Risk Assessment Based on Weak Information using Belief Functions: A Case Study in Water Treatment

Sabrina Démotier[*,†], Walter Schön[*] and Thierry Denœux[*1]

[*] UMR CNRS 6599 Heudiasyc
Université de Technologie de Compiègne
BP 20529 - F-60205 Compiègne cedex, France

[†] Suez Environment,
CIRSEE Information Technology Division,
38, rue du Président Wilson,
78230 Le Pecq, France

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[1]Corresponding author. Email: Thierry.Denoeux@hds.utc.fr. Fax: (33) 3 44 23 44 77.

Abstract

Whereas Probability theory has been very successful as a conceptual framework for risk analysis in many areas where a lot of experimental data and expert knowledge are available, it presents certain limitations in applications where only weak information can be obtained. One such application investigated in this paper is water treatment, a domain in which key information such as input water characteristics and failure rates of various chemical processes is often lacking. An approach to handle such problems is proposed, based on the Dempster-Shafer theory of belief functions. Belief functions are used to describe expert knowledge of treatment process efficiency, failure rates and latency times, as well as statistical data regarding input water quality. Evidential reasoning provides mechanisms to combine this information and assess the plausibility of various noncompliance scenarios. This methodology is shown to boil down to the probabilistic one where data of sufficient quality are available. This case study shows that belief function theory may be considered as a valuable framework for risk analysis studies in ill-structured or poorly informed application domains.
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1 Introduction

Although Probability Theory remains well established as a reasonably well founded conceptual framework for uncertainty management, a number of alternative theories have begun to appear towards the end of the 1970's, including Possibility theory [22] and the so-called Dempster-Shafer theory of belief functions, or Evidence theory [13]. These new theoretical developments have been motivated by the growing recognition that all forms of partial information are not easily amenable to representation in the probabilistic framework. This applies, for instance, to vague linguistic statements often used by experts to express their knowledge, such as "if x is small, then y is very likely to be large". An extreme situation is that of complete ignorance: a quantity x of interest may be totally unknown, and the representation of this lack of information by a probability distribution (even a uniform one) may be shown to lead to paradoxes (see [13] for detailed discussions on this topic). Among existing tools for uncertainty representation, belief functions appear to play a pivotal role as they generalize both probability and possibility distributions, allowing to represent various forms of uncertainty such as randomness, imprecision and vagueness [9, 10]. In recent years, belief functions have received a clear interpretation and an axiomatic justification in the Transferable Belief Model [20, 18], a normative, nonprobabilistic model for reasoning under uncertainty.

In spite of continued interest raised by belief functions since the publication of Shafer's book in 1976, real-life applications have up to now been rather limited and essentially confined to certain technical fields such as expert systems [15][2], pattern recognition [5][6] and data fusion [19][12]. In particular, applications of belief functions or other nonprobabilistic uncertainty representation frameworks in reliability engineering and risk analysis has been, up to now, very limited. This fact may be explained by the outstanding successes of probabilistic methods in such domains as transportation systems, where a lot of experimental data and expert knowledge are available. However, when attempting to transpose standard probabilistic tools to other types of applications, where data are scarce and expert knowledge is partial and imprecise, the analyst is faced with the relative inadequacy of standard probabilistic tools. For instance, only crude approximations of certain parameters of interest such as failure rates or latency times may be available, and treating such approximations as accurate data may impair the whole analysis and make its conclusions unreliable. The main thesis of this paper is that other uncertainty representation frameworks such as Belief Function theory can then provide more flexible and reliable tools.

The particular application considered in this paper concerns risk analysis in drinking water treatment, an issue of growing importance due to the general degradation of water sources, and the increasing public awareness of the impact of many contaminants on human health. To meet contractual requirements and internal objectives, water utilities now have not only to select a combination of treatment processes most appropriate to treat the contaminants found in raw water in normal operating conditions, but also to estimate the residual risk of producing water that momentarily does not meet drinking water standards, due to failure of a treatment process, or accidental pollution of the water source. Application of existing reliability and availability methods to this problem leads to computing the probability of producing noncompliant water, taking into account the quality of the raw water to be treated (i.e., the estimated probability to find a given contaminant level), as well as different characteristics of the treatment unit and its various operating modes [11, 3].

However, a major hurdle here is the difficulty to obtain precise and reliable estimates of key input data needed for reliability analysis such as failure rates, or frequency distribution of raw water quality parameters. In this paper, it is proposed to model the uncertainty on raw water quality, process line efficiency and state of the treatment plant in the belief function framework. Each source of information will be modeled by a belief function and combined to obtain an assessment of the plausibility to produce noncompliant water. In the limit case of precise and certain information, the proposed methodology will be shown to yield exactly the same results as the classical probabilistic approach. In the general case, however, the output belief function that quantifies the uncertainty related to the noncompliance of treated water is not a probability measure, and its imprecision directly reflects the imprecision of input data. These results can be used to estimate a level of confidence that contractual requirements will be met with a given treatment plant technology, and therefore to help treatment plant designers to choose the optimal process line, given an objective level of residual risk.

The rest of the paper is organized as follows. The classical methodology is first recalled in Section 2. The necessary theoretical background on belief functions is then summarized in Section 3, and our approach is described in Section 4. The equivalence with the probabilitic approach is the proved in Section 4.7. Finally, a generalization to multiple quality parameters is presented in Section 5, and simulations are presented in Section 6. Section 7 concludes the paper.

2 Problem description and probabilistic approah

2.1 Problem statement

Drinking water regulations in industrialized countries define maximum admissible levels for several dozens of contaminants, including volatile and synthetic organic compounds, inorganic compounds, and microbial organisms. Standards are also defined for global water quality parameters such as turbidity and color. To comply with these regulations, water treatment facilities must include a series of treatment processes called a “treatment train”. The most commonly used processes include filtration, flocculation and sedimentation, and disinfection. Some treatment trains also include ion exchange and adsorption. The design of a treatment train usually ensures that water standards will be met in normal operating conditions. However, as the quality or water sources is continuously degrading, and regulations are becoming more and more stringent, it is now necessary to take into account “rare” events in the analysis, such as extreme levels of some contaminants due to accidental pollution of the water source, and failure of treatment processes. The problem addressed in this paper is to assess a “degree of confidence” that, at a given time, the output level of a given quality parameter will not exceed the corresponding regulation standard. A solution to this problem in the probabilistic setting is proposed in this section. The rest of the paper is concerned with elaborating a more complete solution in the Belief Function framework.

2.2 Modeling of treatment efficiency

The global efficiency of the treatment plant in the nominal mode with respect to a given contaminant may be modelled by a transfer function, giving the output level $\mathbf{s}$ as a function of the input level $\mathbf{e}$. Such a transfer function may be constructed by studying the impact of each of the treatment processes along the treatment train, on the contaminant (or quality parameter) of interest. In most cases, this transfer function can be assumed to be linear and expressed using a single parameter α_0, called the abatement rate or reduction factor:

$$\mathbf{s} = (1 - \alpha_0)\mathbf{e}.$$

It is also possible to account for nonlinearities by defining different abatement rates according to the input level. However, such a sophistication is not fundamental at the conceptual level, and we will adopt the linear assumption throughout this paper.

In addition to the nominal mode x_0, it is assumed that the plant may be in one of I degraded modes $x_i, i = 1, \ldots, I$. Each degraded mode corresponds to a fault in one of the processes along the treatment train. For each degraded mode x_i, the influence of the whole treatment on the considered quality parameter will still be assumed to be linear, with an abatement rate $\alpha_i < \alpha_0$. The complete model of the plant (with respect to a given water quality parameter) is thus defined by I+1 coefficients α_i, $i = 0, \ldots, I$.

