon statistics depends only on L and may be noted $W(L)$. Let $\widetilde{z}_1, \ldots, \widetilde{z}_N$ be N fuzzy intervals defining fuzzy constraints on the z_i's, et let $\widetilde{P} = \pi(\widetilde{z}_1, \ldots, \widetilde{z}_N)$ be the corresponding fuzzy partial order. The fuzzy Wilcoxon statistic may be defined as the fuzzy interval with β-cut

$$\widetilde{W}^{\beta} = \left[\min_{L \in \Lambda(\widetilde{P}^{(1-\beta)+})} W(L), \max_{L \in \Lambda(\widetilde{P}^{(1-\beta)+})} W(L) \right] \tag{27}$$

for all $\beta \in (0,1]$. As in the case of the Kendall's tau coefficient, the bounds of $\widetilde{W}^{\beta}$ may be approximated by randomly generating linear extensions of $\widetilde{P}^{(1-\beta)+}$ using the algorithm described in Appendix B.

A fuzzy p-value $\widetilde{p}(\widetilde{W})$ can be defined by applying the extension principle to (26). Let U^- and U^+ denote the lower and upper bounds of $W(L) - n(N+1)/2$ for all $L \in \Lambda(\widetilde{P}^{(1-\beta)+})$. The β-cut $\widetilde{p}(\widetilde{W})^{\beta}$ of $\widetilde{p}(\widetilde{W})$ is the closed interval $[p^-(\widetilde{W}^{\beta}), p^+(\widetilde{W}^{\beta})]$ with

$$p^-(\widetilde{W}^{\beta}) = 2\left[1 - \Phi^{-1}\left(\max(U^+, -U^-)\sqrt{\frac{12}{nm(N+1)}}\right)\right] \tag{28}$$

$$p^+(\widetilde{W}^{\beta}) = 2\left[1 - \Phi^{-1}\left(\max(0, U^-, -U^+)\sqrt{\frac{12}{nm(N+1)}}\right)\right]. \tag{29}$$

The possibility of rejection and non-rejection of H_0 are then defined by (21) and (22), respectively.

EXAMPLE 9 Figure 11 shows the fuzzy Wilcoxon statistic and associated 5% critical values for the trapezoidal fuzzy data displayed in Figure 10. The corresponding p-value is shown in Figure 12. As we can see, the null hypothesis is, in that case, both rejected at the 5% significance level with possibility 1, and accepted with possibility 0.45.

6 Conclusions

We have shown that nonparametric statistical procedures based on orderings of observations can be easily extended to fuzzy data, using the concept of fuzzy partial order induced by fuzzy numbers. This fuzzy order can be viewed as defining a fuzzy set of linear orders compatible with the observations, which induces possibilistic constraints on the statistic of interest, as well as on the p-value of an associated signifiance test.

The approach has been demonstrated using two special cases: the Kendall's tau coefficient and the Wilcoxon two-sample rank test. It can be applied without any difficulty to other rank-based nonparametric procedures, such as the Wilcoxon one-sample sign test, the Kruskal-Wallis test (a generalization of the Wilcoxon two-sample test for more than two populations), Kendall's coefficient of concordance between several rankings, etc.

In this paper, the variables under study have been assumed to be continuous, so that ties could not occur. In terms of ordering relation, a consequence of this assumption is that only linear orders have been considered. A generalization to weak orders, allowing to deal effectively with the problem of ties, is left for further study.

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A Proof of Theorem 1

Our proof of Theorem 1 uses the concept of maximal antichains of an interval order. We first introduce this notion and important related result, and then proceed with the proof.

A subset M of U is an antichain of P if $(u, v) \notin P$ and $(v, u) \notin P$ for all u, v in M. A maximal antichain is an antichain not properly contained in any other antichain. Let $MA(P)$ denote the set of maximal antichains of P. The following lemma, proved in [9] and cited in [28], states that there exists a natural linear ordering on $MA(P)$.

Lemma 1 *Let $MA(P)$ be the set of maximal antichains of an interval order P on U. Let $<$ denote the relation on $MA(P)$ defined by*

$$M < M' \Leftrightarrow (M \setminus M')P(M' \setminus M), \quad \forall M, M' \in MA(P),$$

where the relation P is extended to subsets of U. Then $<$ is a linear order.

