Modeling vague beliefs using fuzzy-valued belief structures[1]

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Abstract

This paper presents a rational approach to the representation and manipulation of imprecise degrees of belief in the framework of evidence theory. We adopt as a starting point the non probabilistic interpretation of belief functions provided by Smets' Transferable Belief Model, as well as previous generalizations of evidence theory allowing to deal with fuzzy propositions. We then introduce the concepts of interval-valued and fuzzy-valued belief structures, defined, respectively, as crisp and fuzzy sets of belief structures verifying hard or elastic constraints. We then proceed with a generalization of various concepts of Dempster-Shafer theory including those of belief and plausibility functions, combination rules and normalization procedures. Most calculations implied by the manipulation of these concepts are based on simple forms of linear programming problems for which analytical solutions exist, making the whole scheme computationally tractable. We discuss the application of this framework in the areas of decision making under uncertainty and classification of fuzzy data.

Keywords: Evidence Theory, Belief Functions, Fuzzy Numbers, Uncertainty Representation, Approximate Reasoning, Decision Making, Pattern Recognition.

1 Introduction

One of the main objections against the adoption of Probability theory as a universal model of uncertainty is its "unreasonable requirement for precision" [28]. As convincingly argued by Walley [28], imprecision can hardly be avoided in the representation of beliefs because of *indeterminacy* of actual or ideal beliefs, and *model incompleteness*. Indeterminacy, which may be defined as an absence of preference between different alternatives, is mainly due to lack of information. The inability of the Bayesian model of subjective probabilities to represent states of total ignorance has been pointed out by many authors [20, 22, 28], and has been one of the main incentives for studying non additive uncertainty measures such as belief functions [20]. Such measures may be obtained by distributing fractions of a unit "mass of belief" to certain subsets of a possibility space Ω, the belief in $A \subseteq \Omega$ being then defined as the mass assigned to subsets of A. In this framework, total ignorance is easily represented by the assignment of the total mass of belief to the whole possibility space, while probabilities are recovered as a special case when all belief masses are allocated to singletons.

The second main source of imprecision in belief representation is model incompleteness, which essentially stems from difficulties in analyzing evidence and eliciting numerical degrees of belief. Even though such numbers may in principle exist, their practical determination may be too difficult, too costly, or unnecessary [28]. Within the probabilistic framework, such arguments have lead to theories of imprecise probabilities in which an unknown probability measure is only partially specified by probability intervals assigned to certain propositions. The replacement of hard constraints by elastic ones leads in turn to the notion of fuzzy or linguistic probability, which was introduced by Zadeh [36] and explored by several authors [29, 13, 12].

The approach discussed in this paper attempts to combine the two previous ideas, by defining a theoretical and practical framework in which *imprecise belief masses may be assigned to imprecise propositions*. This idea was already present in a paper by Zadeh [38, p. 15], who related some concepts of Evidence theory to the notions of expected possibility and expected certainty, the expectation being taken with respect to crisp or fuzzy probabilities. Despite its generality and mathematical elegance, this idea seems to have been, so to say, overlooked for many years, which may be due to at least two reasons. The first one is related to the wide acceptance of a particular interpretation of belief functions in terms of lower and upper bounds of a family of probability functions, as originally proposed by Dempster [1]. Under this interpretation, a belief function is viewed as defining probability intervals, and there does not seem to be any reason for allowing additional imprecision. However, as shown by Smets [24], the Dempster model is but one interpretation of the theory of belief functions, and is open to certain criticisms regarding the conditioning and evidence combination mechanisms. These inconsistencies are avoided in the Transferable Belief Model (TBM) introduced by Smets [22, 27], in which belief functions are seen as alternatives to probability functions for *pointwise* representation of the beliefs held by a rational agent. This view is accepted in this paper, which presents what may be seen as an extension of the TBM allowing imprecision in the specification of degrees of belief.

The second factor which seems to have prevented the widespread use of imprecise degrees of belief in evidence theory is the general feeling that the manipulation of such numbers would be intractable. For example, Lamata and Moral [17] attempted to define rules of calculus with linguistic probabilities and beliefs, but they felt that they had to "sacrifice mathematical purity to get an efficient and simple calculus". This lead them to defining heuristic rules and "ad hoc" methods which cannot be widely accepted for lack of clear theoretical justification.

This aim of this paper is to present and discuss a principled approach allowing the representation and manipulation of imprecise degrees of belief in the framework of evidence theory. We adopt as a starting point the TBM interpretation of belief functions, as well as previous generalizations of Evidence Theory allowing to deal with fuzzy propositions. We then introduce the concepts of interval-valued and fuzzy-valued belief structures, and proceed with a generalization of various concepts of Dempster-Shafer theory including those of belief and plausibility functions, combination rules and normalization procedures. We show that, for that matter, mathematical rigor is compatible with computational tractability, and that imprecise degrees of belief may be used efficiently in certain problems such as fuzzy data analysis.

The organization of the paper is as follows. Section 2 presents the necessary background concerning the theory of belief functions and previous "fuzzifications" of this theory. The concept of interval-valued belief structure (IBS) is then introduced in Section 3, where various issues pertaining to the use and manipulation of such mathematical objects are discussed. A further generalization is carried out in Section 4, in which we define a fuzzy-valued belief structure (FBS) as a fuzzy set of belief structures defined by fuzzy constraints. The manipulation of such objects is shown to be quite easy using the techniques introduced in the previous section. Finally, Section 5 discusses two general applications of this framework in the areas of decision making and pattern classification.

2 Background

2.1 Basic definitions

Let Ω denote a finite set of possible answers to a certain question, and let Y be a variable describing the correct (but unknown) answer. We wish to propose a coherent description of the beliefs held by a rational agent (denoted by "You") regarding the value taken by Y, given a certain body of evidence. In the TBM, it is assumed that Your state of belief at a given time may be represented by the distribution of a unit "mass of belief" to certain crisp subsets of Ω [22, 27]. The fraction of the mass (also called a belief number) assigned to $A \subseteq \Omega$ represents the part of Your belief that supports A (i.e., the hypothesis that $Y \in A$), without supporting any more specific subset, because of lack of sufficient information. Mathematically, such an allocation of belief numbers to subsets of Ω defines a function m from 2^Ω to $[0, 1]$, verifying:

$$\sum_{A \subseteq \Omega} m(A) = 1, \tag{1}$$

which is called a *belief structure* (BS), or a *basic belief assignment*. The subsets A of Ω such that $m(A) > 0$ are called the *focal elements* of m. A belief structure m such that $m(\emptyset) = 0$ is said to be normal. The normality condition corresponds to the certainty that Y lies in Ω. If Ω is assumed to be exhaustive (closed-world assumption), then such a condition should be imposed (this is the situation initially considered by Shafer [20]). In the most general case, however, the allocation of a positive belief number to the empty set may be interpreted as quantifying Your belief that $Y \notin \Omega$ [22].

Assuming Your state of belief to be represented by m, Your *total belief* in the proposition $Y \in A$ is represented by a number, called the *credibility* of A, and defined as:

$$\mathrm{bel}_m(A) \triangleq \sum_{\emptyset \neq B \subseteq A} m(B). \tag{2}$$

Such a function $\mathrm{bel}_m : 2^\Omega \mapsto [0,1]$, called a *belief function*, may be shown to have the property of complete monotonicity [20]. Smets recently proposed a set of axioms justifying the use of belief functions for representing degrees of belief [26]. Closely related to the notion of belief function is that of *plausibility function*, defined for each $A \subseteq \Omega$ as:

$$\begin{aligned} \mathrm{pl}_m(A) &\triangleq \sum_{B \cap A \neq \emptyset} m(B) && (3) \\ &= \mathrm{bel}_m(\Omega) - \mathrm{bel}_m(\overline{A}) && (4) \end{aligned}$$

where $\overline{A}$ denotes the complement of A. The quantity $\mathrm{pl}_m(A)$ receives a natural interpretation as the amount of potential support that could be given to A, if further evidence became available. It should be noted that the three functions m, bel_m and pl_m are in one-to-one correspondence, and therefore constitute three equivalent representations of the same information (Your state of belief).

Let us now assume that You collect two distinct pieces of evidence coming from two different sources. Let m_1 and m_2 denote the BSs induced by each of these pieces of evidence considered individually. Then, m_1 and m_2 may be combined in several different ways, depending on Your knowledge concerning the reliability of the two sources. If You know that they both are reliable, then You may combine m_1 and m_2 conjunctively by defining a new BS $m_1 \cap m_2$ as:

$$(m_1 \cap m_2)(A) \triangleq \sum_{B \cap C = A} m_1(B) m_2(C) \quad \forall A \subseteq \Omega. \tag{5}$$

On the other hand, if You only know that *at least* one of the two sources is reliable, then the corresponding BSs should rather be combined in a disjunctive fashion [9, 23], leading to:

$$(m_1 \cup m_2)(A) \triangleq \sum_{B \cup C = A} m_1(B) m_2(C) \quad \forall A \subseteq \Omega. \tag{6}$$

Note that the conjunctive sum as described by (5) may produce a subnormal BS (i.e., it is possible to have $(m_1 \cap m_2)(\emptyset) > 0$). Under the closed-world assumption, some kind of normalization thus has to be performed. The *Dempster normalization procedure* converts a subnormal BS m into a normal BS m^* defined by:

$$m^*(A) \triangleq \begin{cases} \dfrac{m(A)}{1 - m(\emptyset)} & \text{if } A \neq \emptyset \\ 0 & \text{if } A = \emptyset. \end{cases} \tag{7}$$

The conjunctive sum operation with Dempster normalization is the *orthogonal sum* operation (also called Demspter's rule of combination) initially studied by Shafer [20]. To avoid some counterintuitive effects of this rule in case of conflicting evidence, other normalization procedures have been proposed. In particular, Yager [32] proposed to convert a subnormal BS m into a normal one m° by transferring the mass $m(\emptyset)$ to the whole frame of discernment, leading to the following definition:

$$m^\circ(A) \triangleq \begin{cases} m(A) & \text{if } A \in 2^\Omega \setminus \{\emptyset, \Omega\} \\ m(\Omega) + m(\emptyset) & \text{if } A = \Omega \\ 0 & \text{if } A = \emptyset. \end{cases} \quad (8)$$

However, the association of the conjunctive sum with this normalization procedure (hereafter referred to as *Yager normalization*) defines an operation on BSs which is no longer associative.

Another important concept in the TBM is that of *pignistic transformation*. Assume that Your state of belief is described by a BS m, and You have to select an action among a set $\mathcal{A}$. The consequences of these actions only depend on the value of Y, in such a way that, if You choose action α_i while $Y = \omega_j$, You surely gain a reward whose utility is measured by some number $u_{i,j}$. What should be Your decision? Starting from simple rationality requirements, Smets [27] showed that the solution resides in the transformation of m into a *pignistic* probability function, defined for all $A \subseteq \Omega$ as:

$$\mathrm{BetP}(A) \triangleq \sum_{B \subseteq \Omega, B \neq \emptyset} m^*(B) \frac{|A \cap B|}{|B|}, \quad (9)$$

where m^* is the *normalized* form of m according to the Dempster procedure (7).

The selected action is then the one entailing the maximum expected utility, relative to the pignistic probability function. The pignistic transformation thus provides a bridge between a *"credal" level*, where beliefs are represented by belief functions, and a *decision level* where decision are made according to an additive measure.

As argued by Smets [25], a distinctive advantage of the TBM as compared to the classical Bayesian approach based on probability measures resides in its ability to represent every state of partial belief, from absolute certainty up to the state of total ignorance (described by the vacuous belief structure verifying $m(\Omega) = 1$). However, the TBM in its standard form does not allow You to assign degrees of belief to ambiguous propositions such as expressed in many verbal statements. For example, assume that You want to forecast the temperature of the next day. Let $\Omega = \{-10, \ldots, 40\}$ be Your frame of discernment (temperatures being measured in $°C$). You are told by expert E_1 that tomorrow's temperature will be *very high*, whereas another expert E_2 asserts that it will be *medium*. Knowing that You have degrees of confidence of 0.5 in expert E_1 and of 0.8 in expert E_2, what is Your belief that tomorrow's temperature will be *high* ? A possible approach to answer such a question is to extend the theory in order to allow degrees of belief to be assigned to fuzzy subsets of the possibility space. Such extensions have been proposed by several authors and are reviewed in the next section.

2.2 Extension to fuzzy propositions

The idea of extending the concepts of evidence theory to fuzzy sets was first proposed by Zadeh [38], in relation to his work on information granularity and Possibility theory. The approach described in [38] is based on a generalization of the Dempster's model, in which belief functions are introduced as lower probabilities induced by a multivalued mapping [1]. In Zadeh's model, a body of evidence is represented as a probability distribution P_X of a random variable X taking values in an underlying space $U = \{1, \ldots, n\}$, and a possibility distribution $\Pi_{(Y|X)}$ of Y given X. For a given value i of X, the conditional possibility distribution $\Pi_{(Y|X)}$ defines a normal fuzzy subset F_i of Ω such that

$$\Pi_{(Y|X=i)} = F_i. \tag{10}$$

The probability distribution of X induces on Ω a belief structure m whose focal elements are the fuzzy sets F_i, $1 \leq i \leq n$ (this is called a *fuzzy belief structure* by Yager [30]), and

$$m(F_i) = P_X(i) \quad \forall i \in U. \tag{11}$$

For any fuzzy subset A of Ω, let us denote $\Pi(A|F_i)$ the conditional possibility measure of A given that Y is F_i. This quantity is equal to:

$$\begin{aligned} \Pi(A|F_i) &= \mathrm{Poss}(Y \text{ is } A|X = i) && (12)\\ &= \max_{\omega\in\Omega} \mu_A(\omega) \wedge \mu_{F_i}(\omega), && (13) \end{aligned}$$

where μ_A and μ_{F_i} denote the membership functions of A and F_i, respectively, and $\wedge$ denotes the minimum operator. The uncertainty on X being described by a probability measure, it is then natural to consider the expectation of $\mathrm{Poss}(Y \text{ is } A|X)$:

$$\begin{aligned} \mathbb{E}_X[\mathrm{Poss}(Y \text{ is } A|X)] &= \sum_{i=1}^{n} P_X(i)\mathrm{Poss}(Y \text{ is } A|X = i) && (14)\\ &= \sum_{i=1}^{n} m(F_i)\Pi(A|F_i) && (15) \end{aligned}$$

If both A and the F_i are crisp, then $\mathbb{E}_X[\mathrm{Poss}(Y \text{ is } A|X)]$ is equal to the plausibility of A, which allows to propose (15) as a general definition for the plausibility of a fuzzy set A:

$$\mathrm{pl}_m(A) \triangleq \sum_{i=1}^{n} m(F_i)\Pi(A|F_i). \tag{16}$$

When the F_i are crisp but A is fuzzy, then (15) may be written in the following alternative form:

$$\mathrm{pl}_m(A) = \sum_{i=1}^{n} m(F_i) \max_{\omega\in F_i} \mu_A(\omega), \tag{17}$$

which is exactly the upper expectation of the membership function of A with respect to m [21]. In that sense, this definition of the plausibility of a fuzzy event may be seen as generalizing Zadeh's definition of the probability of a fuzzy event proposed in [35].