For each contaminant, the regulation defines a standard level N that should not be exceeded by treated water. This defines two possible states for the produced water: compliance ($\mathbf{s} < N$) and noncompliance ($\mathbf{s} \geq N$). Applying the inverse nominal and degraded mode transfer functions to level N defines $I + 1$ admissible levels $\theta_i =$

$N/(1-\alpha_i)$ for raw water: the raw water level $\mathbf{e}$ must be less than θ_i to respect the regulation, assuming the plant to be in state x_i (see Figure 1).

In the sequel (except in Section 5), the analysis will be carried out for a single quality parameter, which does not pose any problem if the parameters are assumed to be independent. The case of several dependent parameters will be addressed in Section 5.

2.3 Probabilistic analysis

A major step in the classical approach is the Failure Mode Effects and Criticality Analysis (FMECA), in which the different failure modes x_i, $i = 1, \ldots, I$ of the treatment process are listed, and the corresponding failure rates λ_i (failure probability per unit time), latency times T_i (time to restore nominal mode after failure), and degraded transfer functions (degraded abatement rates α_i for $i > 0$) are determined. The compliant water unavailability induced by failure mode x_i is simply the probability to be in that mode, and is equal to $p_i = \lambda_i T_i$. The probability to be in the nominal mode is

$$p_0 = 1 - \sum_{i=1}^{I} p_i. \tag{1}$$

This data provides the basic input to Fault Tree Analysis (FTA), as illustrated in Figure 2. For the considered parameter, the top level event of the fault tree is "Produced water does not comply with standard level N". The first level of the tree is a decomposition between the different operating modes of the treatment plant by the top level OR gate. The plant in a given mode x_i will not accept input level exceeding θ_i for the considered contaminant, which corresponds to the second level AND gate. The probability of the top level event is trivially obtained in that case as:

$$P(\mathbf{s} \geq N) = \sum_{i=0}^{I} p_i P(\mathbf{s} \geq N | x_i) = \sum_{i=0}^{I} p_i P(\mathbf{e} \geq \theta_i) \ . \tag{2}$$

2.4 Limitation of the approach

The above methodology assumes perfect knowledge of transfer functions, failure rates, latency times, and input distribution of the quality parameter under study. This assumption has proved unrealistic in the present application, for several reasons. The approach is to be used at early stages of treatment plant design, where few experimental data, if any, are available. Treatment efficiency can be precisely measured in laboratory experiments, but observed values in operational conditions are often different and vary considerably from one site to another. Due to the scarcity of historical records, water treatment engineers are often not able to provide more than crude estimates of failure rates and latency times. Lastly, laboratory measurement of key quality parameters are costly, and the number of available measurements rarely exceeds a few dozens, providing only partial information on the input distribution of contaminant levels in raw water.

These considerations have motivated the adoption of a more flexible framework, allowing to represent and manage weaker forms of knowledge than assumed in the probabilistic setting. This framework will be described in the next section.

3 The Transferable Belief Model

3.1 Basic notions

The theory of belief functions originates from a series of seminal papers par Dempster [4] and an influential book by Shafer [13]. It was later developed by Smets [20] who clarified semantic issues, introduced many new tools, and provided axiomatic justifications for the use of belief functions and the main combination rules. This has resulted in a coherent framework for representing and manipulating uncertain information, called the Transferable Belief Model (TBM) [20]. In this model, a belief function is understood as representing an agent's state of belief, without resorting to an underlying probability model. Only the essential definitions and specific notations will be given here. A detailed exposition of the TBM may be found in [18].

Let $\mathbf{x}$ denote a variable or quantity of interest defined on a finite domain (or *frame of discernment* X). An agent's belief concerning the value taken by $\mathbf{x}$ is represented by a basic belief assignment (bba) m^X defined as a function from the powerset 2^X of

X to $[0, 1]$ verifying

$$\sum_{A \subseteq X} m^X(A) = 1.$$

Subsets A of X such that $m^X(A) \neq 0$ are called the focal sets of m^X. The quantity $m^X(A)$ is interpreted as a fraction of a unit mass of belief assigned specifically to subset A (i.e., to the hypothesis that A contains the true value of $\mathbf{x}$), given the facts known by the agent at a given time. Total ignorance is therefore represented by the vacuous bba defined by $m^X(X) = 1$, whereas full knowledge corresponds to the case where $m^X(\{x\}) = 1$ for some $x \in X$. Probabilistic knowledge is recovered when all the focal sets are singletons (m^X is then a probability function on X, and is called a Bayesian bba). The normality condition $m(\emptyset) = 0$ is not systematically imposed in the TBM. The quantity $m(\emptyset)$ measures the amount of conflict after combining several information sources (see below), and can sometimes be interpreted as a weight of belief in the hypothesis that the quantity of interest might take its value outside the known set of alternatives (open-world assumption).

Associated with m^X are several set functions which play an important role in the theory. Two such functions are the belief function bel^X and the plausibility function pl^X defined, respectively, as:

$$bel^X(A) = \sum_{\emptyset \neq B \subseteq A} m^X(B), \tag{3}$$

$$pl^X(A) = \sum_{B \cap A \neq \emptyset} m^X(B), \quad \forall A \subseteq X. \tag{4}$$

The quantity $bel^X(A)$ is interpreted as the amount of support actually given to A by available evidence, whereas $pl^X(A) = bel^X(\Omega) - bel^X(\overline{A})$ (where $\overline{A}$ denotes the complement of A) represents an amount of the potential support that could be given to A, if further information became available.

A key feature in any uncertainty management framework is the way pieces of information are combined, leading to a new knowledge state. In the TBM, the basic mechanisms for combining independent pieces of information is the conjunctive sum, also known as the unnormalized Dempster's rule of combination. Let m_1^X and m_2^X be two bba's induced by distinct sources of information. Their conjunctive sum $m^X_{1 \textcircled{\cap} 2} = m^X_1 \textcircled{\cap} m^X_2$ is defined as:

$$m^X_{1 \textcircled{\cap} 2}(A) = \sum_{B \cap C = A} m^X_1(B) m^X_2(C), \quad \forall A \subseteq X. \tag{5}$$

Note that $m^X_{1 \textcircled{\cap} 2}(\emptyset)$ may be nonzero even if m^X_1 and m^X_2 are normalized: the quantity $m^X_{1 \textcircled{\cap} 2}(\emptyset)$ is then interpreted as a degree of conflict between the two sources. The conjunctive sum operation is commutative and associative.

It sometimes occurs that a source of information induces a bba m^X, but we have some doubt regarding the reliability of that source. Such metaknowledge may be represented by discounting [13] m^X by some factor $\delta \in [0, 1]$, which leads to a bba m^X_δ defined as:

$$m^X_\delta(A) = (1 - \delta) m^X(A) \qquad \forall\, A \subseteq X, A \neq X \tag{6}$$

$$m^X_\delta(X) = \delta + (1 - \delta) m^X(X) \tag{7}$$

A discount rate $\delta = 1$ means that the source can certainly not be trusted: the resulting bba is then vacuous. On the contrary, a null discount rate leaves m^X unchanged: this corresponds to the situation where the source is known to be fully reliable.