Example 10 Let us consider the four intervals $\overline{x}_i$, $i = 1, \ldots, 4$ shown in Figure 13. These intervals are assumed to be related to four objects u_i, $i = 1, \ldots, 4$. The induced partial order P has three maximal antichains $M_1 = \{u_1, u_2\}$, $M_2 = \{u_2, u_3\}$ and $M_3 = \{u_3, u_4\}$. We have $M_1 \setminus M_2 = \{u_1\}$, $M_2 \setminus M_1 = \{u_3\}$ and $u_1 P u_3$. Hence, $M_1 < M_2$. Similarly, $M_2 \setminus M_3 = \{u_2\}$, $M_3 \setminus M_2 = \{u_4\}$ and $u_2 P u_4$, therefore $M_2 < M_3$. The linear order on $MA(P)$ is, consequently, $M_1 < M_2 < M_3$.

We now proceed with the proof of Theorem 1.

Part 1: $\lambda(\overline{x}_1 \times \ldots \times \overline{x}_n) \subseteq \Lambda(P)$. Let $L = \lambda(x_1, \ldots, x_n) \in \lambda(\overline{x}_1 \times \ldots \times \overline{x}_n)$ with $x_i \neq x_j$, $\forall i \neq j$. For all $(u_i, u_j) \in P$, we have:

$$\sup \overline{x}_i \leq \inf \overline{x}_j \Rightarrow x_i < x_j \Rightarrow (u_i, u_j) \in L.$$

Hence, L is a linear extension of P. We thus have $\lambda(\overline{x}_1 \times \ldots \times \overline{x}_n) \subseteq \Lambda(P)$.

Part 2: $\Lambda(P) \subseteq \lambda(\overline{x}_1 \times \ldots \times \overline{x}_n)$. A constructive proof may be deduced from Lemma 1. Let L be a linear extension of P. Without loss of generality, let us assume that we have $u_i L u_j$ if and only if $i < j$. We want to define n numbers $x_1 < \ldots < x_n$ such that $x_i \in \overline{x}_i$, $i = 1, \ldots, n$. Let $M_1 < \ldots < M_q$ denote the maximal antichains of P, with $M_k = \{u_i \in U \mid i \in I_k\}$. Consider the following algorithm:

- Begin
- $\ell \leftarrow \{1, \ldots, n\}$
- For $k = 1 : q$,
  - $\ell' \leftarrow \ell \cap I_k$
  - choose card(ℓ') distinct points x_i, $i \in \ell'$, in $\bigcap_{i \in I_k} \overline{x}_i$ such that $i < j \Rightarrow x_i < x_j$ for all i, j in ℓ'.
  - $\ell \leftarrow \ell \setminus \ell'$.
  - EndFor
- End

By construction, the above procedure generates n distinct points $x_1 < \ldots < x_n$ such that $x_i \in \overline{x}_i$, $i = 1, \ldots, n$. Hence $L \in \lambda(\overline{x}_1 \times \ldots \times \overline{x}_n)$, which completes the proof. □

EXAMPLE 11 Let us come back to Example 10 and Figure 13. At step one ($k = 1$), we have $\ell = \{1, 2, 3, 4\}$ and $\ell' = \ell \cap I_1 = \{1, 2\}$. We thus pick two numbers x_1 and x_2, with $x_1 < x_2$, in $\overline{x}_1 \cap \overline{x}_2$. At the second step ($k = 2$), we have $\ell = \{3, 4\}$ and $\ell' = \ell \cap I_2 = \{3\}$. We pick one number x_3 in $\overline{x}_2 \cap \overline{x}_3$. Lastly, at step three ($k = 3$), we have $\ell = \{4\}$ and $\ell' = \ell \cap I_3 = \{4\}$. We thus pick one number x_4 in $\overline{x}_3 \cap \overline{x}_4$. By construction, we have $x_1 < x_2 < x_3 < x_4$.

B Random generation of linear extensions

A good summary of Bubley and Dyer's algorithm [5] and its properties can be found in [14]. This summary is essentially reproduced here.

Let P be a partial order on a set U of n elements. We encode each linear order on U by a permutation σ of $\mathbb{N}_n = \{1, \ldots, n\}$. The algorithm constructs a Markov chain $\mathcal{M}_f = (\sigma_t)_{t \geq 0}$ as follows. At any time step t, toss a fair coin. If the coin lands head, then let $\sigma_{t+1} = \sigma_t$. If the coin lands tail, choose an index $i \in \mathbb{N}_{n-1}$ according to a fixed probability distribution f. If the permutation σ obtained from σ_t by switching the elements i and $i+1$ of σ_t is also a linear extension of P, then let $\sigma_{t+1} = \sigma$. Otherwise, let $\sigma_{t+1} = \sigma_t$.