Similarly, the conditional necessity of the proposition Y is A given that Y is F_i is given by:

$$\begin{aligned} N(A|F_i) &\triangleq \mathrm{Nec}(Y \text{ is } A|X=i) && (18)\\ &= 1-\mathrm{Poss}(Y \text{ is not } A|X=i) && (19)\\ &= \min_{\omega\in\Omega} \mu_A(\omega) \vee \mu_{\overline{F}_i}(\omega), && (20)\end{aligned}$$

where $\vee$ denotes the maximum operator. The expected necessity of A is then defined as

$$\begin{aligned} \mathbb{E}_X[\mathrm{Nec}(Y \text{ is } A|X)] &= \sum_{i=1}^{n} P_X(i)\mathrm{Nec}(Y \text{ is } A|X=i) && (21)\\ &= \sum_{i=1}^{n} m(F_i)N(A|F_i), && (22)\end{aligned}$$

which, when A and the F_i are crisp (and provided $F_i \neq \emptyset, \forall i$), reduces to the credibility of A, and can therefore be proposed as a definition of the credibility of a fuzzy event:

$$\mathrm{bel}_m(A) \triangleq \sum_{i=1}^{n} m(F_i)N(A|F_i). \tag{23}$$

When the F_i are crisp and A is fuzzy, we have:

$$\mathrm{bel}_m(A) = \sum_{i=1}^{n} m(F_i) \min_{\omega\in F_i} \mu_A(\omega), \tag{24}$$

which is equal to the lower expectation of function μ_A [21].

Remark 1 The condition that the fuzzy focal elements of a fuzzy BS m be normal ($\max F_i = 1, \forall i$) generalize the normality condition ($m(\emptyset) = 0$) imposed to classical belief structures under the closed-world assumption. If this condition is relaxed, the definition of the conditional necessity of A given F_i should to be changed to:

$$\begin{aligned} N(A|F_i) &\triangleq \Pi(\Omega|F_i) - \Pi(\overline{A}|F_i) && (25)\\ &= \max_{\omega} \mu_{F_i}(\omega) - \max_{\omega} \mu_A(\omega) \wedge \mu_{\overline{F}_i}(\omega), && (26)\end{aligned}$$

which ensures that the expected necessity is still a valid generalization of the credibility [10].

Although (16) and (23) provide "natural" extensions of the concepts of plausibility and credibility[1], other definitions have been proposed. As remarked by Yager [30], the min and max operations used in the definitions of the conditional possibility and

[1] In particular, the expected possibility and the expected necessity are, respectively, subadditive and superadditive measures, as shown by Zadeh [38].

necessity measures may be replaced by any other t-norm and t-conorm, respectively, leading to more general forms of plausibility and credibility measures. It is not clear, however, how a particular pair of a t-norm and a t-conorm should be selected.

Starting from Dempster's model but following a different path, Yen [34] proposed another generalization of the concepts of belief and plausibility functions, based on a decomposition of each fuzzy focal element F of a fuzzy belief structure m into a collection of crisp focal elements ${}^{\alpha_i}F$ (the α_i-cuts of F) with belief numbers $m(F)(\alpha_i - \alpha_{i-1})$. This leads to the following alternative definitions of the credibility and plausibility of a fuzzy subset A:

$$\mathrm{bel}_m(A) \triangleq \sum_F m(F) \sum_{\alpha_i} (\alpha_i - \alpha_{i-1}) \min_{\omega \in {}^{\alpha_i}F} \mu_A(\omega) \tag{27}$$

$$\mathrm{pl}_m(A) \triangleq \sum_F m(F) \sum_{\alpha_i} (\alpha_i - \alpha_{i-1}) \max_{\omega \in {}^{\alpha_i}F} \mu_A(\omega). \tag{28}$$

As noted by Yen, the belief and plausibility measures defined above are more sensitive to changes in the membership functions of their focal elements, as compared to their counterparts defined in (23) and (16) (or other, more *ad hoc* definitions proposed by Ishizuka [15] and Ogawa [18]). However, given the relative arbitrariness that often prevails in the choice of the membership function of a fuzzy set, this lack of robustness might very well be viewed as a disadvantage of Yen's approach. In the rest of this paper, we shall adopt (23) and (16) as our definitions for the credibility and plausibility of a fuzzy event.

The next step in the generalization of evidence theory to fuzzy events concerns the combination of fuzzy belief structures. As proposed by Yager [31, 33], (5) and (6) may be readily extended to fuzzy belief structures by replacing crisp intersection and union by fuzzy counterparts. More generally, any binary set operator ∇ defines a corresponding operation on BSs such that:

$$(m_1 \nabla m_2)(A) \triangleq \sum_{B \nabla C = A} m_1(B) m_2(C), \tag{29}$$

where A is an arbitrary fuzzy subset of Ω.

Of course, the combination of two normal fuzzy BSs using, for example, the conjunctive sum, produces a fuzzy BS that may not be normal. If the normality condition is enforced, the conversion of an arbitrary fuzzy BS into a normal one may be performed by generalizing either of the Dempster or Yager normalization procedures. Yen [34] and Yager [33, 32] both propose to generalize the Dempster procedure as:

$$m^*(A) \triangleq \frac{\sum_{B^* = A} h_B m(B)}{\sum_{B \in \mathcal{F}(m)} h_B m(B)} \tag{30}$$

where $h_B = \max_\omega \mu_B(\omega)$ denotes the height of B, B^* is the normal fuzzy set defined by $\mu_{B^*}(\omega) = \mu_B(\omega)/h_B$, and $\mathcal{F}(m)$ is the set of focal elements of m (this procedure is called *soft normalization* par Yager). In [32], it is also proposed to generalize the Yager procedure as:

$$m^\circ(A) \triangleq \sum_{B^\circ = A} m(B) \tag{31}$$

where B° is a normal fuzzy set defined by $\mu_{B^\circ}(\omega) = \mu_B(\omega) + 1 - h_B$.

Finally, it may be remarked here that pignistic probabilities may also be defined in the case of fuzzy BSs, by generalizing (9) using the usual definition of the cardinality of a fuzzy set B as $|B| = \sum_{\omega\in\Omega} \mu_B(\omega)$ (the quantity $\frac{|A\cap B|}{|B|}$ may then be interpreted as a degree of subsethood of A in B [16]).

3 Interval-valued belief structures

Fuzzy belief structures and the associated concepts of credibility and plausibility of fuzzy events, recalled in Section 2.2, constitute a very useful generalization of the theory of belief functions, in that they provide a means of representing someone's belief in vague propositions such as produced in natural language. However, a fuzzy belief structure still assigns precise real numbers to each focal element, thereby ignoring the uncertainty attached to elicited belief numbers in many realistic situations. In this section, we go one step further in the generalization of evidence theory, by allowing belief masses to be provided in the form of *intervals*. This allows us to define a more flexible framework in which Your beliefs are no longer described by a unique belief structure, but by a convex set of belief structures verifying certain constraints. Although the concept of interval-valued belief structure defined in this section is interesting its own right [4], it is mainly seen in this paper as a preliminary step towards the complete fuzzification of the TBM (undertaken in Section 4) in which fuzzy belief numbers are allowed to be assigned to fuzzy propositions.

3.1 Definition

Our point of departure in this section is the notion of *interval-valued belief structure* (IBS), defined as a set of belief structures verifying certain inequality constraints. In the rest of this paper, we denote by $[0,1]^\Omega$ the set of fuzzy subsets of Ω, and by $\mathcal{S}_\Omega$ the set of belief structures on Ω. The set of focal elements of a BS m will be noted $\mathcal{F}(m)$. Unless explicitly stated, no distinction shall be made between BSs with crisp and fuzzy focal elements, neither shall we assume the BSs to be normalized. The reader is referred to a companion paper [4] for detailed proofs of most results presented in this section.

Definition 1 (Interval-valued belief structure)
An interval-valued belief structure (IBS) $\mathbf{m}$ *is a non empty subset of* $\mathcal{S}_\Omega$ *such that there exist* n *crisp or fuzzy subsets* $F_1, \ldots, F_n$ *of* Ω*, and* n *intervals* $([a_i, b_i])_{1\leq i\leq n}$ *of* $\mathbb{R}$*, such that* $m \in \mathbf{m}$ *iff*

- $a_i \leq m(F_i) \leq b_i \quad \forall i \in \{1,\ldots,n\}$*, and*
- $\displaystyle\sum_{i=1}^{n} m(F_i) = 1.$ □

Remark 2 As shown in [4], a necessary and sufficient condition for $\mathbf{m}$ to be non empty is that $\sum_{i=1}^n a_i \leq 1$ and $\sum_{i=1}^n b_i \geq 1$.

REMARK 3 If m is a belief structure, then $\{m\}$ is an IBS with $a_i = b_i = m(F_i)$ for all $F_i \in \mathcal{F}(m)$. Hence, the concept of IBS generalizes that of BS.

REMARK 4 The set of all belief structures on Ω with crisp focal elements is an IBS $\mathbf{m}$ with $\{F_1, \ldots, F_n\} = 2^\Omega$ and $[a_i, b_i] = [0, 1]$ for all $i \in \{1, \ldots, n\}$.

An IBS is completely specified by a set of n subsets $\{F_1, \ldots, F_n\}$ of Ω, and a corresponding set of interval $[a_i, b_i]$ for $1 \leq i \leq n$. However, it is important to note that this representation is not unique: since both b_i and $1 - \sum_{j \neq i} a_i$ are upper bounds of $m(F_i)$, it is clear that, whenever $b_i \geq 1 - \sum_{j \neq i} a_i$, b_i may be replaced by a higher bound $b'_i \geq b_i$. To obtain a unique characterization of $\mathbf{m}$, we thus introduce the concepts of *tightest lower and upper bounds* of $\mathbf{m}$, defined for all $A \in [0, 1]^\Omega$ as, respectively:

$$m^-(A) \triangleq \min_{m \in \mathbf{m}} m(A) \tag{32}$$

$$m^+(A) \triangleq \max_{m \in \mathbf{m}} m(A). \tag{33}$$

We may then define the set $\mathcal{F}(\mathbf{m})$ of focal elements of $\mathbf{m}$ as

$$\mathcal{F}(\mathbf{m}) \triangleq \{A \in [0, 1]^\Omega | m^+(A) > 0\}.$$

The tightest bounds may be easily obtained from any set of intervals $[a_i, b_i]$ defining $\mathbf{m}$ by:

$$m^-(F_i) = \max \left[a_i, 1 - \sum_{j \neq i} b_j \right] \tag{34}$$

$$m^+(F_i) = \min \left[b_i, 1 - \sum_{j \neq i} a_j \right] \tag{35}$$

for all $1 \leq i \leq n$, and $m^-(A) = m^+(A) = 0$, for all $A \notin \mathcal{F}(\mathbf{m})$.

For each IBS $\mathbf{m}$, one may define an interval-valued set function $f_\mathbf{m}$ such that

$$f_\mathbf{m}(A) \triangleq [m^-(A), m^+(A)] \tag{36}$$

for all $A \in [0, 1]^\Omega$. Function $f_\mathbf{m}$ may be called the interval-valued mapping associated to $\mathbf{m}$. By abuse of notation, we shall not distinguish between an IBS $\mathbf{m}$ and its corresponding mapping $f_\mathbf{m}$, which will also be denoted as $\mathbf{m}$. Hence, an IBS will be viewed, depending on the context, as a *set* of BSs, or as a *generalized* BS assigning intervals to propositions.

EXAMPLE 1 Let F_1, F_2 and F_3 be three arbitrary subsets of a finite possibility space Ω, and $\mathbf{m}$ the set of BSs m verifying the equality:

$$m(F_1) + m(F_2) + m(F_3) = 1$$

as well as the following inequality constraints:

$$m(F_1) \in [0.38, 0.65] \quad m(F_2) \in [0.23, 0.8] \quad m(F_3) \in [0.06, 0.5].$$

According to Definition 1, $\mathbf{m}$ is an IBS. By applying the formula given by (34) and (35), we obtain the following tightest bounds for $\mathbf{m}$:

$$
\begin{array}{lll}
m^-(F_1) = 0.38 & m^-(F_2) = 0.23 & m^-(F_3) = 0.06 \\
m^+(F_1) = 0.65 & m^+(F_2) = 0.56 & m^+(F_3) = 0.39,
\end{array}
$$

which shows that the constraints $m(F_2) \leq 0.8$ and $m(F_3) \leq 0.5$ were to loose and could never be active. Finally, we may write:

$$
\mathbf{m}(F_1) = [0.38, 0.65] \quad \mathbf{m}(F_2) = [0.23, 0.56] \quad \mathbf{m}(F_3) = [0.06, 0.39].
$$

Such a IBS with at most three focal elements may be very conveniently represented as a set of points in the two-dimensional probability simplex (The same representation was used by Walley [28], and others, for representing imprecise probabilities). This is an equilateral triangle with unit height, in which the masses assigned to each of the three focal elements are identified with perpendicular distances to each side of the triangle. Hence, each BS with corresponding focal elements is uniquely represented by a point in this triangle, while each constraint of the form $m(F_i) \leq m^+(F_i)$ or $m(F_i) \geq m^-(F_i)$ for some i is identified with a line parallel to one side of the triangle, and dividing the simplex in two parts. An IBS is thus represented as a convex polyhedron with sides parallel to sides of the triangle.

The above IBS is represented in this way in Figure 1, in which the original constraints and the tightest bounds are shown as dotted and dashed lines, respectively.