3.2 Link with Possibility Theory

The theory of Belief functions has close links with Possibility theory, another uncertainty management framework [22][8] closely related to Fuzzy Set Theory. A possibility distribution on X is a function $\pi^X : X \to [0, 1]$. It is interpreted as a soft constraint on values that can possibly be taken by $\mathbf{x}$, and is formally equivalent to a fuzzy subset of X. The associated possibility and necessity measures are set functions defined, respectively, as:

$$\Pi^X(A) = \max_{x \in A} \pi^X(x),$$

$$N^X(A) = 1 - \Pi^X(\overline{A}), \quad \forall A \subseteq X.$$

These functions are characterized by the following properties:

$$\Pi^X(A \cup B) = \max(\Pi^X(A), \Pi^X(B)) \tag{8}$$

$$N^X(A \cap B) = \min(N^X(A), N^X(B)), \quad (9)$$

for all $A, B \subseteq X$.

Let m^X be a consonant bba, i.e., a bba with nested focal sets $F_1 \subseteq \ldots \subseteq F_n$. As shown by Shafer [13], the plausibility and belief measures induced by m^X verify (8) and (9), respectively. Hence, pl^X is a possibility measure, and bel^X is the dual necessity measure. The corresponding possibility distribution is the function defined by $\pi^X(x) = pl^X(\{x\})$, for all $x \in X$.

Conversely, let π^X be a possibility distribution on X. The associated possibility and necessity measures Π^X and N^X are, respectively, plausibility and belief functions [8]. The associated consonant bba m^X may be computed as follows. Let $\pi_1 > \ldots > \pi_r$ be the distinct values taken by π^X, arranged in decreasing order, and $\pi_{r+1} = 0$ by convention. Let $A_i = \{x \in X \mid \pi^X(x) \geq \pi_i\}$, $i = 1, \ldots, r$. Then, we have, for any non-empty subset A of X:

$$m^X(A) = \begin{cases} \pi_i - \pi_{i+1} & \text{if } A = A_i, i = 1, \ldots, r \\ 0 & \text{otherwise,} \end{cases} \quad (10)$$

and $m^X(\emptyset) = 1 - \pi_1$.

Although combination rules in Possibility theory and in the TBM are different, the above considerations show that belief functions provide a very general framework, which encompasses both probability and possibility measures are special cases. This generality makes the TBM particularly suitable as a modelling tools for problems in which the available pieces information are provided in different formats, which is often the case when subjective knowledge (often conveniently represented in the form of a possibility distribution) has to be combined with statistical information, which lends itself more naturally to probabilistic representation.

3.3 Coarsenings and Refinements

In applying the TBM framework to a real-world problem, the definition of the frame of discernment is a crucial step. As remarked by Shafer [13], the degree of “granularity” of the frame is always, to some extent, a matter of convention, as any element ω of Ω

representing a "state of nature" can always be split into several possibilities. Hence, it is fundamental to examine how a belief function defined on a frame may be expressed in a finer or, conversely, in a coarser frame [7].

Let X and Y denote two finite sets. A mapping $\rho : 2^Y \to 2^X$ is called a *refining* if it verifies the following properties:

1. The set $\{\rho(\{y\}), y \in Y\} \subseteq 2^X$ is a partition of X.

2. For all $B \subseteq Y$, we have

$$\rho(B) = \bigcup_{y \in B} \rho(\{y\}) . \tag{11}$$

Following the terminology introduced by Shafer, Y is then called a *coarsening* of X, and X is called a *refinement* of Y. Formally, defining a coarsening of a frame amounts to defining a partition of that frame.

A bba m^Y on Y may be tranformed into a bba on a refinement X by tranferring each mass $m^Y(B)$ for $B \subseteq Y$ to $A = \rho(B)$. This operation is called a *vacuous extension* of m^Y to X. Formally:

$$m^X(A) = \begin{cases} m^Y(B) & \text{if } A = \rho(B) \text{ for some } B \subseteq Y, \\ 0 & \text{otherwise.} \end{cases} \tag{12}$$

3.4 Operations on Joint Spaces

Most problems involve several variables, with available knowledge consisting in exact or approximate relations between groups of variables. In probability theory, such knowledge is typically represented by a joint probability distribution over the set of variables, from which marginal and conditional probability distributions can be computed. Similar operations can be defined in Belief Function Theory, as well as other operations which have no equivalent in probability theory [17][1]. These operations are briefly presented in the sequel.

Let m^{XY} denote a bba defined on the Cartesian product $X \times Y$ of two variables $\mathbf{x}$ and $\mathbf{y}$ (the notation $X \times Y$ is replaced by XY in the superscript of bba's to simplify the notation). The marginal bba $m^{XY\downarrow X}$ on X is defined, for all $A \subseteq X$, as

$$m^{XY\downarrow X}(A) = \sum_{\{B\subseteq X\times Y \mid \mathrm{Proj}(B\downarrow X)=A\}} m^{X\times Y}(B) \ , \tag{13}$$

where $\mathrm{Proj}(B \downarrow X)$ denotes the projection of B onto X, defined as

$$\mathrm{Proj}(B \downarrow X) = \{x \in X \mid \exists y \in Y, (x, y) \in B\} \ . \tag{14}$$

Marginalization may be seen as going from a frame $X \times Y$ to a coarsening X. The inverse operation, which has no equivalent in Probability Theory, is a particular instance of vacuous extension. Let m^X be a bba on X. Its vacuous extension on $X \times Y$ is defined as:

$$m^{X\uparrow XY}(B) = \begin{cases} m^X(A) & \text{if } B = A \times Y \text{ for some } A \subseteq X, \\ 0 & \text{otherwise.} \end{cases} \tag{15}$$

Other important notions are those of conditioning, and its inverse operation called the *ballooning extension* [17]. Let m^{XY} denote a bba on $X \times Y$ (with underlying variables $(\mathbf{x}, \mathbf{y})$), and m_y^{XY} the bba on $X \times Y$ with single focal set $X \times \{y\}$. The conditional bba of $\mathbf{x}$ given $\mathbf{y} = y$ is defined as:

$$m^X[y] = (m^{XY} \textcircled{\cap} m_y^{XY})^{\downarrow X} \tag{16}$$

The conditioning operation for belief functions has the same meaning as in Probability Theory. However, it also admits an inverse operation (with no probabilistic counterpart) called the *ballooning extension* [17]. Let $m^X[y]$ denote the conditional bba on X, given y. The ballooning extension of $m^X[y]$ on $X \times Y$ is the least committed bba, whose conditioning on y yields $m^X[y]$ (see [17] for detailed justification). It is obtained as:

$$m^X[y]^{\Uparrow XY}(B) = \begin{cases} m^X[y](A) & \text{if } B = (A \times \{y\}) \cup (X \times (Y \setminus \{y\})) \text{ for some } A \subseteq X, \\ 0 & \text{otherwise.} \end{cases} \tag{17}$$

3.5 Belief networks

A set of belief functions over subsets of a joint space $X_1 \times \ldots \times X_n$ of n variables can be represented graphically by an undirected hypergraph, in which each node is a variable, and each hyperedge linking a subset of nodes is weighted by a belief function defined on the product space of the corresponding variables [14]. Such a graphical model is called a *belief network*. Once a problem has been formalized as a belief network, a solution is found by combining the belief functions, and marginalizing on variables of interest. Local computation algorithms as described in [14], [1] and [16] allow to perform these operations at a considerable savings of time and space.

4 Application of the TBM

4.1 Representation of basic input data

As already mentioned in Section 2.4, one of the main difficulties in applying the probabilistic approach to the problem considered in this paper is the unavailability of precise and reliable input data. The Belief Function formalism, being more general and flexible than the probabilistic one, will allow us to represent and manage weak information available in this application, without making it artificially and deceptively precise.