It can be shown that $\mathcal{M}_f$ si ergodic with uniform stationary distribution. The running time required to obtain a certain precision ε (defined by a measure of distance to the uniform distribution) depends on f. If f is uniform, then this time (called the mixing time) is $O(n^5 \log n + n^4 \log \varepsilon^{-1})$. Bubley and Dyer showed that the mixing time can be reduced to $O(n^3 \log n\varepsilon^{-1})$ by defining f as $f(i) = i(n - i)/K$ with $K = (n^3 - n)/6$.

Tables

Table 1: Data of Example 1.

i	1	2	3	4
x_i	11.5	9.5	15.5	16.5
y_i	10.0	15.5	9.0	13.5

Table 2: Data of Example 3.

i	1	2	3	4
$\overline{x}_i$	[10,13]	[8,11]	[14,17]	[15,18]
$\overline{y}_i$	[9,11]	[15,16]	[8,10]	[13,14]

Table 3: Kendall's correlation coefficients between the linear extensions of P_X and P_Y in Example 3.

τ	$(1,2,3,4)$	$(2,1,3,4)$	$(1,2,4,3)$	$(2,1,4,3)$
$(1,3,4,2)$	1/3	0	0	-1/3
$(3,1,4,2)$	0	-1/3	-1/3	-2/3

Table 4: Fuzzy partial order of Example 5.

$\widetilde{P}$ $\nearrow$	u_1	u_2	u_3	u_4
u_1	0	0	1	1
u_2	2/3	0	1	1
u_3	0	0	0	1/3
u_4	0	0	0	0

Table 5: Data of Example 6.

i	1	2	3	4
$\widetilde{x}_i$	(10,11.5,13)	(8,9.5,11)	(14,15.5,17)	(15,16.5,18)
$\widetilde{y}_i$	(9,10,11)	(15,15.5,16)	(8,9,10)	(13,13.5,14)

Table 6: Fuzzy partial order $\widetilde{P}_Y$ of Example 6.

$\widetilde{P}_Y$ $\nearrow$	u_1	u_2	u_3	u_4
u_1	0	1	0	1
u_2	0	0	0	0
u_3	1/2	1	0	1
u_4	0	1	0	0

Table 7: Computation of the fuzzy Kendall's correlation coefficients between the linear extensions of $\widetilde{P}_X$ and $\widetilde{P}_Y$ in Example 6.

τ	$\frac{(1,2,3,4)}{1/3}$	$\frac{(2,1,3,4)}{1}$	$\frac{(1,2,4,3)}{1/3}$	$\frac{(2,1,4,3)}{2/3}$
$\frac{(1,3,4,2)}{1/2}$	$\frac{1/3}{1/3}$	$\frac{0}{1/2}$	$\frac{0}{1/3}$	$\frac{-1/3}{1/2}$
$\frac{(3,1,4,2)}{1}$	$\frac{0}{1/3}$	$\frac{-1/3}{1}$	$\frac{-1/3}{1/3}$	$\frac{-2/3}{2/3}$

Table 8: Triangular fuzzy data of Example 8. Each column contains the fuzzy values of an attribute reported for 8 objects.

Attributes				
1	2	3	4	5
(6.8,11.2,13.0)	(92.9,94.6,97.4)	(63.6,66.7,76.1)	(87.8,90.7,94.6)	(47.8,60.3,64.2)
(21.4,25.7,34.1)	(80.4,87.0,92.7)	(2.5,5.6,8.3)	(96.2,96.2,100)	(97.6,99.9,99.9)
(75.7,81.6,84.5)	(70.4,78.2,80.2)	(4.8,4.8,9.7)	(38.8,45.2,53.0)	(80.6,86.0,89.7)
(96.2,96.2,99.9)	(0,6.4,13.2)	(77.6,84.7,89.9)	(70.2,75.5,81.7)	(0,5.2,9.9)
(88.6,93.8,97.2)	(83.5,90.5,95.2)	(6.2,9.9,14.8)	(10.3,19.2,25.1)	(77.4,85.6,92.5)
(95.0,95.0,100)	(0.1,0.1,5.8)	(79.2,86.4,93.1)	(12.2,12.2,20.4)	(0.1,0.1,9.1)
(0.1,0.1,0.1)	(68.5,91.5,91.5)	(0.1,0.1,0.1)	(0.1,0.1,0.1)	(80.6,91.1,95.0)
(97.4,99.9,99.9)	(100,100,100)	(100,100,100)	(95.4,99.9,99.9)	(30.0,41.5,58.7)