3.2 Interval-valued evidential functions

Given an IBS $\mathbf{m}$, and a crisp or fuzzy subset A of Ω, let us now consider the problem of determining the possible values of $\mathrm{bel}_m(A)$ (defined by (23)), where m ranges over $\mathbf{m}$. Since this quantity is a linear combination of belief numbers constrained to lie in closed intervals, its range is itself a closed interval. We thus have:

$$
\{x \in \mathbb{R} | \exists m \in \mathbf{m}, x = \mathrm{bel}_m(A)\} = [\mathrm{bel}^-_m(A), \mathrm{bel}^+_m(A)]
$$

where

$$
\mathrm{bel}^-_m(A) \triangleq \min_{m \in \mathbf{m}} \mathrm{bel}_m(A)
$$

and

$$
\mathrm{bel}^+_m(A) \triangleq \max_{m \in \mathbf{m}} \mathrm{bel}_m(A).
$$

The interval $[\mathrm{bel}^-_m(A), \mathrm{bel}^+_m(A)]$ will be called the *credibility interval* of A induced by $\mathbf{m}$, and will be noted $\mathbf{bel_m}(A)$. The interval-valued function

$$
\mathbf{bel_m} : A \mapsto \mathbf{bel_m}(A)
$$

will be called the interval-valued belief function induced by $\mathbf{m}$. Without any risk of confusion, the same notation will also be used to refer to the set of belief functions bel verifying $\mathrm{bel}(A) \in \mathbf{bel_m}(A)$ for all subset A of Ω.

REMARK 5 It must be well understood at this point that the set $\mathbf{bel_m}$ *is not* the set $\mathcal{B}_\mathbf{m}$ of belief functions induced by some IBS in $\mathbf{m}$. However, we obviously have the inclusion

$$\mathcal{B}_\mathbf{m} \subseteq \mathbf{bel_m},$$

which allows to regard $\mathbf{bel_m}$ as an approximation to $\mathcal{B}_\mathbf{m}$ (it is in fact the smallest interval-valued belief function containing $\mathcal{B}_\mathbf{m}$).

The practical determination of the credibility intervals involves the resolution of a particular class of linear programming (LP) problems, in which the goal is to find the minimum and maximum of a linear function of n variables $x_1, \ldots, x_n$, under one linear equality constraint and a set of box constraints. A general solution to this problem was proposed by Dubois and Prade [6, 7, 11] who proved the following theorem:

THEOREM 1 (DUBOIS AND PRADE, 1981)
Let $x_1, \ldots, x_n$ be n variables linked by the following constraints:

$$\sum_{i=1}^{n} x_i = 1$$

$$a_i \leq x_i \leq b_i \quad 1 \leq i \leq n$$

and let f be a function defined by $f(x_1, \ldots, x_n) = \sum_{i=1}^{n} c_i x_i$ with

$$0 \leq c_1 \leq c_2 \leq \ldots \leq c_n.$$

Then

$$\begin{aligned}
\min f &= \max_{k=1,n} \left(\sum_{j=1}^{k-1} b_j c_j + \left(1 - \sum_{j=1}^{k-1} b_j - \sum_{j=k+1}^{n} a_j \right) c_k + \sum_{j=k+1}^{n} a_j c_j \right) \\
\max f &= \min_{k=1,n} \left(\sum_{j=1}^{k-1} a_j c_j + \left(1 - \sum_{j=1}^{k-1} a_j - \sum_{j=k+1}^{n} b_j \right) c_k + \sum_{j=k+1}^{n} b_j c_j \right) \square
\end{aligned}$$

Hence, an exact determination of $\mathrm{bel}^-_\mathbf{m}(A)$ and $\mathrm{bel}^+_\mathbf{m}(A)$ may be obtained without resorting to an iterative procedure. In particular, when both A and the focal elements of $\mathbf{m}$ are crisp, then the coefficients c_i in Theorem 1 are all equal to 0 or 1, and one obtains without difficulty the following expressions for the bounds of $\mathbf{bel_m}(A)$:

$$\mathrm{bel}^-(A) = \max \left[\sum_{\emptyset \neq B \subseteq A} m^-(B), 1 - \sum_{B \not\subseteq A} m^+(B) - m^+(\emptyset) \right] \tag{37}$$

$$\mathrm{bel}^+(A) = \min \left[\sum_{\emptyset \neq B \subseteq A} m^+(B), 1 - \sum_{B \not\subseteq A} m^-(B) - m^-(\emptyset) \right] \tag{38}$$

Needless to say, the approach adopted above to define the credibility interval of a fuzzy event A may easily be transposed to the definition of the plausibility and pignistic probability intervals of A, denoted, respectively, as $\mathbf{pl_m}(A)$ and $\mathbf{BetP_m}(A)$:

$$
\begin{aligned}
\mathbf{pl_m}(A) &\triangleq [\min_{m\in\mathbf{m}} \mathrm{pl}_m(A), \max_{m\in\mathbf{m}} \mathrm{pl}_m(A)] \\
\mathbf{BetP_m}(A) &\triangleq [\min_{m\in\mathbf{m}} \mathrm{BetP}_m(A), \max_{m\in\mathbf{m}} \mathrm{BetP}_m(A)].
\end{aligned}
$$

The bounds of $\mathbf{pl_m}(A)$ are easily obtained using Theorem 1, in exactly the same way as explained above for the bounds of $\mathbf{bel_m}(A)$. The calculation of $\mathrm{BetP}^-_{\mathbf{m}}(A)$ and $\mathrm{BetP}^+_{\mathbf{m}}(A)$ is not so straightforward in the general case, because (9) defines a linear function of the *normalized* belief numbers. When the IBS $\mathbf{m}$ is not normal, i.e. $\max_\omega F_i < 1$ for some $F_i \in \mathcal{F}(\mathbf{m})$, some normalization procedure has to be defined. This issue will be addressed in a Section 3.4.

3.3 Combination of IBSs

3.3.1 Definition

As shown in Section 2.2, any binary set operation ∇ induces a binary operation on belief structures (also denoted ∇ for simplicity) through (29). In this section, we go one step further in the generalization process by extending any binary operation in $\mathcal{S}_\Omega$ to IBSs. This will be achieved by considering the lower and upper bounds of $(m_1 \nabla m_2)(A)$, for all $A \in [0,1]^\Omega$.

Definition 2 (Combination of two IBSs)
Let $\mathbf{m}_1$ and $\mathbf{m}_2$ be two IBSs on the same frame Ω, and let ∇ be a binary operation on BSs. The combination of $\mathbf{m}_1$ and $\mathbf{m}_2$ by ∇ is defined as the IBS $\mathbf{m} = \mathbf{m}_1 \nabla \mathbf{m}_2$ with bounds:

$$
\begin{aligned}
m^-(A) &= \min_{(m_1,m_2)\in\mathbf{m}_1\times\mathbf{m}_2} (m_1 \nabla m_2)(A) \\
m^+(A) &= \max_{(m_1,m_2)\in\mathbf{m}_1\times\mathbf{m}_2} (m_1 \nabla m_2)(A)
\end{aligned}
$$

for all $A \in [0,1]^\Omega$. □

Remark 6 A more natural definition for the combination of $\mathbf{m}_1$ and $\mathbf{m}_2$ by ∇ could have been to consider the set $M_{1,2}$ of all BSs obtained by combining one BS in $\mathbf{m}_1$ with one BS in $\mathbf{m}_2$:

$$
M_{1,2} = \{m \in \mathcal{S}_\Omega | \exists (m_1, m_2) \in \mathbf{m}_1 \times \mathbf{m}_2, m = m_1 \nabla m_2\} \tag{39}
$$

Unfortunately, $M_{1,2}$ is not, in general, an IBS, as shown by the following counterexample [4]. Note, however, we have obviously $M_{1,2} \subseteq \mathbf{m}_1 \nabla \mathbf{m}_2$. The IBS $\mathbf{m}_1 \nabla \mathbf{m}_2$ is thus the smallest IBS containing $M_{1,2}$.

Example 2 Let us $\mathbf{m}_1$ and $\mathbf{m}_2$ be two IBSs such that $\mathcal{F}(\mathbf{m}_1) = \{A, \Omega\}$, $\mathcal{F}(\mathbf{m}_2) = \{B, \Omega\}$, with $C = A \cap B \notin \{A, B\}$, and:

$$
\begin{aligned}
\mathbf{m}_1(A) = [0, 0.5] \quad \mathbf{m}_1(\Omega) = [0.5, 1] \\
\mathbf{m}_2(B) = [0, 0.5] \quad \mathbf{m}_2(\Omega) = [0.5, 1]
\end{aligned}
$$

The conjunctive sum $\mathbf{m} = \mathbf{m}_1 \cap \mathbf{m}_2$ is immediately obtained as:

$$\begin{aligned}
\mathbf{m}(A) &= [m_1^-(A)m_2^-(\Omega), m_1^+(A)m_2^+(\Omega)] = [0, 0.5] \\
\mathbf{m}(B) &= [m_1^-(\Omega)m_2^-(B), m_1^+(\Omega)m_2^+(B)] = [0, 0.5] \\
\mathbf{m}(C) &= [m_1^-(A)m_2^-(B), m_1^+(A)m_2^+(B)] = [0, 0.25] \\
\mathbf{m}(\Omega) &= [m_1^-(\Omega)m_2^-(\Omega), m_1^+(\Omega)m_2^+(\Omega)] = [0.25, 1]
\end{aligned}$$

Let $m \in \mathbf{m}$ defined by $m(A) = 0.4$, $m(B) = 0.2$, $m(C) = 0.1$ and $m(\Omega) = 0.3$. Let us show that it is impossible to find $m_1 \in \mathbf{m}_1$ and $m_2 \in \mathbf{m}_2$ such that $m = m_1 \cap m_2$. Let $x = m_1(A)$ and $y = m_2(B)$. These quantities must be solutions of a system of four equations:

$$\begin{cases}
x(1-y) &= 0.4 \\
(1-x)y &= 0.2 \\
xy &= 0.1 \\
(1-x)(1-y) &= 0.3
\end{cases}$$

It is easy to see that this system is incompatible. Hence $m \notin M_{1,2}$.

Remark 7 It may also be shown by counterexamples [4] that the extension of the ∇ operation from BSs to IBSs performed according to Definition 2 does not, in general, preserve the associativity property, i.e., we may have

$$(\mathbf{m}_1 \nabla \mathbf{m}_2) \nabla \mathbf{m}_3 \neq \mathbf{m}_1 \nabla (\mathbf{m}_2 \nabla \mathbf{m}_3)$$

for some $\mathbf{m}_1$, $\mathbf{m}_2$ and $\mathbf{m}_3$. To avoid any influence of the order in which n IBSs are combined, it is therefore necessary to combine them at once using an n-ary combination operator introduced in the following definition.

Definition 3 (Combination of n IBSs)
Let $\mathbf{m}_1, \ldots, \mathbf{m}_n$ be n IBSs on the same frame Ω, and let ∇ be a transitive operation on BSs. The combination of $\mathbf{m}_1, \ldots, \mathbf{m}_n$ by ∇ is defined as the IBS $\mathbf{m} = \mathbf{m}_1 \nabla \ldots \nabla \mathbf{m}_n$ with bounds:

$$\begin{aligned}
m^-(A) &= \min_{(m_1,\ldots,m_n) \in \mathbf{m}_1 \times \ldots \times \mathbf{m}_n} (m_1 \nabla \ldots \nabla m_n)(A) \\
m^+(A) &= \max_{(m_1,\ldots,m_n) \in \mathbf{m}_1 \times \ldots \times \mathbf{m}_n} (m_1 \nabla \ldots \nabla m_n)(A)
\end{aligned}$$

for all $A \in [0,1]^\Omega$. □

It may be shown [4] that, for any IBSs $\mathbf{m}_1$, $\mathbf{m}_2$ and $\mathbf{m}_3$, we have:

$$(\mathbf{m}_1 \nabla \mathbf{m}_2) \nabla \mathbf{m}_3 \supseteq \mathbf{m}_1 \nabla \mathbf{m}_2 \nabla \mathbf{m}_3.$$

Hence, given a sequence of n IBS $\mathbf{m}_1, \ldots, \mathbf{m}_n$ the strategy of combining them one by one using the binary operator introduced in Definition 2 leads to pessimistic lower and upper bounds for the belief intervals introduced more rigorously in Definition 3.

Techniques for computing the combination of two or more BSs were introduced in [4]. They are briefly recalled in Appendix A.

3.4 Normalization of an IBS

As already mentioned in Section 2.1, a reasonable condition to impose on a BS m with crisp focal elements, in the case where the variable Y of interest is known with absolute certainty, is $m(\emptyset) = 0$. When the focal elements of m are fuzzy, this normality condition may be generalized to $h_F = 1$ for all $F \in \mathcal{F}(m)$. By analogy, an IBS $\mathbf{m}$ will be said to be normal if it contains only normal BSs, which can be expressed as $h_F = 1$ for all $F \in \mathcal{F}(\mathbf{m})$. The aim of this section is to introduce extensions of the Dempster and Yager normalization procedures, allowing to convert subnormal IBSs into normal ones. As it involves only linear transformations, the Yager procedure is considerably simpler, and will therefore be examined first.

3.4.1 Yager normalization

Let $\mathbf{m}$ be an IBS with crisp or fuzzy focal elements. The normalized form of $\mathbf{m}$, according to the Yager procedure, will be defined as the IBS $\mathbf{m}^\circ$ with bounds:

$$m^{\circ -}(A) \triangleq \min_{m \in \mathbf{m}} m^\circ(A) \tag{40}$$

$$m^{\circ +}(A) \triangleq \max_{m \in \mathbf{m}} m^\circ(A), \tag{41}$$

for all A in $[0,1]^\Omega$, $m^\circ(A)$ being defined by (31). The bounds of $\mathbf{m}^\circ$ may therefore be found as the solutions to very simple linear programming problems. It is easy to see that:

$$m^{\circ -}(A) = \max\left(\sum_{F^\circ = A} m^-(F), 1 - \sum_{F^\circ \neq A} m^+(F)\right) \tag{42}$$

$$m^{\circ +}(A) = \min\left(\sum_{F^\circ = A} m^+(F), 1 - \sum_{F^\circ \neq A} m^-(F)\right). \tag{43}$$

3.4.2 Dempster normalization

Conceptually, the extension of the Dempster normalization procedure to IBSs may be performed in exactly the same way as for Yager's procedure. Given an arbitrary IBS $\mathbf{m}$, its normalized form, according to the Dempster procedure, will be defined as the IBS $\mathbf{m}^*$ with bounds:

$$m^{*-}(A) \triangleq \min_{m \in \mathbf{m}} m^*(A) \tag{44}$$

$$m^{*+}(A) \triangleq \max_{m \in \mathbf{m}} m^*(A), \tag{45}$$

for all A in $[0,1]^\Omega$, with $m^*(A)$ defined by (30). However, because of the non linearity of this equation, the practical determination of $\mathbf{m}^*$ is significantly more difficult than that of $\mathbf{m}^\circ$. This problem was solved exactly in [4] for the case where all focal elements of $\mathcal{F}(\mathbf{m})$ are crisp. We showed that:

$$m^{*-}(A) = \frac{m^-(A)}{1 - \max\left[m^-(\emptyset), 1 - \sum_{B \neq A, B \neq \emptyset} m^+(B) - m^-(A)\right]} \quad (46)$$

$$m^{*+}(A) = \frac{m^+(A)}{1 - \min\left[m^+(\emptyset), 1 - \sum_{B \neq A, B \neq \emptyset} m^-(B) - m^+(A)\right]} \quad (47)$$

for all $A \in \mathcal{F}(\mathbf{m}^*) = \mathcal{F}(\mathbf{m}) \setminus \emptyset$.