The basic quantities of interest in our problem are abatement rates α_i ($i = 0, \ldots, I$), failure rates λ_i and latency times T_i associated to each degraded mode x_i ($i = 1, \ldots, I$), and the frequency distribution of the raw water quality parameter under study. Whereas these quantities were assumed to be known in Section 2, we will now consider the more realistic situation were they are only partially specified.

Defining precise abatement rates α_i hardly makes any sense, because the efficiency of a treatment process depends on changing characteristics of raw water, and the linearity of the plant input-output transfer function is only a simplifying assumption. Indeed, the experts felt more comfortable to provide, for each rate α_i ($i = 0, \ldots, I$), a lower bound α_i^-, an upper bound α_i^+, and a point estimate α_i^0. This data can be modeled by the triangular possibility distribution π_i on the real line with support $[\alpha_i^-, \alpha_i^+]$ and mode α_i^0.

Failure rates and latency times were also elicited from experts. Here again, only imprecise assessements can be obtained. For simplicity, only upper and lower bounds of these parameters were considered, leading to intervals $[\lambda_i^-, \lambda_i^+]$ and $[T_i^-, T_i^+]$ for $i = 1, \ldots, I$.

Lastly, information about the distribution of quality parameter in raw water will be assumed to consist either in a set of n values $\mathbf{e}(t_1), \ldots, \mathbf{e}(t_n)$ measured in samples taken at n time steps $t_1, \ldots, t_n$, or partial probability assessments provided by experts.

The representation of this input data in the TBM framework will be described in the following sections.

4.2 Discretization

The theory outlined in Section 3 assumes belief functions to be defined on finite spaces (a generalization to continuous spaces is possible, but it requires considerably more mathematical sophistication). We thus have to discretize variables $\mathbf{e}$ and $\mathbf{s}$ representing the input and output levels.

For that purpose, let us define $K-1$ thresholds σ_k, $(k = 1, \ldots, K-1)$ for the output level $\mathbf{s}$, which induce K possible states $s_k = [\sigma_{k-1}, \sigma_k)$ $(k = 1, \ldots, K)$ with $\sigma_0 = 0$ and $\sigma_K = \infty$ by convention. To recover the classical limit, one of the output thresholds must be the standard level ($N = \sigma_k$ for some k).

For a given mode x_i of the treatment plant, each output threshold σ_k defines an input threshold $\theta_{k,i} = \sigma_k/(1 - \alpha_i^0)$ (the input concentration must be less than $\theta_{k,i}$ for the output concentration to be less than σ_k when the treatment plant is in mode x_i). For this mode x_i it would be sufficient to consider the K raw water states $A_{k,i} = [\theta_{k-1,i}, \theta_{k,i})$, $1 \leq k \leq K$, with $\theta_{0,i} = 0$ and $\theta_{K,i} = \infty$, but we need a discretization which allows to take into account all $I+1$ plant operating modes. We therefore consider the whole set of $\theta_{k,i}$ for all modes x_i and all output thresholds σ_k, and we note η_j $(1 \leq j \leq J-1)$ this set of values arranged in increasing order. Note that we defined $K-1$ thresholds for each of the $I+1$ modes, so that $J-1 \leq (K-1)(I+1)$ (the upper bound may not be strict because some of the thresholds may be equal). We finally have J raw water states $e_j = [\eta_{j-1}, \eta_j)$ $(1 \leq j \leq J$ with $\eta_0 = 0$ and $\eta_J = \infty)$.

Note that the set of states $E = \{e_1, \ldots, e_J\}$ can be thought of as the coarsest common refinement of the $I+1$ sets $\{A_{k,i} | 1 \leq k \leq K\}$ $(0 \leq i \leq I)$. This discretization scheme is represented in Figure 3.

We thus have three discrete underlying variables: the discretized input level taking values in $E = \{e_1, \ldots, e_J\}$, the discretized output level taking values in $S = \{s_1, \ldots, s_K\}$, and the plant state in $X = \{x_0, \ldots, x_I\}$. The available pieces of information will now be modeled as belief functions on these domains, leading to the belief network shown in Figure 4.

4.3 Belief on X

In the probabilistic case, knowledge of failure rates λ_i and latency times T_i induce a probability function on X, as shown in Section 2.3. Since λ_i and T_i are now only known to lie in given intervals, it is natural to define a probability interval $[p_i^-, p_i^+]$ for each of the failure mode as:

$$p_i^- = \lambda_i^- T_i^-, \quad p_i^+ = \lambda_i^+ T_i^+, \quad i = 1, \ldots, I. \tag{18}$$

The probability interval $[p_0^-, p_0^+]$ for the nominal mode can be deduced from (1) and (18) to be:

$$p_0^- = \max\left(0, 1 - \sum_{i=1}^{I} p_i^+\right), \tag{19}$$

$$p_0^+ = 1 - \sum_{i=1}^{I} p_i^-, \tag{20}$$

assuming that

$$\sum_{i=1}^{I} p_i^- \leq 1, \tag{21}$$

which ensures the feasability of the constraints. Since the different failure probabilities p_i, $i = 1, \ldots, I$ are small, we can further assume that

$$\sum_{i=1}^{I} p_i^+ \leq 1. \tag{22}$$

Equation (19) then simplifies to $p_0^- = 1 - \sum_{i=1}^{I} p_i^+$.

Let $P^-(A)$ and $P^+(A)$ denote, respectively, the lower and upper bounds on the probability of $A \subseteq X$, under the constraints (18). In general, the imprecise probability framework [21] is distinct from the belief function framework, and the concept of coherent lower probability measure as defined by Walley [21] is strictly more general than that of belief function. In the special case considered here, however, it will be shown that, under conditions (21) and (22), P^- is a belief function, P^+ being the associated plausibility function. The corresponding bba m^X happens to have at most $2I + 1$ focal sets of cardinality 1 or 2.

Proposition 1 *Let $\mathcal{P}$ denote the set of probability measures on X defined by contraints (18), under conditions (21) and (22).*

1. *We have, for all $A \subseteq X$:*

$$P^-(A) = \min_{P \in \mathcal{P}} P(A) = \begin{cases} \sum_{x_i \in A} p_i^- & \text{if } x_0 \notin A \\ 1 - \sum_{x_i \notin A} p_i^+ & \text{otherwise,} \end{cases}$$

$$P^+(A) = \max_{P \in \mathcal{P}} P(A) = \begin{cases} \sum_{x_i \in A} p_i^+ & \text{if } x_0 \notin A \\ 1 - \sum_{x_i \notin A} p_i^- & \text{otherwise,} \end{cases}$$

2. *P^- is a belief function with associated plausibility function P^+ and bba m^X defined by:*

$$m^X(\{x_i\}) = p_i^- \quad i = 0, \ldots, I \tag{23}$$

$$m^X(\{x_0, x_i\}) = p_i^+ - p_i^- \quad i = 1, \ldots, I. \tag{24}$$

Proof: See Appendix A.