Figures

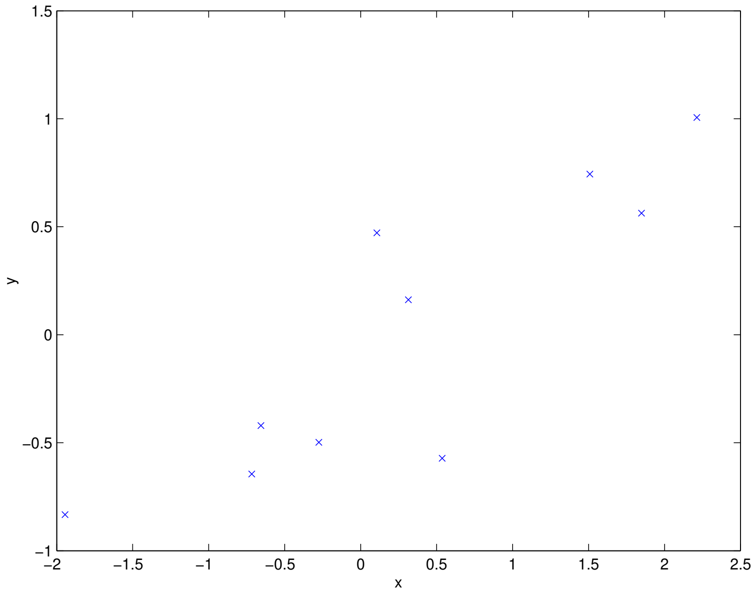

Figure 1: Data set of Example 2.

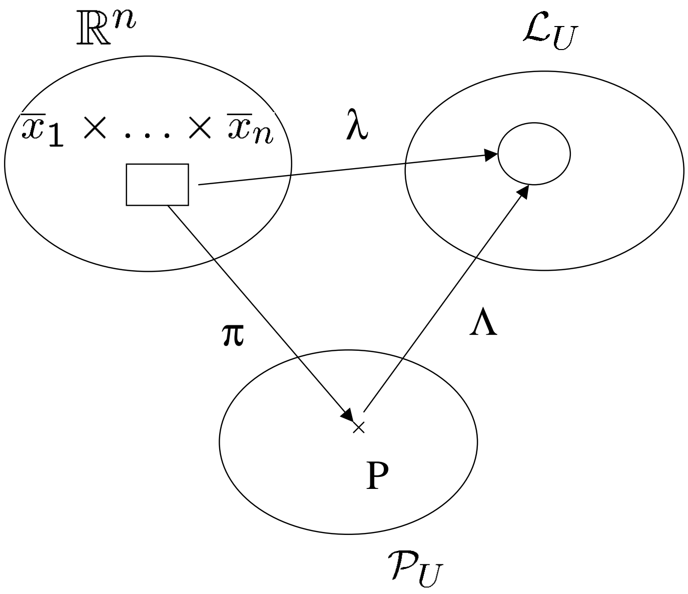

Figure 2: Relationship between mappings λ, π and Λ.