In the more general case in which some focal elements of $\mathbf{m}$ are fuzzy, the bounds of $\mathbf{m}^*$ are the solutions of non linear programming problems for which no analytic solution is, to our knowledge, available. These values thus have to be computed numerically using an iterative non linear optimization procedure.

4 Fuzzy-valued belief structures

4.1 Definition

In many applications, the degrees of belief in various hypotheses are either directly obtained through verbal statements such as "high", "very low", "around 0.8", or are inferred from "vague" evidence expressed linguistically in a similar way. In such situations, it is difficult to avoid arbitrariness in the assignment of a precise number, or even an interval, to each hypothesis. *Fuzzy numbers* have been proposed as a suitable formalism for handling such kind of ambiguity in modeling subjective probability judgments [29, 13, 7, 12]. Mathematically, a fuzzy number may be defined as a normal fuzzy subset $\widetilde{x}$ of $\mathbb{R}$ with compact support, and whose α-cuts are closed intervals [11, 16] (Dubois and Prade make a distinction between *fuzzy intervals* and *fuzzy numbers* depending on the multiplicity or uniqueness of modal values. We shall use the term "fuzzy number" in its most general sense in this paper.). A fuzzy number may be viewed as an elastic constraint acting on a certain variable which is only known to lie "around" a certain value. It generalizes both concepts of real number and closed interval.

In this section, we introduce the new concept of a *fuzzy-valued belief structure* (FBS), which will be defined as a fuzzy set of belief structures on Ω, whose belief masses are restricted by fuzzy numbers.

Definition 4 (Fuzzy-valued belief structure)
A fuzzy-valued belief structure (FBS) is a normal fuzzy subset $\widetilde{\mathbf{m}}$ of $\mathcal{S}_\Omega$ such that there exist n elements $F_1, \ldots, F_n$ of $[0,1]^\Omega$, and n non null fuzzy numbers $\widetilde{m}_i, 1 \leq i \leq n$, with supports from $[0,1]$, such that, for every $m \in \mathcal{S}_\Omega$,

$$\mu_{\widetilde{\mathbf{m}}}(m) \triangleq \begin{cases} \min_{1 \leq i \leq n} \mu_{\widetilde{m}_i}(m(F_i)) & \text{if } \sum_{i=1}^n m(F_i) = 1 \\ 0 & \text{otherwise} \end{cases} \qquad \square$$

REMARK 8 This definition obviously reduces to Definition 1 when the $\widetilde{m}_i$ are crisp intervals. Hence, the concept of FBS generalizes that of IBS, which can be expressed schematically as:

$$\text{fuzzy-valued BS} \supset \text{interval-valued BS} \supset \text{(precise) BS}.$$

REMARK 9 The assumption that $\widetilde{\mathbf{m}}$ is a *normal* fuzzy set imposes certain conditions on fuzzy numbers $\widetilde{m}_i$. More precisely, the fact that $\mu_{\widetilde{\mathbf{m}}}(m) = 1$ for some m implies that

$$\mu_{\widetilde{m}_i}(m(F_i)) = 1$$

for every $i \in \{1, \dots, n\}$. Hence, for all i, $m(F_i)$ belongs to the core ${}^1\widetilde{m}_i$ of $\widetilde{m}_i$. The BS m thus belongs to an IBS with bounds $[{}^1\widetilde{m}_i^-, {}^1\widetilde{m}_i^+]$. According to Remark 2, this implies that

$$\sum_{i=1}^{n} {}^1\widetilde{m}_i^- \leq 1 \quad \text{and} \quad \sum_{i=1}^{n} {}^1\widetilde{m}_i^+ \geq 1.$$

As suggested in the above remark, each BS m belonging to the *core* of a FBS $\widetilde{\mathbf{m}}$ constrained by fuzzy numbers $\widetilde{m}_i$, belongs to an IBS bounded by the cores of the $\widetilde{m}_i$. Conversely, it is obvious that a BS m such that $m(F_i) \in {}^1\widetilde{m}_i$ for all i and $\sum_{i=1}^n m(F_i) = 1$ has full membership to $\widetilde{\mathbf{m}}$. Hence, we may deduce that the core of a FBS $\widetilde{\mathbf{m}}$ constrained by fuzzy numbers $\widetilde{m}_i$ is an IBS ${}^1\widetilde{\mathbf{m}}$ bounded by the cores of the $\widetilde{m}_i$. This result may be extended to any α-cut of $\widetilde{\mathbf{m}}$, which happen to have a very simple characterization in terms of the α-cuts of the fuzzy numbers constraining $\widetilde{\mathbf{m}}$, as stated in the following proposition.

PROPOSITION 1
Let $\widetilde{\mathbf{m}}$ be a FBS defined by n elements $F_1, \dots, F_n$ of $[0,1]^\Omega$ and n fuzzy numbers $\widetilde{m}_1, \dots, \widetilde{m}_n$. For any $\alpha \in]0,1]$, the α-cut of $\widetilde{\mathbf{m}}$ is an IBS ${}^\alpha\widetilde{\mathbf{m}}$ with bounds ${}^\alpha\widetilde{m}_i$ for all $i \in \{1, \dots, n\}$. □

Proof: Let α be any real number in $]0,1]$, and ${}^\alpha\widetilde{\mathbf{m}}$ the α-cut of $\widetilde{\mathbf{m}}$. We have

$$\begin{aligned}
{}^\alpha\widetilde{\mathbf{m}} &= \{m \in \mathcal{S}_\Omega | \mu_{\widetilde{\mathbf{m}}}(m) \geq \alpha\} \\
&= \{m \in \mathcal{S}_\Omega | \min_{1 \leq i \leq n} \mu_{\widetilde{m}_i}(m(F_i)) \geq \alpha \text{ and } \sum_{i=1}^{n} m(F_i) = 1\} \\
&= \{m \in \mathcal{S}_\Omega | \mu_{\widetilde{m}_i}(m(F_i)) \geq \alpha \ \forall i \text{ and } \sum_{i=1}^{n} m(F_i) = 1\} \\
&= \{m \in \mathcal{S}_\Omega | m(F_i)) \in {}^\alpha\widetilde{m}_i \ \forall i \text{ and } \sum_{i=1}^{n} m(F_i) = 1\}
\end{aligned}$$

Since the $\widetilde{m}_i$ are fuzzy numbers, their α-cuts are closed intervals. Hence, ${}^\alpha\widetilde{\mathbf{m}}$ is an IBS. □

REMARK 10 Following the same line of reasoning, it is simple to show that the support of a FBS $\widetilde{\mathbf{m}}$ constrained by fuzzy numbers $\widetilde{m}_i$ is an IBS ${}^{0+}\widetilde{\mathbf{m}}$ constrained by the supports of the $\widetilde{m}_i$.

As in the case of interval-valued belief structures, it is useful to define a unique representation of a FBS $\widetilde{\mathbf{m}}$, in the form of fuzzy numbers assigned to each of the focal elements. This may be achieved by considering the upper and lower bounds of all its α-cuts. More precisely, let us denote:

$$\begin{aligned} {}^{\alpha}\widetilde{m}^{-}(F_i) &\triangleq \min_{m \in {}^{\alpha}\widetilde{\mathbf{m}}} m(F_i) \\ {}^{\alpha}\widetilde{m}^{+}(F_i) &\triangleq \max_{m \in {}^{\alpha}\widetilde{\mathbf{m}}} m(F_i) \end{aligned}$$

The fuzzy set $\widetilde{\mathbf{m}}(F_i)$ with α-cuts ${}^{\alpha}\widetilde{\mathbf{m}}(F_i) = [{}^{\alpha}\widetilde{m}^{-}(F_i), {}^{\alpha}\widetilde{m}^{+}(F_i)]$ satisfies all the axioms of a fuzzy number. Hence, a FBS may be seen a fuzzy mapping assigning a fuzzy number to each $A \in [0,1]^{\Omega}$ (with $\widetilde{\mathbf{m}}(A) = 0$ for all $A \notin \{F_1, \ldots, F_n\}$).

As for IBSs, we may define a focal element of a FBS $\widetilde{\mathbf{m}}$ as a crisp or fuzzy subset of Ω that receives a positive mass of belief from at least one BS with non zero membership to $\widetilde{\mathbf{m}}$. The set $\mathcal{F}(\widetilde{\mathbf{m}})$ of focal elements of $\widetilde{\mathbf{m}}$ is thus identical to $\mathcal{F}({}^{0+}\widetilde{\mathbf{m}})$, the set of focal elements of the support of $\widetilde{\mathbf{m}}$.

Example 3 Assume that, in view of certain evidence, an expert assigns the following fuzzy belief numbers to three crisp or fuzzy subsets F_1, F_2 and F_3 of a possibility space Ω:

$$\widetilde{m}_1 \triangleq \text{'around 0.2'} \quad \widetilde{m}_2 \triangleq \text{'around 0.5'} \quad \widetilde{m}_3 \triangleq \text{'around 0.3'}$$

where *'around x'* denotes the triangular fuzzy number with modal value x and support $[\max(0, x - 0.1), \min(1, x + 0.1)]$. According to Definition 4, these fuzzy constraints define a FBS $\widetilde{\mathbf{m}}$, whose support is the IBS ${}^{0+}\widetilde{\mathbf{m}}$ defined by

$${}^{0+}\widetilde{\mathbf{m}}(F_1) = [0.1, 0.3] \quad {}^{0+}\widetilde{\mathbf{m}}(F_2) = [0.4, 0.6] \quad {}^{0+}\widetilde{\mathbf{m}}(F_3) = [0.2, 0.4]$$

and whose core is the BS ${}^{1}\widetilde{\mathbf{m}}$ defined by

$${}^{1}\widetilde{\mathbf{m}}(F_1) = 0.2 \quad {}^{1}\widetilde{\mathbf{m}}(F_2) = 0.5 \quad {}^{1}\widetilde{\mathbf{m}}(F_3) = 0.3$$

Let m denote the BS defined by

$$m(F_1) = 0.25 \quad m(F_2) = 0.48 \quad m(F_3) = 0.27$$

We have

$$\mu_{\widetilde{m}_1}(0.25) = 0.5 \quad \mu_{\widetilde{m}_2}(0.48) = 0.8 \quad \mu_{\widetilde{m}_3}(0.27) = 0.7$$

Hence, $\mu_{\widetilde{\mathbf{m}}}(m) = 0.5$.

The membership function $\mu_{\widetilde{\mathbf{m}}}$ of a FBS with at most three focal elements may be visualized as a surface in the probability simplex representation. For any $\alpha \in]0,1]$, the α-level contour of $\mu_{\widetilde{\mathbf{m}}}$ is the convex polyhedron representing the α-cut ${}^{\alpha}\widetilde{\mathbf{m}}$ of $\widetilde{\mathbf{m}}$. Such a representation is shown for the above example in Figure 2, for 10 different values of α.

4.2 Fuzzy credibility and plausibility

The fuzzy credibility and the fuzzy plausibility of a crisp or fuzzy subset A of Ω induced by a FBS may be defined by applying the extension principle to (23) and (16), respectively. Generally speaking, the extension principle provides a canonical way of finding the range of a function f whose arguments are restricted by a certain possibility distribution [36, 6]. In the case where each variable $\widetilde{x}_i$ is restricted by a possibility distribution $\mu_{\widetilde{x}_i}$, and where the variables are constrained to lie within a domain D, their image $\widetilde{z} = f(\widetilde{x}_1, \ldots, \widetilde{x}_n)$ under f is defined as a fuzzy set with membership function:

$$\mu_{\widetilde{z}}(w) \triangleq \sup_{u_1,\ldots,u_n} \min_i \mu_{\widetilde{x}_i}(u_i) \tag{48}$$

under the constraints $w = f(u_1, \ldots, u_n)$ and $(u_1, \ldots, u_n) \in D$.

By applying this principle to (23), we may define the fuzzy credibility of A as a fuzzy set $\widetilde{\mathbf{bel}}(A)$ with membership function:

$$\mu_{\widetilde{\mathbf{bel}}(A)}(w) \triangleq \sup_{\{m\in D|\mathrm{bel}_m(A)=w\}} \min_{1\leq i\leq n} \mu_{\widetilde{\mathbf{m}}(F_i)}(m(F_i)) \tag{49}$$

where D is the set of BSs m such that $\sum_{i=1}^{n} m(F_i) = 1$, and $\mathrm{bel}_m(A)$ is defined according to (23). Since, by definition, for any $m \in D$,

$$\mu_{\widetilde{\mathbf{m}}}(m) = \min_{1\leq i\leq n} \mu_{\widetilde{\mathbf{m}}(F_i)}(m(F_i)),$$

(49) may be written more simply as:

$$\mu_{\widetilde{\mathbf{bel}}(A)}(w) \triangleq \sup_{\{m|\mathrm{bel}_m(A)=w\}} \mu_{\widetilde{\mathbf{m}}}(m). \tag{50}$$

As shown by Dubois and Prade [6], (48) defines a fuzzy number when the x_i are fuzzy numbers and the domain D is defined by linear equality constraints. Hence, $\widetilde{\mathbf{bel}}(A)$ defined by (50) is a fuzzy number. Its α-cut is given by:

$${}^{\alpha}\widetilde{\mathbf{bel}}(A) = \min_{m\in{}^{\alpha}\widetilde{\mathbf{m}}} \mathrm{bel}_m(A),$$

which is nothing but the credibility interval induced by the IBS ${}^{\alpha}\widetilde{\mathbf{m}}$. Similarly, using (16) as a definition for the plausibility $\mathrm{pl}_m(A)$ of a crisp or fuzzy subset A of Ω, induced by a BS m, the fuzzy plausibility of A may be defined as a fuzzy number with membership function:

$$\mu_{\widetilde{\mathbf{pl}}(A)}(w) \triangleq \sup_{\{m|\mathrm{pl}_m(A)=w\}} \mu_{\widetilde{\mathbf{m}}}(m). \tag{51}$$

Its α-cut is the plausibility interval induced by ${}^{\alpha}\widetilde{\mathbf{m}}$.