Example 1 Assume that we have two degraded modes x_1 and x_2. Mode x_1 happens with a frequency comprised between 1 and 3 hours per year ($\lambda_1 \in [1.14 \times 10^{-4}\ \text{h}^{-1}, 3.42 \times 10^{-4}\ \text{h}^{-1}]$), and has a latency time of between 48 and 96 hours ($T_1 \in [48\ \text{h}, 96\ \text{h}]$), whereas mode x_2 happens between 2 and 5 hours per year ($\lambda_2 \in [2.28 \times 10^{-4}\ \text{h}^{-1}, 5.70 \times 10^{-4}\ \text{h}^{-1}]$), and has a latency time of between 24 and 48 hours

($T_2 \in [24 \text{ h}, 48 \text{ h}]$). We thus have (with two significant digits):

$$\begin{aligned} p_1 &\in [0.0055, 0.033] \\ p_2 &\in [0.0055, 0.027] \\ p_0 &\in [0.94, 0.99], \end{aligned}$$

which leads to the following bba:

$$\begin{aligned} m^X(\{x_0\}) &\approx 0.94 \\ m^X(\{x_1\}) &\approx 0.0055 \\ m^X(\{x_2\}) &\approx 0.0055 \\ m^X(\{x_0, x_1\}) &\approx 0.027 \\ m^X(\{x_0, x_2\}) &\approx 0.022. \end{aligned}$$

4.4 Belief on E

Information about the frequency distribution of contaminant levels in raw water may be obtained either from data, or from expert opinion. The most favorable situation is that where a large amount of data is available, allowing to build a histogram with J classes $e_1, \ldots, e_J$, which then constitutes a good estimate of the true probability distribution. In that case, $m^E(\{e_j\})$ may simply be defined as the relative frequency of data in class $e_j = [\eta_{j-1}, \eta_j)$, and m^E is then a Bayesian bba (the associated belief function is a probability measure).

A usual rule of thumb to build a histogram is to impose the absolute frequency of each class to be at least equal to five. If this is not the case, the sample size is too small to build a histogram with J classes. A useful strategy is then to merge some classes, which amounts to defining a coarsening E' of E with $J' < J$ elements. Building a histogram with the classes defined by E' leads to a Bayesian bba $m^{E'}$ on E', which can be vacuously extended on E using (12). The resulting bba m^E is then no longer Bayesian: it has J' focal sets which form a partition of E.

Example 2 Let $E = \{e_1, \ldots, e_6\}$, and assume that the absolute frequencies of $n = 50$ observations in each class are 2, 5, 14, 20, 8, 1. The frequencies of the two extreme classes are two small to provide reliable estimates of the probabilities of these classes. We then merge e_1 and e_2, as well as e_5 and e_6. We the obtain a coarsened frame $E' = \{e'_1, e'_2, e'_3, e'_4\}$ linked to E by a refining ρ such that $\rho(\{e'_1\}) = \{e_1, e_2\}$, $\rho(\{e'_2\}) = \{e_3\}$, $\rho(\{e'_3\}) = \{e_4\}$, $\rho(\{e'_4\}) = \{e_5, e_6\}$. The bba on E' is then defined as:

$$\begin{array}{ll} m^{E'}(\{e'_1\}) = 7/50, & m^{E'}(\{e'_2\}) = 14/50 \\ m^{E'}(\{e'_3\}) = 20/50, & m^{E'}(\{e'_4\}) = 9/50. \end{array}$$

The vacuous extension of $m^{E'}$ on E yields:

$$\begin{array}{ll} m^{E}(\{e_1, e_2\}) = 7/50, & m^{E}(\{e_3\}) = 14/50 \\ m^{E}(\{e_4\}) = 20/50, & m^{E}(\{e_5, e_6\}) = 9/50. \end{array}$$

When no data is available, we must resort to expert knowledge to obtain at least some imprecise specification of the distribution of input water quality parameter values. Typically, an expert will not be able to estimate all class frequencies, but he or she may be able to provide a limited number of approximate cumulative probability values $F_j = P(\{e_1, \ldots, e_j\})$ for $j = j_1, \ldots, j_r$. This information may be encoded exactly as a bba m_F^E defined as

$$m_F^E(\{e_1, \ldots, e_{j_1}\}) = F_{j_1},$$

$$m_F^E(\{e_{j_{k-1}+1}, \ldots, e_{j_k}\}) = F_{j_k} - F_{j_{k-1}}, \quad k = 2, \ldots, r,$$

$$m_F^E(\{e_{j_r+1}, \ldots, e_J\}) = 1 - F_{j_r}.$$

Additional information concerning the reliability of the expert's estimates may be expressed in the form of a degree of confidence $1 - \delta$ between 0 (no confidence at all) and 1 (full confidence). It may be included in the evaluation by discounting m^E with a discount rate δ, using (6)(7).

Example 3 Let us consider the same domain E as in Example 2, and let us assume that the following information has been elicited from an expert:

$$P(\{e_1, e_2\}) \approx 0.2, \qquad P(\{e_1, \ldots, e_4\}) \approx 0.8.$$

Assume furthermore that the expert has a degree of confidence of $1 - \delta = 0.8$ on his or her assessements. The cumulative probability assessments lead to:

$$m_F^E(\{e_1, e_2\}) = 0.2, \qquad m_F^E(\{e_3, e_4\}) = 0.8 - 0.2 = 0.6,$$
$$m_F^E(\{e_5, e_6\}) = 1 - 0.8 = 0.2.$$

Applying a discount rate of $1 - 0.8 = 0.2$ to this bba leads to the following final result:

$$m^E(\{e_1, e_2\}) = 0.2 \times 0.8 = 0.16, \quad m^E(\{e_3, e_4\}) = 0.6 \times 0.8 = 0.48,$$
$$m^E(\{e_5, e_6\}) = 0.2 \times 0.8 = 0.16 \quad m^E(E) = 1 - 0.16 - 0.48 - 0.16 = 0.2.$$

4.5 Belief on $X \times E \times S$

The triangular possibility distribution $\pi_i(\alpha)$ on α_i defined in Section 4.1 may be seen as constraining the set of possible values for the pair of variables $(\mathbf{e}, \mathbf{s})$, when the plant is in state x_i. Since we then have $\mathbf{s}/\mathbf{e} = 1 - \alpha_i$, the induced possibility distribution on variables $(\mathbf{e}, \mathbf{s})$ may be expressed as a function of the ratio $\beta = \mathbf{s}/\mathbf{e}$ as:

$$\pi[x_i](\mathbf{e}, \mathbf{s}) = \pi_i(1 - \beta) = \begin{cases} \dfrac{1 - \beta - \alpha_i^-}{\alpha_i^0 - \alpha_i^-} & \text{if } \alpha_i^- \leq 1 - \beta \leq \alpha_i^0 \\ \dfrac{\alpha_i^+ - 1 + \beta}{\alpha_i^+ - \alpha_i^0} & \text{if } \alpha_i^0 < 1 - \beta \leq \alpha_i^+ \\ 0 & \text{otherwise.} \end{cases} \tag{25}$$

In (25), the notation $[x_i]$ indicates that this possibility distribution is conditional on the plant being in state x_i.

The above possibility is defined on the continuous space $\mathbb{R}_+^2$. Since the spaces of input and output levels have been discretized (Section 4.2), we have to define the induced possibility distribution on the discrete space $E \times S$. Applying standard rules of Possibility Theory, the simplest and most natural way to define this distribution would be as:

$$\pi^{ES}[x_i](e_j, s_k) = \max_{\mathbf{e} \in e_j, \mathbf{s} \in s_k} \pi[x_i](\mathbf{e}, \mathbf{s}) \quad j = 1, \ldots, J, \; k = 1, \ldots, K. \tag{26}$$

However, this discretization scheme has the undesirable effect that introducing even an infinitesimal amount of imprecision in the possibility distribution $\pi_i(\alpha)$ from $\alpha_i^- = \alpha_i^+$ to $\alpha_i^+ - \alpha_i^- = d\alpha$ results in drastic change in the joint possibility distribution $\pi^{ES}[x_i]$, as some possibility values jump from 0 to 1. To avoid this effect, the following "smooth" discretization procedure was adopted. The possibility given to each rectangle $R_{k,k',i} = A_{k,i} \times s_{k'}$ was computed as the height of the intersection between the quadratic possibility distribution defined by (25) and a "pyramidal" possibility distribution with support (the base of the pyramid) equal to $R_{k,k',i}$ and with kernel (the point with maximum possibility value) equal to the center of $R_{k,k',i}$ (Figure 5). This method yields the same results as the simple one (26) in the "precise" case ($\alpha_i^- = \alpha_i^+$), but it has the desirable property that the possibility associated to rectangles varies continuously, ensuring the continuity at the classical limit.