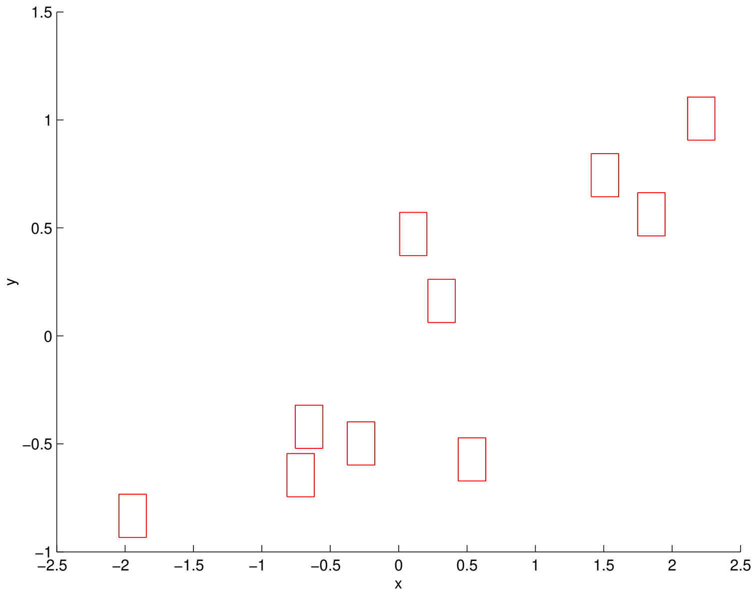

Figure 3: First data set of Example 4.

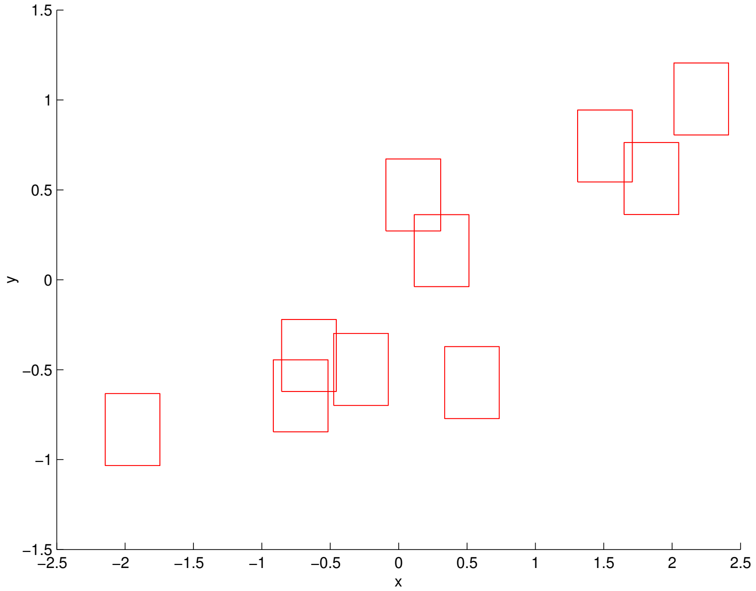

Figure 4: Second data set of Example 4.

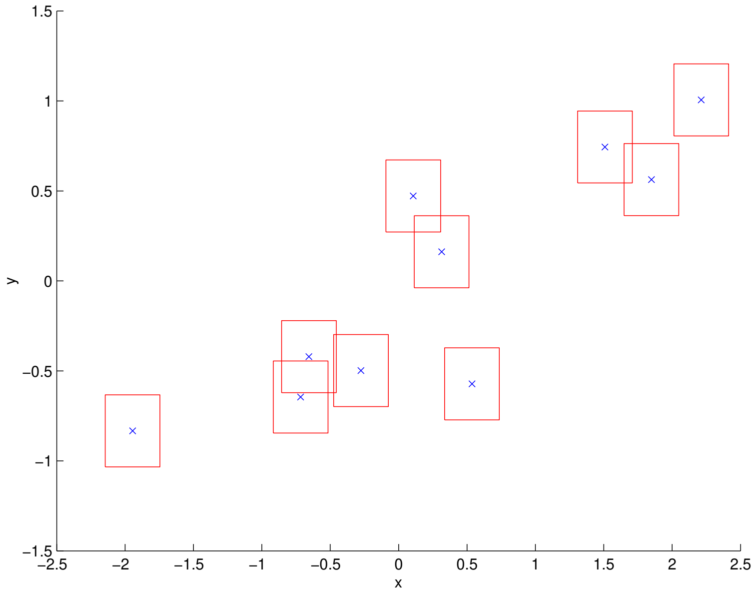

Figure 5: Fuzzy data set of Example 7.