REMARK 11 It is well known that a plausibility function pl with nested focal elements $F_1 \subset F_2 \ldots \subset F_n$ is a possibility measure [8], since

$$\mathrm{pl}(A \cup B) = \max(\mathrm{pl}(A), \mathrm{pl}(B)) \quad \forall A, B \subseteq \Omega.$$

The function π on Ω defined by

$$\pi(\omega) = \mathrm{pl}(\{\omega\}) \quad \forall \omega \in \Omega$$

is then a possibility distribution, which, as remarked by Zadeh [37], may be viewed as the membership function of a fuzzy set F. These observations may be extended to the case of fuzzy belief numbers. Let $\widetilde{\mathbf{m}}$ be a FBS with nested focal elements $F_1 \subset F_2 \ldots \subset F_n$ in 2^Ω. By analogy with the crisp case, the associated fuzzy plausibility function $\widetilde{\mathbf{pl}}$ may be termed a fuzzy possibility function. One may then define a fuzzy possibility distribution $\widetilde{\pi}$ defined as:

$$\widetilde{\pi}(\omega) = \widetilde{\mathbf{pl}}(\{\omega\})$$

for any $\omega \in \Omega$. Function $\widetilde{\pi}$ is formally equivalent to a type 2 fuzzy subset of Ω, i.e., a fuzzy set with fuzzy membership values [36].

Remark 12 The manipulation of fuzzy numbers may be considerably simplified by using the LL parameterization introduced by Dubois and Prade [11]. A fuzzy number $\widetilde{x}$ is of type LL if its membership function is of the form:

$$\mu_{\widetilde{x}}(u) \triangleq \begin{cases} L\left(\dfrac{\alpha - u}{\gamma}\right) & \forall u \leq \alpha \\ 1 & \forall u \in [\alpha, \beta] \\ L\left(\dfrac{u - \beta}{\delta}\right) & \forall u \geq \beta \end{cases},$$

where L is a left-continuous, non increasing mapping from $\mathbb{R}_+$ to $[0, 1]$. We may then write, without ambiguity,

$$\widetilde{x} \triangleq (\alpha, \beta, \gamma, \delta)_{LL},$$

or $\widetilde{x} \triangleq (\alpha, \gamma, \delta)_{LL}$ when $\alpha = \beta$. When the fuzzy masses $\widetilde{\mathbf{m}}(F)$ assigned by $\widetilde{\mathbf{m}}$ are fuzzy numbers of type LL, then $\widetilde{\mathbf{bel}}(A)$ and $\widetilde{\mathbf{pl}}(A)$ defined by (50) and (51) are also LL fuzzy numbers [11, p. 55]. Their parameters may be calculated by applying the formula in Theorem 1 to the core and support of $\widetilde{\mathbf{m}}$. □

Example 4 Consider again the FBS $\widetilde{\mathbf{m}}$ of Example 3, assuming F_1, F_2 and F_3 to be three subsets of a possibility space $\Omega = \{1, \ldots, 10\}$ defined as:

$$F_1 \triangleq \{1, \ldots, 5\} \quad F_2 \triangleq \{0.1/2, 0.5/3, 1/4, 0.5/5, 0.1/6\}$$

$$F_3 \triangleq \{0.1/3, 0.5/4, 1/5, 0.5/6, 0.1/7\}.$$

F_1 is thus a crisp subset of Ω corresponding to the proposition "Y is strictly smaller than 6" (where Y denote the unknown variable of interest), while F_2 and F_3 are fuzzy subsets that might correspond to such fuzzy propositions as, respectively, "Y is around 4", and "Y is around 5". Given that piece of evidence, what is the credibility and plausibility of the following propositions: (1) $A \triangleq$ "Y is equal to 3 or 4", (2) $B \triangleq$ "Y is around 3 or 4" ? To answer these questions, let us first identify propositions A and B with the following subsets of Ω:

$$A \triangleq \{3, 4\} \quad B \triangleq \{0.1/1, 0.5/2, 1/3, 1/4, 0.5/5, 0.1/6\}.$$

We then have, for any BS m such that $\mathcal{F}(m) = \{F_1, F_2, F_3\}$:

$$\begin{aligned}
\mathrm{bel}_m(A) &= \sum_{i=1}^{n} m(F_i) \min_{\omega \in \Omega} \mu_A(\omega) \vee \mu_{\overline{F}_i}(\omega) \\
&= m(F_1) \times 0 + m(F_2) \times 0.5 + m(F_3) \times 0 \\
\mathrm{pl}_m(A) &= \sum_{i=1}^{n} m(F_i) \max_{\omega \in \Omega} \mu_A(\omega) \wedge \mu_{F_i}(\omega) \\
&= m(F_1) \times 1 + m(F_2) \times 1 + m(F_3) \times 0.5.
\end{aligned}$$

Since the fuzzy belief masses $\widetilde{\mathbf{m}}(F_1)$ are all triangular LL fuzzy numbers, the corresponding credibility and plausibility values $\widetilde{\mathbf{bel}}(A)$ and $\widetilde{\mathbf{pl}}(A)$ are also triangular LL fuzzy numbers. Note that the calculation of $\widetilde{\mathbf{bel}}(A)$ is straightforward in this case, since it involves only one fuzzy belief number:

$$\widetilde{\mathbf{bel}}(A) = 0.5 \times \widetilde{\mathbf{m}}(F_2) = (0.25, 0.05, 0.05)_{LL}.$$

The calculation of $\widetilde{\mathbf{pl}}(A)$ is, in principle, a little more delicate, since it involves three interactive fuzzy numbers. However, in this case, the cores of the $\widetilde{\mathbf{m}}(F_i)$ are reduced to single numbers, so that the core of $\widetilde{\mathbf{pl}}(A)$ may be obtained without any difficulty. Theorem 1 is then needed only for the calculation of the support of $\widetilde{\mathbf{pl}}(A)$. We finally obtain:

$$\widetilde{\mathbf{pl}}(A) = (0.85, 0.05, 0.05)_{LL}.$$

Similarly, we have:

$$\begin{aligned}
\mathrm{bel}_m(B) &= m(F_1) \times 0.1 + m(F_2) \times 0.5 + m(F_3) \times 0.5 \\
\mathrm{pl}_m(B) &= m(F_1) \times 1 + m(F_2) \times 1 + m(F_3) \times 0.5,
\end{aligned}$$

which leads to:

$$\begin{aligned}
\widetilde{\mathbf{bel}}(B) &= (0.42, 0.04, 0.04)_{LL} \\
\widetilde{\mathbf{pl}}(B) &= (0.85, 0.05, 0.05)_{LL}.
\end{aligned}$$

Note that we have a higher degree of belief in a less precise proposition, which is an illustration of the usual opposition between imprecision and uncertainty [11]. □

4.3 Combination of fuzzy belief structures

A binary operation ∇ on BSs may also be generalized to FBSs by applying the extension principle to (29). Given to FBSs $\widetilde{\mathbf{m}}_1$ and $\widetilde{\mathbf{m}}_2$, their combination by ∇ may be defined as a FBS $\widetilde{\mathbf{m}}$ assigning to each $A \in [0,1]^\Omega$ a fuzzy number $\widetilde{\mathbf{m}}(A) \triangleq (\widetilde{\mathbf{m}}_1 \nabla \widetilde{\mathbf{m}}_2)(A)$ with membership function

$$\mu_{\widetilde{\mathbf{m}}(A)}(w) \triangleq \sup_{\{m_1, m_2\}} \min \left[\min_{B \in \mathcal{F}(\widetilde{\mathbf{m}}_1)} \mu_{\widetilde{\mathbf{m}}_1(B)}(m_1(B)), \min_{B \in \mathcal{F}(\widetilde{\mathbf{m}}_2)} \mu_{\widetilde{\mathbf{m}}_2(B)}(m_2(B)) \right]$$

under the constraints

$$\begin{aligned}\sum_{B\in\mathcal{F}(\widetilde{\mathbf{m}}_1)} m_1(B) &= 1\\ \sum_{B\in\mathcal{F}(\widetilde{\mathbf{m}}_2)} m_2(B) &= 1\\ (m_1\nabla m_2)(A) &= w,\end{aligned}$$

which may also be written in more compact form as:

$$\mu_{\widetilde{\mathbf{m}}(A)}(w) = \sup_{\{m_1,m_2 | (m_1\nabla m_2)(A)=w\}} \min[\mu_{\widetilde{\mathbf{m}}_1}(m_1), \mu_{\widetilde{\mathbf{m}}_2}(m_2)]. \tag{52}$$

The α-cut of $\widetilde{\mathbf{m}}(A)$ is an interval ${}^\alpha\widetilde{\mathbf{m}}(A) = [{}^\alpha\widetilde{\mathbf{m}}(A)^-, {}^\alpha\widetilde{\mathbf{m}}(A)^+]$ with

$$\begin{aligned}{}^\alpha\widetilde{\mathbf{m}}(A)^- &= \min_{(m_1,m_2)\in{}^\alpha\widetilde{\mathbf{m}}_1\times{}^\alpha\widetilde{\mathbf{m}}_2} (m_1\nabla m_2)(A)\\ {}^\alpha\widetilde{\mathbf{m}}(A)^+ &= \max_{(m_1,m_2)\in{}^\alpha\widetilde{\mathbf{m}}_1\times{}^\alpha\widetilde{\mathbf{m}}_2} (m_1\nabla m_2)(A)\end{aligned}$$

It is therefore equal to $({}^\alpha\widetilde{\mathbf{m}}_1\nabla{}^\alpha\widetilde{\mathbf{m}}_2)(A)$, which may be calculated using one of the techniques described in Appendix A. This important result gives us a method for computing the combination of two FBSs to any degree of accuracy, by combining any number of α-cuts using the techniques developed for IBSs.

Note that, because the calculation of $(m_1\nabla m_2)(A)$ involves multiplications, $\widetilde{\mathbf{m}}(A)$ is not, in general, of type LL. However, when the precise form of the membership function of $\widetilde{\mathbf{m}}(A)$ is not regarded as important, it may be sufficient to approximate it by an LL fuzzy number with the same core and support, as suggested by Dubois and Prade [11].

Example 5 Let $\widetilde{\mathbf{m}}$ denote the FBS already considered in Examples 3 and 4, and $\widetilde{\mathbf{m}}'$ the FBS defined by the trapezoidal fuzzy belief numbers

$$\widetilde{\mathbf{m}}'(F_1') \triangleq (0.5, 0.6.0.3, 0.3)_{LL} \quad \widetilde{\mathbf{m}}'(F_2') \triangleq (0.3, 0.5.0.3, 0.3)_{LL}$$

with

$$F_1' \triangleq \{0.1/2, 0.5/3, 1/4, 1/5, 0.5/6, 0.1/7\} \quad F_2' \triangleq \Omega.$$

Let $\widetilde{\mathbf{m}}'' \triangleq \widetilde{\mathbf{m}} \cap \widetilde{\mathbf{m}}'$ be the conjunctive sum of $\widetilde{\mathbf{m}}$ and $\widetilde{\mathbf{m}}'$. It is a FBS with focal elements:

$$F_1'' = F_1 \cap F_1', \quad F_2'' = F_1, \quad F_3'' = F_2, \quad F_4'' = F_3,$$

where $\cap$ denotes standard fuzzy set intersection. For any $\alpha \in]0,1]$ and any focal element F_i'', the α-cut of $\widetilde{\mathbf{m}}''(F_i'')$ may easily be computed as:

$${}^\alpha\widetilde{\mathbf{m}}''(F_i'') = ({}^\alpha\widetilde{\mathbf{m}} \cap {}^\alpha\widetilde{\mathbf{m}}')(F_i''),$$

using the formula given in Section A.2. The resulting fuzzy belief numbers are shown in Figure 3.

4.4 Normalization of a FBS

The Dempster and Yager normalization procedures that were extended to IBSs in Section 3.4 may be further generalized to FBSs using, once again, the extension principle.

For example, let $\widetilde{\mathbf{m}}$ be a FBS with crisp focal elements. Its normalization using the Dempster procedure yields a normal FBS $\widetilde{\mathbf{m}}^*$ with focal elements $\mathcal{F}(\widetilde{\mathbf{m}}^*) = \mathcal{F}(\widetilde{\mathbf{m}}) \setminus \emptyset$, such that

$$\mu_{\widetilde{\mathbf{m}}^*(A)}(w) \triangleq \sup_{\{m | m(A)/(1-m(\emptyset))=w\}} \mu_{\widetilde{\mathbf{m}}}(m)$$

The α-cut of $\widetilde{\mathbf{m}}^*(A)$ is obviously an interval $[{}^\alpha\widetilde{\mathbf{m}}^*(A)^-, {}^\alpha\widetilde{\mathbf{m}}^*(A)^+]$, with

$$\begin{aligned} {}^\alpha\widetilde{\mathbf{m}}^*(A)^- &= \min_{m \in {}^\alpha\widetilde{\mathbf{m}}} \frac{m(A)}{1 - m(\emptyset)} \\ {}^\alpha\widetilde{\mathbf{m}}^*(A)^+ &= \max_{m \in {}^\alpha\widetilde{\mathbf{m}}} \frac{m(A)}{1 - m(\emptyset)} \end{aligned}$$

These bounds may be computed using (46) and (47). Note that, even when the masses $\widetilde{\mathbf{m}}(B)$ for $B \in \mathcal{F}(\widetilde{\mathbf{m}})$ are LL fuzzy numbers, $\widetilde{\mathbf{m}}^*(A)$ is not because its calculation involves a division. However, an LL fuzzy number with the same core and support as $\widetilde{\mathbf{m}}^*(A)$ may here again easily be computed as an approximation.

The same approach may be used to extend the soft normalization and Yager normalization procedures to FBSs. Yager normalization has the computational advantage of being based only on additions and subtractions, which allows to perform exact computations with LL parameterization of fuzzy numbers.

Example 6 Let us consider a FBS $\widetilde{\mathbf{m}}$ defined on a possibility space $\Omega = \{1, 2, 3, 4\}$, with the following three subnormal fuzzy focal elements:

$$F_1 = \{0.1/1, 0.4/2, 0.5/3, 0.2/4\} \quad F_2 = \{0.3/1, 0.6/2, 0.7/3, 0.4/4\}$$

$$F_3 = \{0.1/1, 0.3/2, 0.2/3, 0.1/4\}$$

and the following triangular LL fuzzy belief numbers:

$$\widetilde{\mathbf{m}}(F_1) = (0.5, 0.3, 0.3)_{LL} \quad \widetilde{\mathbf{m}}(F_2) = (0.2, 0.2, 0.3)_{LL}$$

$$\widetilde{\mathbf{m}}(F_3) = (0.3, 0.2, 0.2)_{LL}.$$

The soft Dempster normalization procedure leads to a normal FBS $\widetilde{\mathbf{m}}^*$ with focal elements

$$F_1^* = \{0.2/1, 0.8/2, 1/3, 0.4/4\} \quad F_2^* = \{0.43/1, 0.86/2, 1/3, 0.57/4\}$$

$$F_3^* = \{0.33/1, 1/2, 0.67/3, 0.33/4\}$$

and the fuzzy belief numbers plotted in Figure 4.