The above procedure yields $I + 1$ conditional possibility distributions which can be converted into conditional bba's $m^{ES}[x_i]$, $i = 0, \ldots, I$ using (10). Each of these bba's represents one's belief in the joint values of input and output concentrations, when the plant is in state x_i. These conditional bba's can be then be converted into joint bba's m_i^{XES} using the ballooning extension (17).

$$m_i^{XES} = m^{ES}[x_i]^{\Uparrow XES}, \quad i = 0, \ldots, I.$$

Remark 1 In our model, expert knowledge concerning transfer functions in different states is initially described in the language of Possibility theory, using (25). The corresponding possibility distributions are then discretized, and finally translated into the belief function formalism. This is essentially a consequence of our choice to represent expert assessments concerning abatement rates α_i as triangular possibility distributions, a convenient model of expert opinion about a parameter (this method is referred to as the "method of consonant intervals" in [1, page 241]). We could as well have avoided to mention Possibility Theory at all, and presented a triangular possibility distribution as a continuous belief function with a mass density over a family of consonant intervals. From our experience, however, we have come to the conclusion that the semantics of a possibility distribution as a flexible constraint over some unkown quantity is quite useful when eliciting expert opinions. Nevertheless, Possibility Theory is far less suitable for representing probabilistic information such as frequency distributions. The ability of the TBM to incorporate both kinds of knowledge is, in our view, a major advantage of this framework.

4.6 Combination and marginalization

The final step is to combine all the available evidence, and marginalize on S. For that purpose, bel^X and bel^E must be extended to the product space $X \times E \times S$ using the vacuous extension (15). The resulting belief functions are combined using Dempster's rule, and the result is marginalized on S. Formally, the final bba m^S is thus defined as:

$$m^S = \left(\left(\textcircled{\cap}_{i=0}^{I} m_i^{XES} \right) \textcircled{\cap} m^{X \uparrow XES} \textcircled{\cap} m^{E \uparrow XES} \right)^{\downarrow S}. \tag{27}$$

Note that these operations can be performed very efficiently using local computation algorithms, as mentioned in Section 3.5.

Several interesting functions can be computed from m^S, including the cumulative belief and plausibility functions, which we define as follows. First, let $\mathcal{S}$ denote the refinement of S induced by the following refining ρ:

$$\rho(\{s_k\}) = [\sigma_{k-1}, \sigma_k),$$

and let $m^{\mathcal{S}}$ denote the vacuous extension of m^S on $\mathcal{S}$. The focal sets of $m^{\mathcal{S}}$ are real intervals. The cumulative belief function $F_{bel}^{\mathcal{S}}$ is then defined for all $s \geq 0$ as

$$\begin{aligned} F_{bel}^{\mathcal{S}}(s) &= bel^{\mathcal{S}}((-\infty, s)) \\ &= \sum_{\{A \subseteq S \ | \ \rho(A) \subseteq (-\infty, s)\}} m^S(A) \\ &= \sum_{\{A \subseteq S \ | \ \sup \rho(A) \leq s\}} m^S(A). \end{aligned}$$

By construction, $F_{bel}^{\mathcal{S}}$ is a right-continuous step function with jumps at each cutpoint σ_k (see Section 6 for graphical displays of this function). Similarly, we may define the cumulative plausibility function $F_{pl}^{\mathcal{S}}$ as

$$\begin{aligned} F_{pl}^{\mathcal{S}}(s) &= pl^{\mathcal{S}}((-\infty, s)) \\ &= \sum_{\{A \subseteq S \ | \ \rho(A) \cap (-\infty, s) \neq \emptyset\}} m^S(A) \\ &= \sum_{\{A \subseteq S \ | \ \inf \rho(A) < s\}} m^S(A), \end{aligned}$$

for all $s \geq 0$. Function $F_{pl}^{\mathcal{S}}$ is also a step function with jumps at each cutpoint σ_k, but it is continous from the left.

4.7 Equivalence with the probabilistic approach

The following proposition states that our method is a valid generalization of the probabilistic approach, i.e., that it yields the same results as the classical method when all necessary data are precisely known.

PROPOSITION 2 *Under the following conditions:*

1. *The abatement rates are exactly known:* $\alpha_i^- = \alpha_i^+$ *for* $i = 0, \dots, I$*;*

2. *The failure rates and latency times of all degraded modes are exactly known:* $\lambda_i^- = \lambda_i^+$ *and* $T_i^- = T_i^+$ *for* $i = 1, \dots, I$*;*

3. m^E *is a Bayesian bba, and it adequately reflects the probability distribution of* $\mathbf{e}$*:* $m^E(\{e_j\}) = P(\eta_{j-1} \leq \mathbf{e} < \eta_j)$ *for* $j = 1, \dots, J$*;*

4. $N = \sigma_{k_0 - 1}$ *for some* $k_0 \in \{2, \dots, K\}$*,*

then the degree of belief in the event $\mathbf{s} \geq N$ *computed using the TBM approach is equal to the probability of that event computed using Equation (2).*

Proof. See Appendix B.

5 Multiparameter extension

The analysis carried out up to now concerned only one quality parameter. This approach, however, can easily be extended to the more realistic situations where several parameters have to be taken into account in the risk analysis.

For simplicity and without loss of generality, let us assume that we have two quality parameters $\mathbf{e}_1$ and $\mathbf{e}_2$. If these two parameters are statistically independant, then the above analysis can be carried out independently for $\mathbf{e}_1$ and $\mathbf{e}_2$, leading to the model depicted in Figure 6. In this model, E_1 and E_2 are the discrete spaces for input levels of parameters $\mathbf{e}_1$ and $\mathbf{e}_2$, S_1 and S_2 are the discrete spaces for output levels of the same parameters, and X is, as before, the set of states of the treatment plant. An additional variable R indicates whether the produced water complies with the standards regarding the both parameters. If $C_1 \subseteq S_1$ and $C_2 \subseteq S_2$ denote, respectively, the sets of compliant states for output levels of parameters 1 and 2, the relation between S_1, S_2 and R can be encoded as a bba $m^{RS_1S_2}$ defined as:

$$m^{RS_1S_2}\left((\{1\} \times C_1 \times C_2) \cup (\{0\} \times \overline{C_1} \times \overline{C_2})\right) = 1.$$

A belief function on R integrating all the available information may be obtained, as before, by combining all the belief functions using Dempster's rule, and marginalizing on R. Alternativeley, more detailed information on the output levels for both parameters may be obtained by marginalizing on $S_1 \times S_2$.

When the two parameters are not independent, then a joint bba $m^{E_1E_2}$ has to be constructed from a two dimensional histogram of observations of both parameters, or from expert opinion as explained in Section 4.4, the rest of the analysis being carried out in the same way. A graphical illustration of such model is shown in Figure 7.

In the most general case, the parameters may be clustered into several classes, with the assumption of independence between classes. The above two-parameter models can then be generalized in an obvious way.

6 Simulation results

As an illustration of the above approach, we study in this section a simple, but realistic example of a treatment plant with a normal state x_0 and one degraded state x_1. In a first step (Sections 6.1 and 6.2), only one parameter (turbidity) will be considered, and the cases of precise and imprecise knowledge will be successively examined. In a second step (Section 6.3), the approach will be extended to two correlated parameters (turbidity and oxydability).