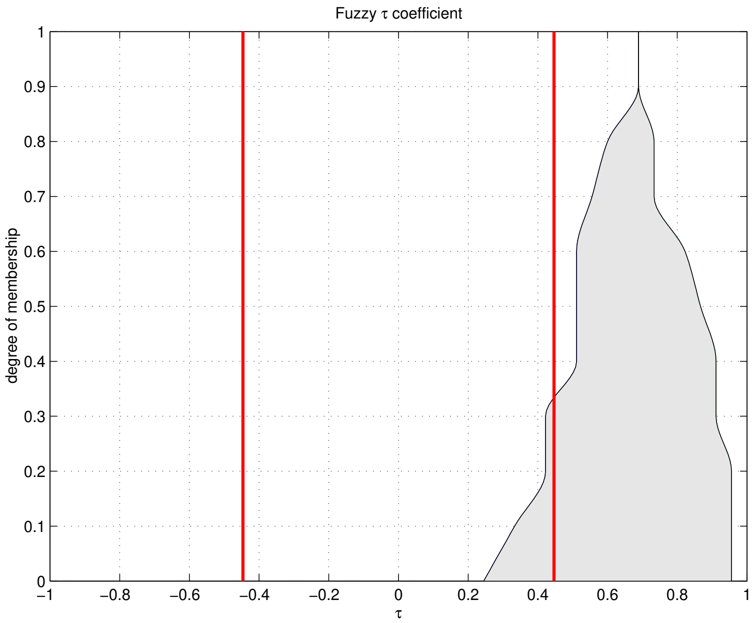

Figure 6: Fuzzy τ coefficient (Example 7). The vertical lines define the critical region of the 5% significance test (the null hypothesis is rejected if τ lies outside these thresholds).

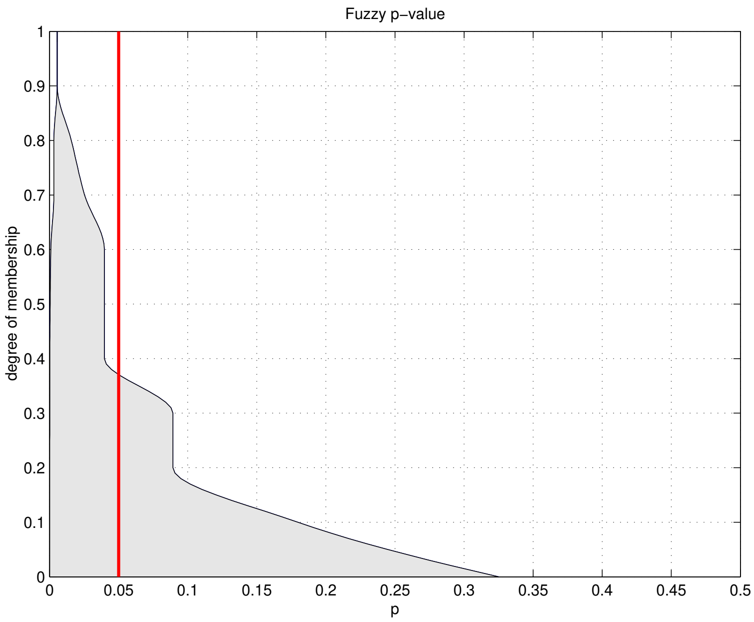

Figure 7: Fuzzy p-value (Example 7). The vertical line corresponds to the 5% significance level.

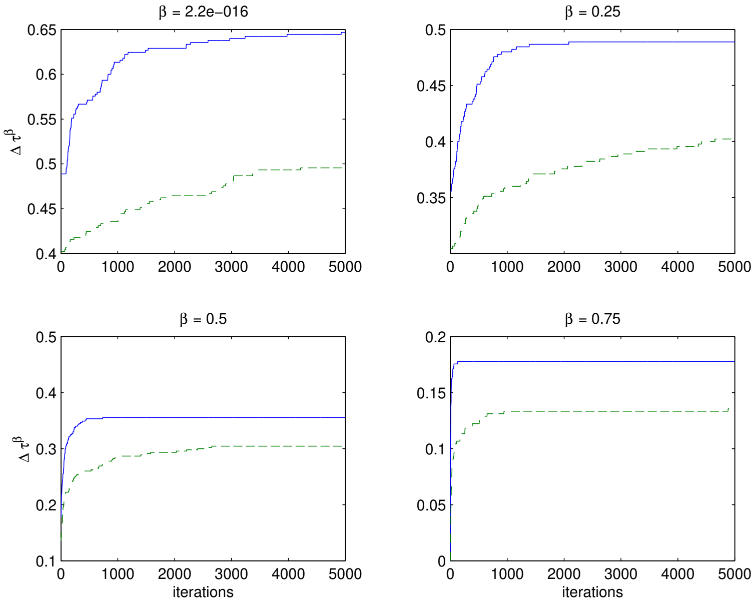

Figure 8: Estimated length of the $\widetilde{\tau}^{\beta}$ interval for the data of Example 7, as a function of the number of iterations in the Monte-Carlo simulation, using the method proposed in this paper (solid lines) and by samping from the $\widetilde{x}_i^{\beta}$ and $\widetilde{y}_i^{\beta}$ intervals (dotted lines), for four different values of β.