In contrast, the application of the Yager procedure yields to a FBS $\widetilde{\mathbf{m}}^\circ$ with only two focal elements, since

$$F_1^\circ = F_2^\circ = \{0.6/1, 0.9/2, 1/3, 0.7/4\} \quad F_3^\circ = \{0.8/1, 1/2, 0.9/3, 0.8/4\}.$$

The corresponding belief masses are triangular LL fuzzy numbers as shown in Figure 5.

5 Applications

5.1 Decision analysis

Decision making under uncertainty is a fundamental problem, whose importance in a wide range of applications cannot be overestimated. As already mentioned in Section 2.1, this problem in handled in the TBM by means of the transformation of a belief function describing Your state of belief into a probability function. Assuming the consequences of each action to be quantified by a utility function, the decision strategy is then based on the principle of expected utility maximization as in the classical Bayesian decision theory. In this section, we extend this approach to the case where beliefs are represented by a FBS. In the spirit of previous applications of Fuzzy Set theory to decision analysis [29, 13, 7, 12], we shall also allow the utilities to be described in terms of fuzzy numbers.

Let $\mathcal{A}$ denote the finite set of actions, $\Omega = \{1, \ldots, M\}$ the possibility space, and

$$u : \mathcal{A} \times \Omega \mapsto \mathbb{R}$$

a (crisp) utility function quantifying the consequences of each action under each state of nature (or value of the unknown quantity of interest Y).

Let us first assume Your beliefs to be represented by a BS with crisp or fuzzy focal elements $F_1, \ldots, F_n$. As already mentioned in Section 2.2, the concept of pignistic probability distribution originally defined by (9) may be generalized in this case by replacing crisp intersection and cardinality by their fuzzy counterparts, leading to:

$$\mathrm{BetP}(j) = \sum_{i=1}^{n} m(F_i) \frac{\mu_{F_i}(j)}{|F_i|} \quad \forall j \in \Omega \tag{53}$$

with $|F_i| = \sum_{k=1}^{n} \mu_{F_i}(k)$. The expected utility of each action $a \in \mathcal{A}$ is then equal to

$$\begin{aligned} U(\alpha) &= \sum_{j=1}^{M} \mathrm{BetP}(j) u(a,j) && (54) \\ &= \sum_{j=1}^{M} \sum_{i=1}^{n} m(F_i) \frac{\mu_{F_i}(j)}{|F_i|} u(a,j) && (55) \\ &= \sum_{i=1}^{n} m(F_i) x_i && (56) \end{aligned}$$

with

$$x_i = \frac{1}{|F_i|} \sum_{j=1}^{M} \mu_{F_i}(j) u(a,j)$$

Using the extension principle, (56) can now be easily generalized to the case where both utilities and belief masses are fuzzy numbers. Let $\widetilde{x}_i$ be the fuzzy number defined by

$$\widetilde{x}_i = \frac{1}{|F_i|} \sum_{j=1}^{M} \mu_{F_i}(j) \widetilde{u}(a,j),$$

where $\widetilde{u}(a,j)$ denotes the fuzzy utility of action a when $Y=j$, and let $\widetilde{\mathbf{m}}$ be a FBS. The fuzzy expected utility $\widetilde{U}(a)$ of action a is then defined as:

$$\mu_{\widetilde{U}(a)}(w) = \sup_{\substack{u_1,\ldots,u_n\\ v_1,\ldots,v_n}} \min_i \min\left(\mu_{\widetilde{x}_i}(u_i), \mu_{\widetilde{\mathbf{m}}(F_i)}(v_i)\right), \tag{57}$$

where the supremum is taken under the constraints

$$\begin{cases} \sum_{i=1}^n u_i v_i = w \\ \sum_{i=1}^n v_i = 1. \end{cases}$$

As shown by Dubois and Prade [11, p. 56], this membership function defines a fuzzy number, whose α-cuts may be computed using a variant of Theorem 1. More precisely, ${}^\alpha\widetilde{U}(a)$ is obtained as:

$${}^\alpha\widetilde{U}(a) = \left[\min\sum_{i=1}^n m(F_i)^\alpha x_i^-, \max\sum_{i=1}^n m(F_i)^\alpha x_i^+\right]$$

where the minimum and maximum are taken under the constraint $m \in {}^\alpha\widetilde{\mathbf{m}}$.

The final step in the decision process consists in the comparison of the obtained fuzzy expected utilities. As remarked by Klir [16], many methods for total ordering of fuzzy numbers have been suggested in the literature, without any method unquestionably emerging as the best one in all cases. A cautious (though not always applicable) approach might be to restrict oneself to a partial ordering, such as $\widetilde{x} \geq \widetilde{y}$ iif ${}^\alpha\widetilde{x}^- \geq {}^\alpha\widetilde{y}^-$ and ${}^\alpha\widetilde{x}^+ \geq {}^\alpha\widetilde{y}^+$, and admit indeterminacy when two fuzzy expected utilities are not comparable.

Example 7 Let $\widetilde{\mathbf{m}}$ be a FBS on $\Omega = \{1,2,3\}$ with focal elements

$$F_1 = \{1/1, 0.5/2, 0/3\} \quad F_2 = \{0.6/1, 1/2, 0.3/3\} \quad F_3 = \{0.1/1, 0.4/2, 1/3\}$$

and triangular fuzzy belief numbers

$$\widetilde{\mathbf{m}}(F_1) = (0.5, 0.2, 0.2)_{LL} \quad \widetilde{\mathbf{m}}(F_2) = (0.3, 0.2, 0.2)_{LL} \quad \widetilde{\mathbf{m}}(F_3) = (0.2, 0.2, 0.2)_{LL}.$$

Let $\mathcal{A} = \{a_1, a_2, a_3\}$ be the set of actions, and assume the fuzzy utilities to be defined as follows:

$$\begin{array}{lll}
\widetilde{u}_{1,1} = (1, 0.25, 0.25)_{LL} & \widetilde{u}_{1,2} = (0.5, 0.25, 0.25)_{LL} & \widetilde{u}_{1,3} = (0, 0, 0.25)_{LL} \\
\widetilde{u}_{2,1} = (0.75, 0.25, 0.25)_{LL} & \widetilde{u}_{2,2} = (0, 0, 0.25)_{LL} & \widetilde{u}_{2,3} = (0.75, 0.25, 0.25)_{LL} \\
\widetilde{u}_{3,1} = (0, 0, 0.25)_{LL} & \widetilde{u}_{3,2} = (0.5, 0.25, 0.25)_{LL} & \widetilde{u}_{3,3} = (1, 0.25, 0.25)_{LL}
\end{array}$$

where $\widetilde{u}_{i,j} = \widetilde{u}(a_i, j)$.

The application of the above method leads to the three fuzzy expected utilities plotted in Figure 6. It is clear in this case that

$$\widetilde{U}(\alpha_1) \geq \widetilde{U}(\alpha_2) \geq \widetilde{U}(\alpha_3)$$

which leads to the prescription of action α_1.

5.2 Pattern classification

Pattern classification is a very general task whose goal is to assign entities, represented by feature vectors, to predefined groups or categories. In the classical approach to statistical pattern recognition (including neural network techniques), a classification rule is learnt automatically from a set of N patterns with known classification. Mathematically, such training data may be represented by a set

$$\mathcal{T} = \{(\mathbf{x}^i, y^i) | 1 \leq i \leq N\},$$

where $\mathbf{x}^i \in \mathbb{R}^d$ is a real-valued feature vector, and $y^i \in \Omega = \{1, \ldots, M\}$ denotes the class of the corresponding entity.

In some applications, however, the nature of the available knowledge does not allow the construction of such a training set. For example, when the training samples are labeled by an expert and only partial information is available, the expert may only be able to provide *imprecise* statements concerning the class membership of example i, such as: $y^i \in A^i$, where A^i is a subset of Ω. We then have a training set of a different kind, in which each sample is no longer labeled with a single class, but with a set of classes.

In previous papers [2, 3], we have proposed a solution to this problem using the theory of belief functions. In this approach, each training sample $z^i = (\mathbf{x}^i, A^i)$ is considered as an item of evidence regarding the class membership of each new vector $\mathbf{x}$ to be classified. This evidence is represented by a BS $m(\cdot|z^i)$ defined as a function of the dissimilarity (according to some relevant measure δ) between vectors $\mathbf{x}$ and $\mathbf{x}^i$:

$$m(A|z^i) = \begin{cases} \varphi(\delta(\mathbf{x}, \mathbf{x}^i)) & \text{if } A = A^i \\ 1 - \varphi(\delta(\mathbf{x}, \mathbf{x}^i)) & \text{if } A = \Omega \\ 0 & \text{otherwise} \end{cases} \tag{58}$$

where φ is a decreasing function verifying $\varphi(0) \leq 1$ and $\lim_{d \to \infty} \varphi(d) = 0$. When δ denotes the squared Euclidean distance, a rational choice for φ was shown in [5] to be:

$$\varphi(d) = \beta \exp(-\gamma d) \tag{59}$$

where β and γ are parameters that may be learnt form training data [40]. The BSs induced by each learning sample are then combined using the conjunctive sum operation (with or without normalization):

$$m = m(\cdot|z^1) \cap \ldots \cap m(\cdot|z^N). \tag{60}$$

This approach may easily be extended to the case where the available knowledge concerning the class membership of training patterns is expressed by linguistic terms and is properly described by fuzzy class labels [39]. For example, if Ω is the base set of a variable Y describing the degree of gravity of a disease, one may only know that example i corresponds to a "serious" case, and represent this information in the form a fuzzy subset A^i of Ω. In that case (58) may still be used, with the difference that $m(\cdot|z^i)$ and the resulting BS m are now BSs with fuzzy focal elements.

The concept of FBS introduced in this paper allows us to model an even more general situation, in which *some components of the pattern vectors are themselves* *tainted with imprecision or uncertainty.* Such a situation may occur, for example, in sensor fusion applications where heterogeneous data coming from a variety of sensors (including humans or knowledge-base systems encapsulating human expertise) have to be taken into account [14]. In other applications, fuzzy numbers may also arise as imprecise estimates of missing features computed by a fuzzy system [19].

More precisely, let us consider a training set

$$\mathcal{T} = \{z^i = (\widetilde{\mathbf{x}}^i, A^i) | 1 \le i \le N\}$$

of N fuzzy feature vectors $\widetilde{\mathbf{x}}^i = (\widetilde{x}^i_1, \ldots, \widetilde{x}^i_d)^t$ and associated fuzzy class labels $A^i \in [0,1]^\Omega$, and assume that we wish to determine our beliefs concerning the class membership of a new entity (or test pattern) described by a fuzzy feature vector $\widetilde{\mathbf{x}}$. According to the extension principle, the dissimilarity between fuzzy vectors $\widetilde{\mathbf{x}}$ and $\widetilde{\mathbf{x}}^i$ may now be defined as a fuzzy number[2] $\widetilde{\delta}^i = \delta(\widetilde{\mathbf{x}}, \widetilde{\mathbf{x}}^i)$ with membership function:

$$\mu_{\widetilde{\delta}^i}(w) = \sup_{\substack{x^i_1,\ldots,x^i_d \\ x_1,\ldots,x_d}} \min_j \min(\mu_{\widetilde{x}^i_j}(x^i_j), \mu_{\widetilde{x}_j}(x_j)),$$

where the supremum is taken under the constraint $\delta(\mathbf{x}^i, \mathbf{x}) = w$. The application of function φ to $\widetilde{\delta}^i$ defines a new fuzzy number $\varphi(\widetilde{\delta}^i)$ that may itself be introduced into (58), yielding a FBS $\widetilde{\mathbf{m}}(\cdot|z^i)$. Assuming the exponential form of (59) for φ, the α-cut of $\widetilde{\mathbf{m}}(\cdot|z^i)$ is given by:

$$^\alpha\widetilde{\mathbf{m}}(A|z^i) = \begin{cases} \left[\beta \exp(-\gamma\ ^\alpha\widetilde{\delta}^{i+}), \beta \exp(-\gamma\ ^\alpha\widetilde{\delta}^{i-})\right] & \text{if } A = A^i \\ \left[1 - \beta \exp(-\gamma\ ^\alpha\widetilde{\delta}^{i-}), 1 - \beta \exp(-\gamma\ ^\alpha\widetilde{\delta}^{i+})\right] & \text{if } A = \Omega \\ 0 & \text{otherwise} \end{cases} \tag{61}$$

where $\widetilde{\delta}^i$ is the fuzzy squared Euclidean distance between $\widetilde{\mathbf{x}}$ and $\widetilde{\mathbf{x}}^i$. Our final belief regarding the class of $\widetilde{\mathbf{x}}$ may then be described by the FBS obtained as the conjunctive sum of the $\widetilde{\mathbf{m}}(\cdot|z^i)$:

$$\widetilde{\mathbf{m}} = \widetilde{\mathbf{m}}(\cdot|z^1) \cap \ldots \cap \widetilde{\mathbf{m}}(\cdot|z^N). \tag{62}$$

Note that each of the FBS $\widetilde{\mathbf{m}}(\cdot|z^i)$ is simple (it has only one focal element in addition of Ω). Hence, the combination of these FBSs may be computed very effectively using the formula given in Section A.2 of the Appendix.

If the ultimate goal of the classification task is to assign vector $\widetilde{\mathbf{x}}$ to a single class, then one may adopt the decision-making approach described in the previous section, after the consequences of different kinds of correct or wrong classifications have been quantified by a (possibly fuzzy) utility function.

EXAMPLE 8 To illustrate this approach, let us take as an example the simple data set shown in Figure 7 and described in Table 1. It is a three-class problem composed of six examples with triangular fuzzy features and fuzzy class labels. We consider

[2] This fuzzy number may be interpreted as a fuzzy constraint acting on the dissimilarity between feature vectors $\mathbf{x}$ and $\mathbf{x}^i$, whose components are themselves constrained by fuzzy numbers. This notion should not be confused with that of distance between two fuzzy numbers, which was introduced to quantify the dissimilarity between two membership functions.

the problem of determining the class membership of a test fuzzy feature vector $\widetilde{x}$ also shown in Figure 7. The results obtained with our method (applied with $\beta = 0.9$ and $\gamma = 1$) are shown in Figures 8 and 9. Figure 8 depicts the final FBS resulting from the combination of the elementary FBSs induced by each of the six examples. The fuzzy pignistic probabilities of each class are then represented in Figure 9. It is clear that, in this case, the unknown pattern should be assigned to class 3.