6.1 One parameter, precise data

In this first case, only one quality parameter (turbidity) is considered, and the available knowledge in assumed to consist in:

- the abatement rates for the normal mode $\alpha_0 = 0.8$ and the degraded mode $\alpha_1 = 0.2$;

- the failure rate and latency time of the failure mode: $\lambda_1 = 1e^{-3}$ h^{-1} and $T_1 = 2 \times 24$ h;

- 366 daily measurements of raw water turbidity, assumed to reflect the frequency distribution of that parameter in the whole population of measurements.

The output turbidity level was discretized in 11 states, with cut-points $\sigma_k = k$, $k = 1, \ldots, 10$. The induced thresholds η_j on the input turbidity level result in 18 input states $E = \{e_1, \ldots, e_{18}\}$. The histogram of turbidity values with these 18 classes is shown in Figure 8.

Figure 9 shows to the conditional possibility distribution on $E \times S$ given the nominal state x_0 (left) and the degraded state x_1 (right). As expected, only "diagonal" rectangles (i.e. those rectangles whose diagonal is crossed by the line $\mathbf{s}/\mathbf{e} = 1 - \alpha_i$) receive a possibility value equal to 1. The corresponding mass functions $m^{ES}[x_i]$ have a unique focal set equal to the union of these rectangles.

The cumulative belief and plausibility functions for the output turbidity level are displayed in Figure 10, together with the linearly interpolated cumulative probability function obtained using the probabilistic approach. The three functions coincide at each cutpoint σ_k, which confirms the equivalence property proved in Section 4.7. The probability of not exceeding the standard level $N = 6$ computed using this approach is 0.96 and, equivalently, the probability of exceeding N is 0.04.

6.2 One parameter, imprecise data

The above simulations were repeated, this time introducing some imprecision on input data:

$$\alpha_0^0 = 0.8, \quad \alpha_0^- = 0.7, \quad \alpha_0^+ = 0.9,$$

$$\alpha_1^0 = 0.2, \quad \alpha_1^- = 0.1, \quad \alpha_1^+ = 0.3,$$

$$\lambda_1^- = 0.8e^{-3}, \quad \lambda_1^+ = 1.2e^{-3},$$

$$T_1^- = 24, \quad T_1^+ = 3 \times 24.$$

Additionnally, a more reliable, but less precise estimation of the frequency distribution of input parameter values was computed by grouping the first three classes $\{s_1, s_2, s_3\}$ and the last five ones $\{s_{14}, \ldots, s_{18}\}$, leading to the twelve-class histogram shown in Figure 11.

Figure 12 shows the conditional possibility distributions on $E \times S$, related to the nominal state x_0 (left) and the degraded state x_1 (right). These possibility distributions are "blurred" versions of the crips relations of Figure 9 obtained under the assumption of precision of input data, each pair (e_j, s_k) being now assigned a degree of possibility.

The global result (combination of all available information and marginalization on S) is represented in Figure 13 in the form of the cumulative belief and plausibility functions $F_{bel}^{\mathcal{S}}$ and $F_{pl}^{\mathcal{S}}$. The belief of the event $\mathbf{s} < N$ is now 0.90, and the plausibility of that event is 0.99. Equivalently, the belief and plausibility of not meeting the quality standard are, respectively, $1 - 0.99 = 0.01$ and $1 - 0.90 = 0.10$, which indicates that the corresponding probability, for certain values of the input parameters within the range provided by experts, could be as high as 0.10, a much higher value than that computed using the classical approach.

Figure 14 illustrates the kind of sensitivity analysis that can be performed using the belief function approach. Here, the belief and plausibility of not exceeding the standard level ($N = 6$) is drawn against the relative degree of imprecision ϵ of input data: for instance, the value $\epsilon = 0.1$ corresponds to 10 % imprecision on input data α_0, α_1, λ_1 and T_1. The lower curve, corresponding to the degree of belief of not exceeding the standard level, is a more conservative assessment than the value computed using the probabilistic approach, which does not take into account the imperfectness of the available knowledge. As can be seen from this figure, this degree of belief decreases rapidly with ϵ, particularly for $\epsilon > 0.12$. As a relative uncertainty of 20 % on input data is not unrealistic in this application, the quantity $bel(\mathbf{s} < N)$ could be as low as 0.87, a significantly lower value than the optimistic assessment of 0.96 obtained when this source of uncertainty is ignored.

6.3 Two parameters, imprecise data

To illustrate the application of our method in the multiparameter case, the analysis was extended to take into account a second quality parameter: oxydability. A scatterplot for the 366 paired observations of these two parameters is shown in Figure 15. Also represented in this figure are the discretization thresholds for the input levels of these two parameters, and the corresponding estimated joint probability density.

The abatement rates for parameter $\mathbf{e}_1$ (turbidity) were given the same values as in Section 6.2. For parameter $\mathbf{e}_2$ (oxydability), the values were 0.7 (normal state) and 0.2 (degraded state), with an uncertainty of ± 0.1. The interval-valued failure rate and latency time for the degraded mode were fixed as above.

Tables 1 and 2 show, respectively, the belief and plausibility values for the event ($\mathbf{s}_1 < N_1$ and $\mathbf{s}_2 < N_2$) for possible values of the standard levels N_1 and N_2. These values are to be compared with the probabilities of the same events shown in Table 3, computed using the standard probabilistic approach, without taking into account the imprecision of basic data. For instance, the belief-plausibility interval for the event $\mathbf{e}_1 < 6$ and $\mathbf{e}_2 < 5$ is $[0.86, 0.98]$, whereas the optimistically biased estimated probability of that event is 0.94.

7 Conclusion

A case study concerning the application of the Transferable Belief Model to a risk assessment problem has been presented. The belief function formalism was shown to be flexible enough to encode such diverse input data as fuzzy values, interval-valued probabilities and statistical data in a single representation format. The result of such analysis is a belief network composed of nodes (one for each variable in the problem), and distinct belief functions connecting subsets of nodes. Combination of this information and marginalization on variables of interest can then be performed using local computational procedures, allowing to compute the belief and plausibility of various events relevant to the risk analysis.

This approach was shown to be more conservative than the classical probabilistic approach, which typically neglects the imprecision and uncertainty of input data. Whereas this assumption is reasonable in domains where a lot of experimental data and expert knowledge are available, it leads to overoptimistic conclusions when such good quality information is lacking. The belief function methodology exemplified in this paper can then lead to more reliable conclusions, while maintaining the consistency with the probabilistic approach when all necessary information can be collected. Consequently, this methodology appears to be particulary adequate in situations where only partial knowledge and scarce statistical data are available. Future work will aim at validating the approach on a wider range of applications. Important methodological issues such as the elicitation of belief functions from experts, or their induction from statistical data also remain to be fully investigated.

A Proof of Proposition 1 (sketch)

A.1 Proof of Part 1 (expression of $P^-(A)$ and $P^+(A)$)

Let $A \subseteq X$ and $P \in \mathcal{P}$. We have $P(A) \geq \sum_{x_i \in A} p_i^-$ and $P(A) = 1 - P(\overline{A}) \geq 1 - \sum_{x_i \notin A} p_i^+$. Hence,

$$P^-(A) \geq \max \left(\sum_{x_i \in A} p_i^-, 1 - \sum_{x_i \notin A} p_i^+ \right).$$

Similarly,

$$P^+(A) \leq \min \left(\sum_{x_i \in A} p_i^+, 1 - \sum_{x_i \notin A} p_i^- \right).$$

To see how these expressions simplify, two cases can be considered.