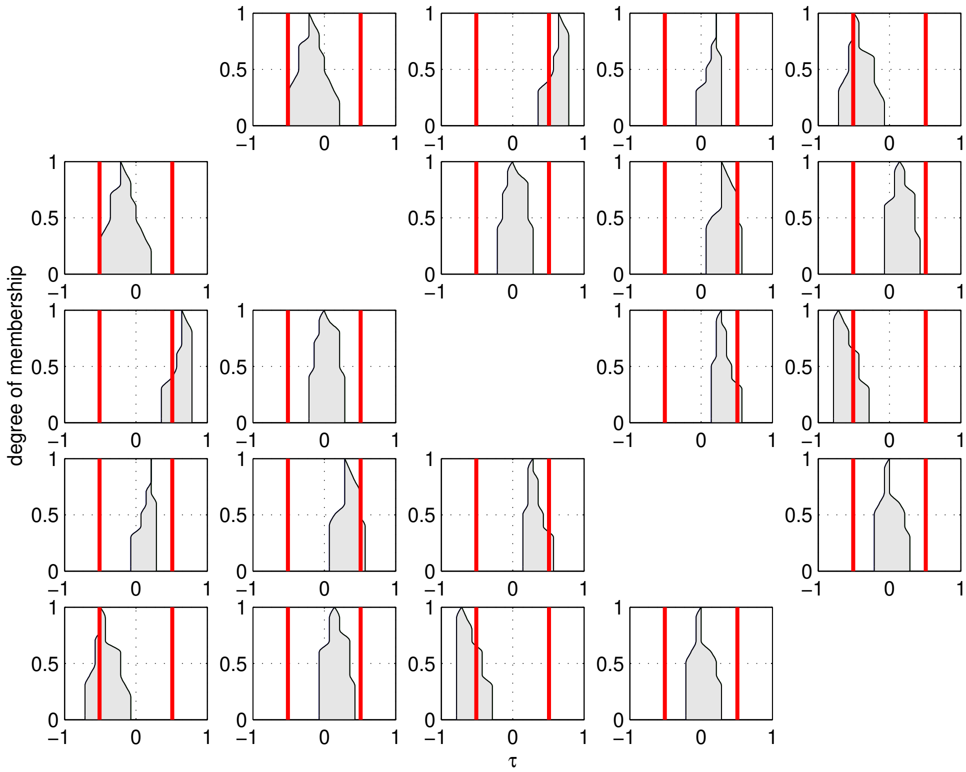

Figure 9: Fuzzy tau coefficients between each pair of attributes for the data of Example 8, together with 5 % critical values.

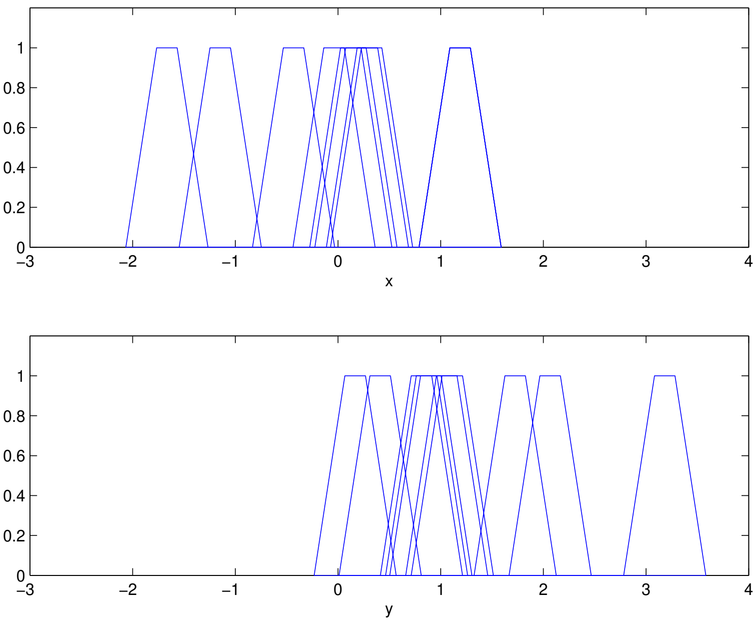

Figure 10: Fuzzy data set of Example 9.