6 Conclusions

This paper has introduced the new concepts of interval-valued and fuzzy-valued belief structures, defined, respectively, as crisp and fuzzy sets of belief structures verifying hard or elastic constraints. These mathematical objects may be seen as generalized belief structures for which extensions of the classical notions of credibility, plausibility, conjunctive or disjunctive sum, and normalization procedures may be defined. These concepts constitute a very flexible framework allowing to express, and reason with partially specified degrees of belief assigned to imprecise propositions. Most calculations implied by the manipulation of these concepts are based on simple forms of linear programming problems for which analytical solutions exist, making the whole scheme computationally tractable. An interesting application area concerns the development of new tools for fuzzy data analysis, allowing to process complex data such as sets of examples described by fuzzy feature vectors and fuzzy class labels. The possibility to describe states of belief using verbal statements is also expected to be useful in all situations involving the elicitation of degrees of belief from experts, such as encountered in the development and operation of diagnosis and decision support systems.

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A Procedures for combining IBSs

A.1 Problem formulation

Let $\mathbf{m}_1$ and $\mathbf{m}_2$ be two IBSs defined on Ω, and $\mathbf{m} = \mathbf{m}_1 \nabla \mathbf{m}_2$ for some set operation ∇. The practical determination of $m^-(A)$ and $m^+(A)$ for an arbitrary subset A of Ω requires to search for the extrema of

$$\varphi_A(m_1, m_2) = \sum_{B \nabla C = A} m_1(B) m_2(C) \tag{63}$$

under the constraints:

$$\begin{aligned} \sum_{B \in \mathcal{F}(\mathbf{m}_1)} m_1(B) &= 1 \\ \sum_{C \in \mathcal{F}(\mathbf{m}_2)} m_2(C) &= 1 \end{aligned}$$

$$\begin{array}{ccccc} m_1^-(B) & \leq & m_1(B) & \leq & m_1^+(B) \quad \forall B \in \mathcal{F}(\mathbf{m}_1) \\ m_2^-(C) & \leq & m_2(C) & \leq & m_2^+(C) \quad \forall C \in \mathcal{F}(\mathbf{m}_2). \end{array}$$

The solution of this problem is trivial when the sum in the right-hand side of (63) contains only one term, since we then simply have a product of two non interactive variables. A more general result with high practical interest was obtained in [4], in which we derived an analytical expression for $(\mathbf{m}_1 \cap \mathbf{m}_2)(A)$ in the case where $\mathbf{m}_2$ is a *simple* IBS, i.e., $\mathcal{F}(\mathbf{m}_2) = \{F, \Omega\}$ for some $F \in [0,1]^\Omega$. This result is given in the next section.

A.2 Conjunctive sum of an arbitrary IBS with a simple IBS

Let us consider the simple case where an arbitrary IBS $\mathbf{m}_1$ is combined (according to the conjunctive sum operation) with a *simple* IBS $\mathbf{m}_2$ with $\mathcal{F}(\mathbf{m}_2) = \{A, \Omega\}$ $(A \subset \Omega)$. Let us denote $\mathbf{m} = \mathbf{m}_1 \cap \mathbf{m}_2$. For any $B \subseteq \Omega$, $m_1 \in \mathbf{m}_1$ and $m_2 \in \mathbf{m}_2$, we then have:

$$\begin{aligned} m(B) &= \sum_{C \cap D = B} m_1(C) m_2(D) && (64) \\ &= m_2(A) \sum_{A \cap C = B} m_1(C) + m_2(\Omega) m_1(B) && (65) \end{aligned}$$

To find the minimum and maximum of $m(B)$, let us consider two cases.

Case 1: $B \not\subseteq A$. In that case, the first term in (65) vanishes, and we have:

$$m(B) = m_2(\Omega) m_1(B)$$

It is then obvious that

$$\begin{aligned} m^-(B) &= m_2^-(\Omega) m_1^-(B) && (66) \\ m^+(B) &= m_2^+(\Omega) m_1^+(B) && (67) \end{aligned}$$

Case 2: $B \subseteq A$. As proved in [4], the bounds of m in (65) are given by:

$$\begin{aligned} m^-(B) &= \max(X_1, X_2, X_3) \\ m^+(B) &= \min(Y_1, Y_2, Y_3) \end{aligned}$$

with

$$\begin{aligned} X_1 &= m_1^-(B) + m_2^-(A) \sum_{\substack{C: A\cap C=B \\ C\neq B}} m_1^-(C) \\ X_2 &= m_1^-(B) + m_2^-(A)\left(1 - \sum_{A\cap C\neq B} m_1^+(C) - m_1^-(B)\right) \\ X_3 &= 1 - \sum_{A\cap C\neq B} m_1^+(C) + (m_2^-(A) - 1) \sum_{\substack{C: A\cap C=B \\ C\neq B}} m_1^+(C) \end{aligned}$$

and

$$\begin{aligned} Y_1 &= m_1^+(B) + m_2^+(A) \sum_{\substack{C: A\cap C=B \\ C\neq B}} m_1^+(C) \\ Y_2 &= m_1^+(B) + m_2^+(A)\left(1 - \sum_{A\cap C\neq B} m_1^-(C) - m_1^+(B)\right) \\ Y_3 &= 1 - \sum_{A\cap C\neq B} m_1^-(C) + (m_2^+(A) - 1) \sum_{\substack{C: A\cap C=B \\ C\neq B}} m_1^-(C) \end{aligned}$$

A.3 General case

In the most general case, an explicit solution to the quadratic programming problem described in Section A.1 is difficult to obtain, and one has to resort to iterative numerical procedures. Whereas general non linear optimization algorithms may, of course, be used, the particular form of this problem suggests to apply the following alternate directions scheme [4].

Consider for example the *minimization* of $\varphi_A(m_1, m_2)$. Let us fix m_1 and m_2 to some admissible values $m_1^{(0)}$ and $m_2^{(0)}$, respectively. Then $\varphi_A(m_1, m_2^{(0)})$ is a linear function of the $m_1(B)$, for $B \in \mathcal{F}(m_1)$:

$$\varphi_A(m_1, m_2^{(0)}) = \sum_{B\in\mathcal{F}(m_1)} m_1(B)\left(\sum_{B\nabla C=A} m_2^{(0)}(C)\right)$$

The search for m_1 minimizing this expression is a linear programming problem that may be solved directly using Theorem 1. Let $m_1^{(1)}$ be a solution (if $m_1^{(0)}$ was already a solution, then we pose $m_1^{(1)} = m_1^{(0)}$). We then proceed by searching $m_2^{(1)}$ minimizing $\varphi_A(m_1^{(1)}, m_2)$. The procedure is iterated until a fixed point has been found, i.e., until we have reached k such that $m_1^{(k)} = m_1^{(k-1)}$ and $m_2^{(k)} = m_2^{(k-1)}$, which was shown in [4] to happen in any case after a finite number of iterations. This method proved experimentally to be significantly faster for that particular problem than general purpose constrained optimization algorithms such as sequential quadratic programming techniques. The generalization of this algorithm to the combination of n IBS is straightforward [4].

References

[1] A. P. Dempster. Upper and lower probabilities induced by a multivalued mapping. *Annals of Mathematical Statistics*, AMS-38:325–339, 1967.

[2] T. Denœux. A k-nearest neighbor classification rule based on Dempster-Shafer theory. *IEEE Transactions on Systems, Man and Cybernetics*, 25(05):804–813, 1995.

[3] T. Denœux. Analysis of evidence-theoretic decision rules for pattern classification. *Pattern Recognition*, 30(7):1095–1107, 1997.

[4] T. Denœux. Reasoning with imprecise belief structures. Technical Report 97/44 (`http://www.hds.utc.fr/~tdenoeux/`), Université de Technologie de Compiègne, Heudiasyc Laboratory, September 1997.

[5] T. Denœux. Application du modèle des croyances transférables en reconnaissance de formes. *Traitement du Signal*, 14(5):443–451, 1998.

[6] D. Dubois and H. Prade. Addition of interactive fuzzy numbers. *IEEE Transactions on Automatic Control*, 26(4):926–936, 1981.

[7] D. Dubois and H. Prade. The use of fuzzy numbers in decision analysis. In M. M. Gupta and E. Sanchez, editors, *Fuzzy Information and Decision Processes*, pages 309–321. North-holland, New-York, 1982.

[8] D. Dubois and H. Prade. Evidence measures based on fuzzy information. *Automatica*, 21(5):547–562, 1985.

[9] D. Dubois and H. Prade. A set-theoretic view of belief functions: logical operations and approximations by fuzzy sets. *International Journal of General Systems*, 12:193–226, 1986.

[10] D. Dubois and H. Prade. An alternative approach to the handling of subnormal possibility distributions. *Fuzzy Sets and Systems*, 24:123–126, 1987.

[11] D. Dubois and H. Prade. *Possibility Theory: An approach to computerized processing of uncertainty.* Plenum Press, New-York, 1988.

[12] D. Dubois and H. Prade. Decision evaluation under uncertainty and imprecision. In J. Kacprzyk and M. Fedrizzi, editors, *Combining Fuzzy Imprecision with Probabilistic Uncertainty in decision Making*, pages 48–64. Springer-Verlag, 1989.

[13] A. N. S. Freeling. Fuzzy sets and decision analysis. *IEEE Transactions on Systems, Man and Cybernetics*, 10(7):341–354, 1980.

[14] R. J. Hathaway, J. C. Bezdek, and W. Pedrycz. A parametric model for fusing heterogeneous fuzzy data. *IEEE Transactions on Fuzzy Systems*, 4(3):270–281, 1996.

[15] M. Ishizuka and K. S. Fu. Inference procedures under uncertainty for the problem-reduction method. *Information Sciences*, 28:179–206, 1982.

[16] G. J. Klir and B. Yuan. *Fuzzy sets and fuzzy logic. Theory and applications.* Prentice-Hall, Upper Saddle River, NJ., 1995.

[17] M. Lamata and S. Moral. Calculus with linguistic probabilities and beliefs. In R. R. Yager, M. Fedrizzi, and J. Kacprzyk, editors, *Advances in the Dempster-Shafer theory of evidence*, pages 133–152. Wiley, New-York, 1994.

[18] H. Ogawa, K. S. Fu, and J. T. P. Yao. An inexact inference for damage assessment of existing structures. *Int. J. Man-Machine Studies*, 22:295–306, 1985.

[19] S. Petit-Renaud and T. Denœux. A fuzzy-neuro system for reconstruction of multi-sensor information. In *Proceedings of Fuzzy-Neuro Systems'98*, Munich, March 1998.

[20] G. Shafer. *A mathematical theory of evidence.* Princeton University Press, Princeton, N.J., 1976.

[21] P. Smets. The degree of belief in a fuzzy event. *Information Sciences*, 25:1–19, 1981.

[22] P. Smets. The combination of evidence in the Transferable Belief Model. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 12(5):447–458, 1990.

[23] P. Smets. Belief functions: the disjunctive rule of combination and the generalized Bayesian theorem. *International Journal of Approximate Reasoning*, 9:1–35, 1993.

[24] P. Smets. What is Dempster-Shafer's model ? In R. R. Yager, M. Fedrizzi, and J. Kacprzyk, editors, *Advances in the Dempster-Shafer theory of evidence*, pages 5–34. Wiley, New-York, 1994.

[25] P. Smets. The application of the Transferable Belief Model to diagnosis problems. Technical Report TR/IRIDIA/97-2, IRIDIA, Bruxelles, 1997.

[26] P. Smets. The normative representation of quantified beliefs by belief functions. *Artificial Intelligence*, 92(1–2):229–242, 1997.

[27] P. Smets and R. Kennes. The Transferable Belief Model. *Artificial Intelligence*, 66:191–243, 1994.

[28] P. Walley. *Statistical Reasoning with Imprecise Probabilities.* Chapman and Hall, London, 1991.

[29] S. R. Watson, J. J. Weiss, and M. L. Donnell. Fuzzy decision analysis. *IEEE Transactions on Systems, Man and Cybernetics*, 9(1):1–9, 1979.

[30] R. R. Yager. Generalized probabilities of fuzzy events from fuzzy belief structures. *Information Sciences*, 28:45–62, 1982.

[31] R. R. Yager. Arithmetic and other operations on Dempster-Shafer structures. *Int. J. Man-Machines Studies*, 25:357–366, 1986.

[32] R. R. Yager. On the normalization of fuzzy belief structure. *International Journal of Approximate Reasoning*, 14:127–153, 1996.

[33] R. R. Yager and D. P. Filev. Including probabilistic uncertainty in fuzzy logic controller modeling using Dempster-Shafer theory. *IEEE Transactions on Systems, Man and Cybernetics*, 25(8):1221–1230, 1995.

[34] J. Yen. Generalizing the Dempster-Shafer theory to fuzzy sets. *IEEE Transactions on Systems, Man and Cybernetics*, 20(3):559–569, 1990.

[35] L. A. Zadeh. Probability measures of fuzzy events. *J. Math. Analysis and Appl.*, 10:421–427, 1968.

[36] L. A. Zadeh. The concept of a linguistic variable and its application to approximate reasoning (Part 1). *Information Sciences*, 8:199–249, 1975.

[37] L. A. Zadeh. Fuzzy sets as a basis for a theory of possibility. *Fuzzy Sets and Systems*, 1:3–28, 1978.

[38] L. A. Zadeh. Fuzzy sets and information granularity. In R. K. Ragade M. M. Gupta and R. R. Yager, editors, *Advances in Fuzzy Sets Theory and Applications*, pages 3–18. North-Holland, Amsterdam, 1979.

[39] L. M. Zouhal and T. Denœux. Generalizing the evidence-theoretic k-NN rule to fuzzy pattern recognition. In *Proceedings of the Second International Symposium on Fuzzy Logic and Applications ISFL'97*, pages 294–300, Zurich, February 1997. ICSC Academic Press.

[40] L. M. Zouhal and T. Denœux. An evidence-theoretic k-NN rule with parameter optimization. *IEEE Transactions on Systems, Man and Cybernetics – Part C*, 28(2):263–271, 1998.