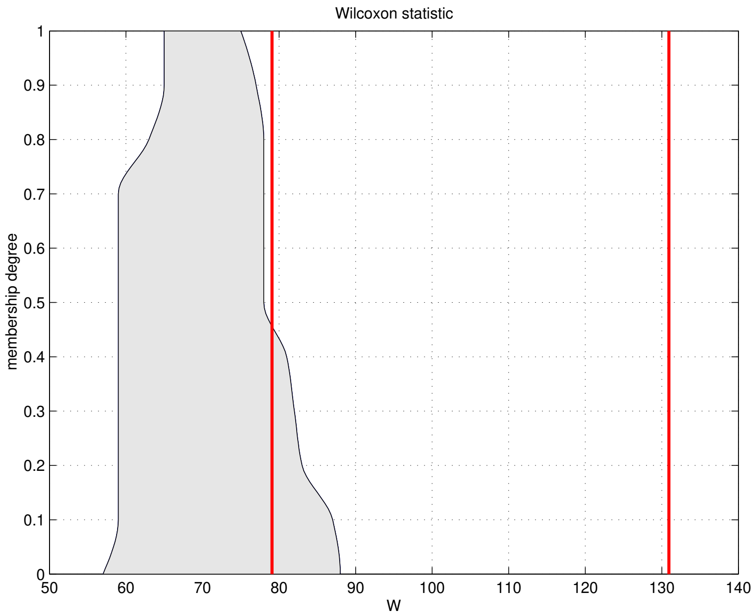

Figure 11: Fuzzy two-sample rank Wilcoxon statistic (Example 9). The vertical lines define the critical region of the 5% significance test (the null hypothesis is rejected if τ lies outside these thresholds).

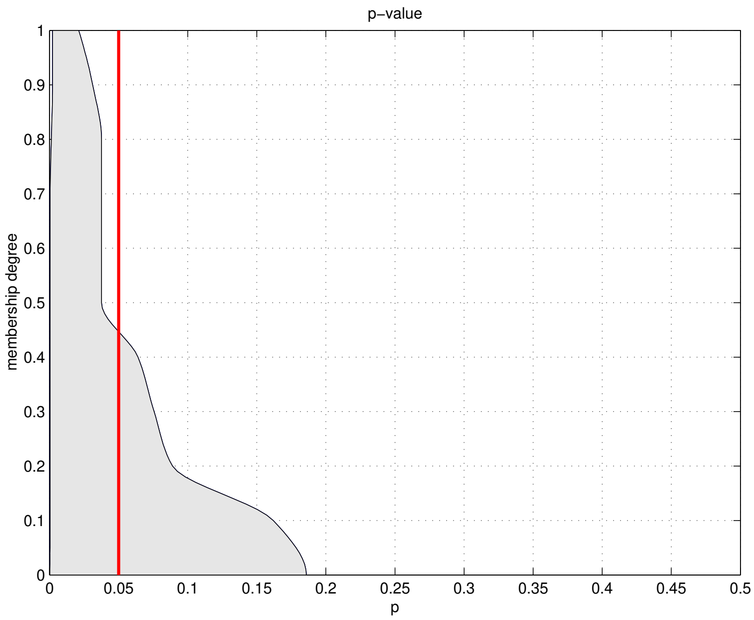

Figure 12: Fuzzy p-value (Example 9). The vertical line corresponds to the 5% significance level.

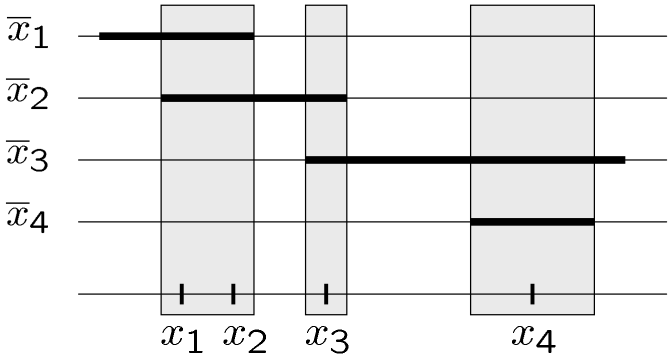

Figure 13: Data of Examples 10 and 11.