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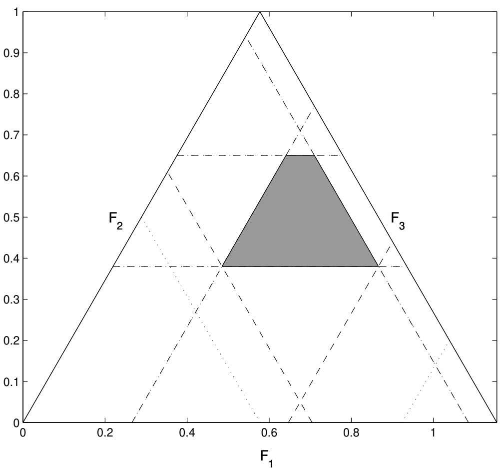

Figure 1: Representation of an IBS (gray region) in the probability simplex. Original inequality constraints and tightest bounds are represented as dotted and dashed lines, respectively

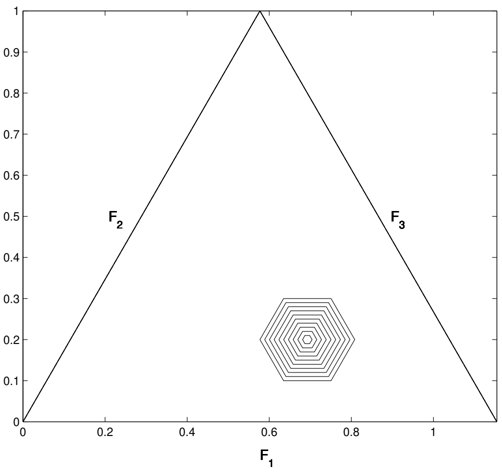

Figure 2: Representation of a FBS in the probability simplex.

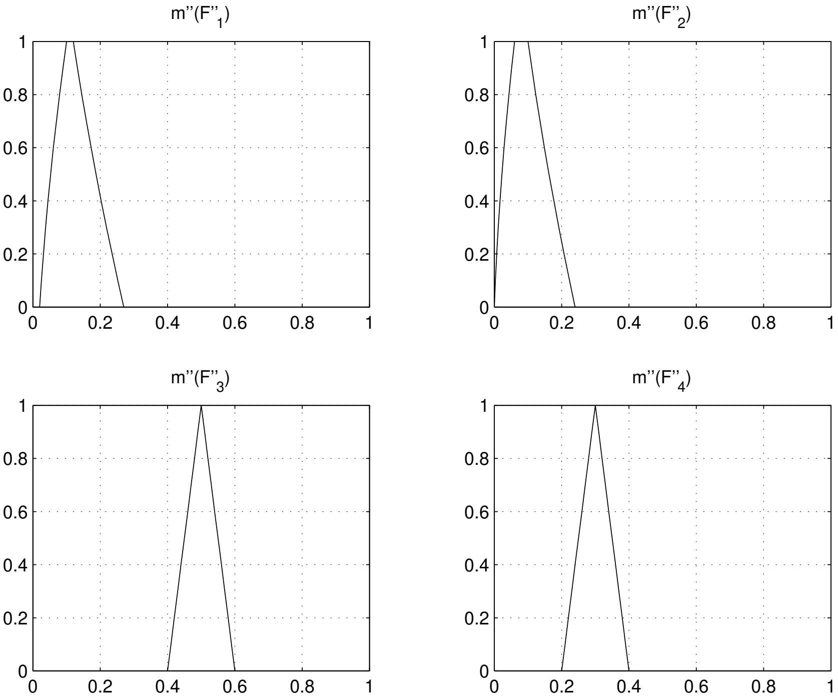

Figure 3: Result of the combination of two FBSs.

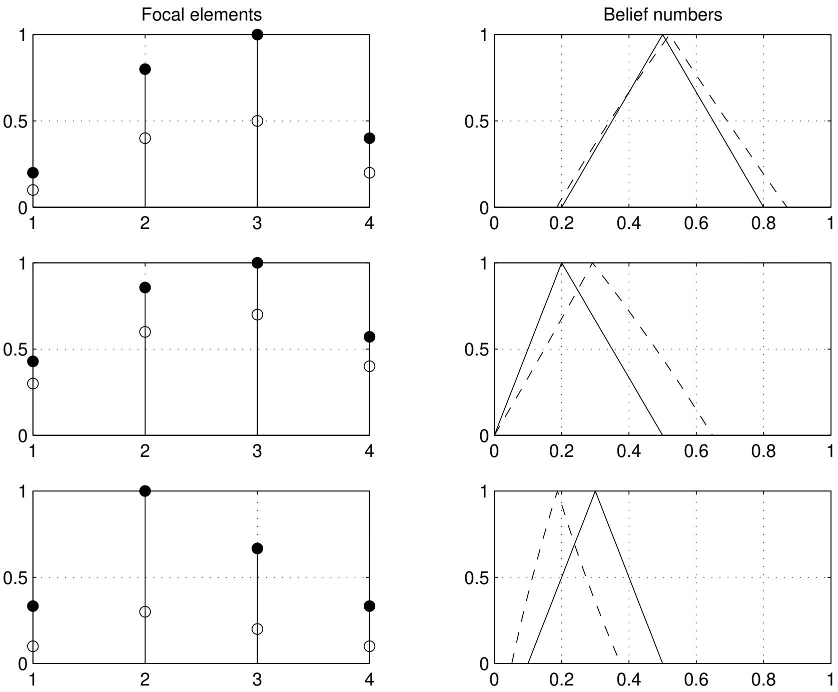

Figure 4: Soft Dempster normalization of a FBS. Left: original subnormal focal elements F_i (white circles), and normalized focal elements F_i^* (black circles). Right: fuzzy belief numbers before (solid lines) and after (dashed lines) normalization.

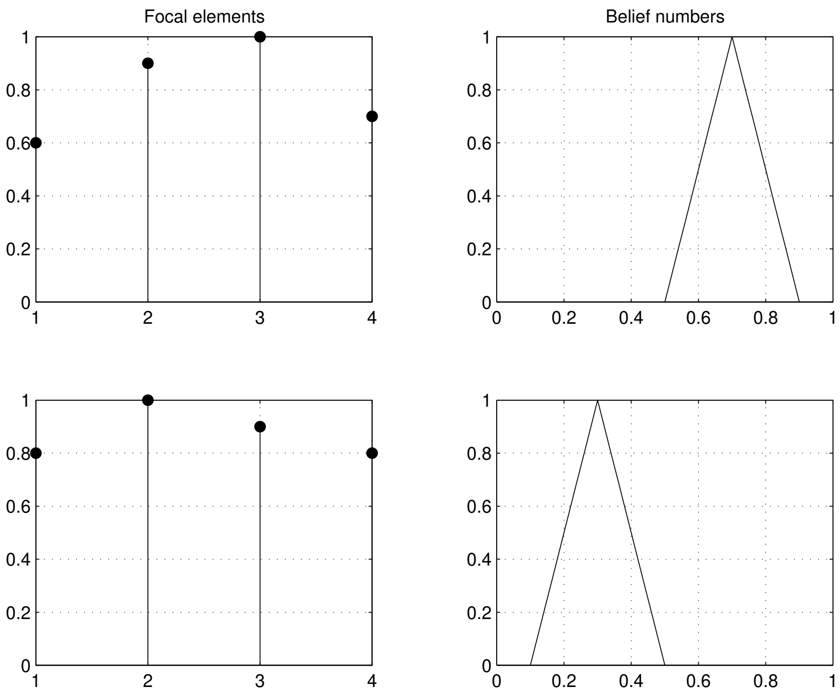

Figure 5: Yager normalization of the FBS shown in Figure 4. Left: normalized focal elements F_i°. Right: fuzzy belief numbers $\widetilde{\mathbf{m}}^\circ(F_i^\circ)$ after normalization.

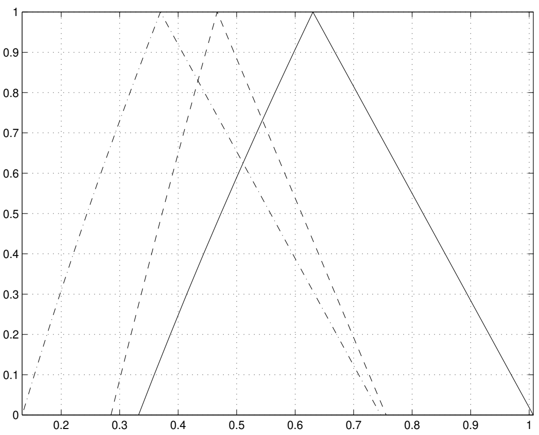

Figure 6: Fuzzy expected utilities for actions α_1 (-), α_2 (- -) and α_3 (-.).

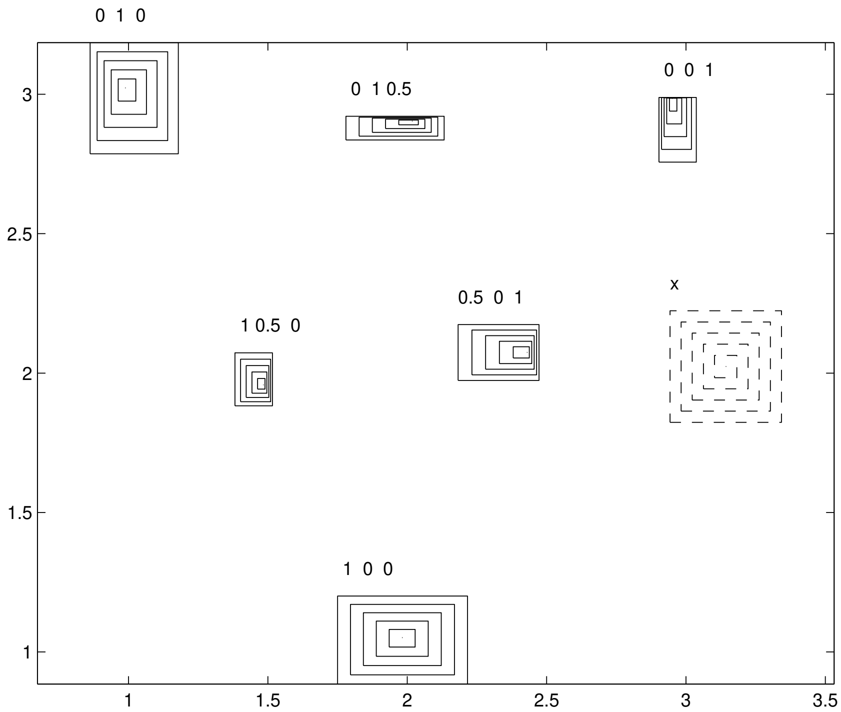

Figure 7: Example data set. Each fuzzy training vector is represented by 6 α-cuts of the Cartesian product of its components. The fuzzy labels are shown for each training pattern as the degrees of membership to each of the three classes. The test pattern is represented by rectangles with dashed lines.

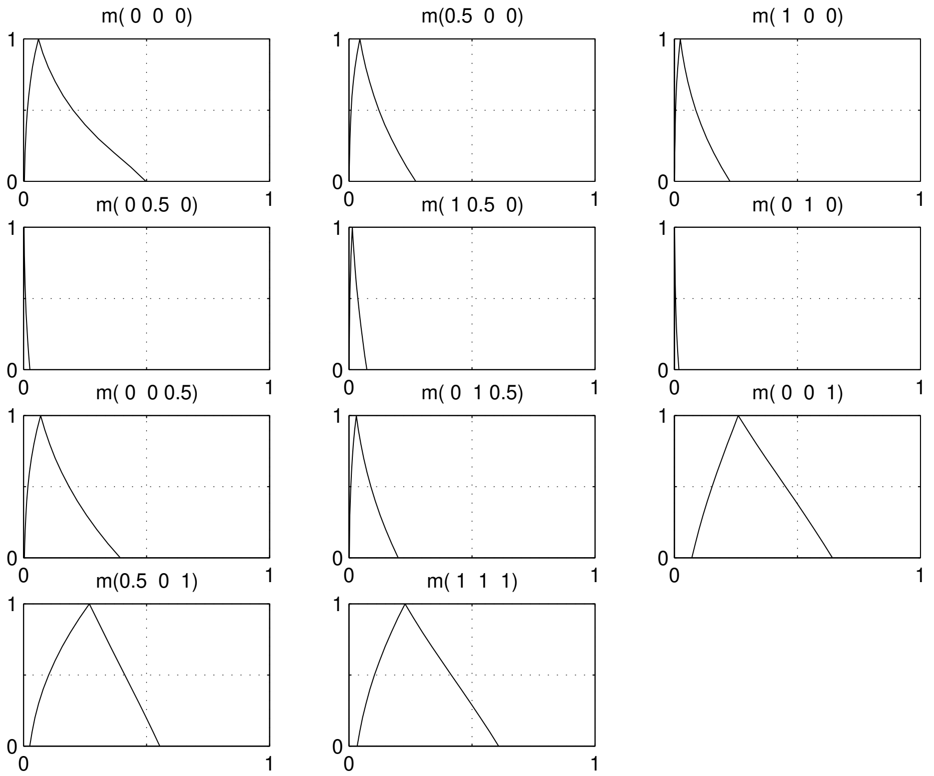

Figure 8: Resulting FBS computed for the test pattern. Each curve is the membership function (approximated with 11 α-cuts) of a fuzzy belief number assigned to a fuzzy focal element (indicated above each figure).

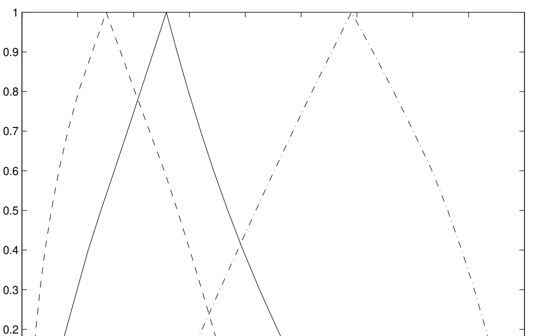

Figure 9: Fuzzy pignistic probabilities of class 1 (-), class 2 (- -) and class 3 (-.) computed for the test pattern.

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Table 1: Data set (the last line corresponds to the test pattern).

$\widetilde{x}_1^i$	$\widetilde{x}_2^i$	A^i
$(1.98, 0.23, 0.23)_{LL}$	$(1.05, 0.17, 0.15)_{LL}$	$\{1\}$
$(1.48, 0.10, 0.03)_{LL}$	$(1.96, 0.08, 0.12)_{LL}$	$\{0.5/1, 0.5/2\}$
$(0.99, 0.13, 0.19)_{LL}$	$(3.02, 0.24, 0.16)_{LL}$	$\{2\}$
$(2.02, 0.24, 0.12)_{LL}$	$(2.90, 0.07, 0.02)_{LL}$	$\{1/2, 0.5/3\}$
$(2.95, 0.05, 0.09)_{LL}$	$(2.99, 0.23, 0.00)_{LL}$	$\{3\}$
$(2.43, 0.25, 0.04)_{LL}$	$(2.07, 0.10, 0.10)_{LL}$	$\{0.5/1, 1/3\}$
$(3.14, 0.20, 0.20)_{LL}$	$(2.02, 0.20, 0.20)_{LL}$	