Case 1: $x_0 \notin A$**.** Then, we have $1 - \sum_{x_i \notin A} p_i^+ = \sum_{i=1}^I p_i^- - \sum_{x_i \notin A, i \neq 0} p_i^+$, and $\sum_{x_i \in A} p_i^- = \sum_{i=1}^I p_i^- - \sum_{x_i \notin A, i \neq 0} p_i^-$. Hence, $\sum_{x_i \in A} p_i^- \geq 1 - \sum_{x_i \notin A} p_i^+$, and we can retain $\sum_{x_i \in A} p_i^-$ as a lower bound for $P(A)$. This bound can be attained by setting $p_i = p_i^-$, $i = 1, \ldots, I$ and $p_0 = p_0^+$ Hence, $P^-(A) = \sum_{x_i \in A} p_i^-$. A similar line of reasoning holds for the upper bound, by exchanging plus and minus superscripts and changing the direction of inequalities. This leads to $\sum_{x_i \in A} p_i^+ \leq 1 - \sum_{x_i \notin A} p_i^-$, and $P^+(A) = \sum_{x_i \in A} p_i^+$.

Case 2: $x_0 \in A$. Then, $\sum_{x_i \in A} p_i^- = 1 - \sum_{i=1}^I p_i^+ + \sum_{x_i \in A, i \neq 0} p_i^-$ and $1 - \sum_{x_i \notin A} p_i^+ = 1 - \sum_{i=1}^I p_i^+ + \sum_{x_i \in A, i \neq 0} p_i^+$. Hence, $1 - \sum_{x_i \notin A} p_i^+ \geq \sum_{x_i \in A} p_i^-$ The bound $1 - \sum_{x_i \notin A} p_i^+$ can be attained by setting $p_i = p_i^+$, $i = 1, \ldots, I$ and $p_0 = p_0^-$. Hence, $P^-(A) = 1 - \sum_{x_i \notin A} p_i^+$. Once again, exchanging plus and minus signs and changing the direction of inequalities allows to show that $P^+(A) = 1 - \sum_{x_i \notin A} p_i^-$.

A.2 Proof of part 2 (expression of m^X)

First of all, it is easy to check that the masses in (23) and (24) sum to one. Next, let us consider the belief and plausibility values given to singletons. We have obviously $bel^X(\{x_i\}) = m^X(\{x_i\}) = p_i^- = P^-(\{x_i\})$ for $i = 0, \ldots, I$, and

$$pl^X(\{x_0\}) = m^X(\{x_0\}) + \sum_{i=1}^I m^X(\{x_0, x_i\}) = 1 - \sum_{i=1}^I p_i^- = p_0^+$$

$$pl^X(\{x_i\}) = m^X(\{x_i\}) + m^X(\{x_0, x_i\}) = p_i^- + p_i^+ - p_i^- = p_i^+ \quad i = 1, \ldots, I.$$

Hence bel^X and pl^X coincide with P^- and P^+, respectively, on singletons. To check that this is also the case for any $A \subseteq X$, $|A| > 1$, we consider again two cases.

Case 1: $x_0 \notin A$. We have $bel^X(A) = \sum_{x_i \in A} m^X(\{x_i\}) = \sum_{x_i \in A} p_i^- = P^-(A)$, and

$$\begin{aligned} pl^X(A) &= \sum_{x_i \in A} m^X(\{x_i\}) + \sum_{x_i \in A} m^X(\{x_0, x_i\}) \\ &= \sum_{x_i \in A} p_i^- + \sum_{x_i \in A} (p_i^+ - p_i^-) = \sum_{x_i \in A} p_i^+ = P^+(A). \end{aligned}$$

Case 2: $x_0 \in A$. Then,

$$\begin{aligned} bel^X(A) &= \sum_{x_i \in A} m^X(\{x_i\}) + \sum_{x_i \in A, i \neq 0} m^X(\{x_0, x_i\}) \\ &= p_0^- + \sum_{x_i \in A, i \neq 0} p_i^- + \sum_{x_i \in A, i \neq 0} (p_i^+ - p_i^-) = 1 - \sum_{x_i \notin A} p_i^+ = P^-(A), \end{aligned}$$

and

$$\begin{aligned} pl^X(A) &= \sum_{x_i \in A} m^X(\{x_i\}) + \sum_{i=1}^{I} m^X(\{x_0, x_i\}) \\ &= p_0^- + \sum_{x_i \in A, i \neq 0} p_i^- + \sum_{i=1}^{I}(p_i^+ - p_i^-) = 1 - \sum_{x_i \notin A} p_i^- = P^+(A). \end{aligned}$$

We have shown that $bel^X(A) = P^-(A)$ and $pl^X(A) = P^+(A)$ for all $A \subseteq X$, which completes the proof.

B Proof of Proposition 2

When the abatement rate α_i is known, there is a deterministic relationship between E and S. Working with the set of input states $\{A_{k,i} | 1 \leq k \leq K\}$ which is the relevant coarsening of $E = \{e_j | 1 \leq j \leq J\}$ for mode x_i, one can assert with certainty that if raw water is in state $A_{k,i}$ with station in mode x_i, then treated water is in state s_k. Consequently, the "diagonal rectangles" (i.e. whose diagonal is crossed by the line representing the transfer function) $A_{k,i} \times s_k$ must have a possibility value 1, all other ones a possibility value 0 (see Figure 3). This is indeed the result given by the discretization procedure described in Section 4: the possibility distribution defined by Equation (25) is equal to 1 on the line $\mathbf{s} = (1 - \alpha_i)\mathbf{e}$, and 0 elsewhere. The only one rectangles having a non empty intersection with this line are the diagonal ones, for which the line passes through the summit of the pyramid.

The corresponding mass function $m^{ES}[x_i]$ thus has a unique focal set in that case:

$$m^{ES}[x_i](B_i) = 1, \tag{28}$$

where B_i is defined as $B_i = \bigcup_{k=1}^{K} A_{k,i} \times \{s_k\}$.

The ballooning extension of $m^{ES}[x_i]$ yields:

$$m[x_i]^{ES \Uparrow XES}((\{x_i\} \times B_i) \cup (\{\overline{x_i}\} \times E \times S)) = 1. \tag{29}$$

In order to combine these belief functions for all x_i, let us calculate the intersection of the unique focal element of two such functions:

$$((\{x_i\} \times B_i) \cup (\{\overline{x_i}\} \times E \times S)) \cap ((\{x_j\} \times B_j) \cup (\{\overline{x_j}\} \times E \times S)) =$$

$$(\{x_i\} \times B_i) \cup (\{x_j\} \times B_j) \cup (\{\overline{x_i, x_j}\} \times E \times S).$$

Therefore, by immediate recurrence:

$$\bigcap_{i=0}^{I} (\{x_i\} \times B_i \cup \{\overline{x_i}\} \times E \times S) = \bigcup_{i=0}^{I} (\{x_i\} \times B_i). \tag{30}$$

Hence, if we note $m_a^{XES} = \textcircled{\cap}_{i=0}^{I} m^{ES}[x_i]^{\Uparrow XES}$, we obtain:

$$m_a^{XES} \left(\bigcup_{i=0}^{I} \{x_i\} \times B_i \right) = 1. \tag{31}$$

In a second step, we have to express our beliefs concerning the raw water and plant states. In the classical case, m^X and m^E are probability functions defined by $m^X(\{x_i\}) = p_i$ $(i = 0, \ldots, I)$ and $m^E(\{e_j\}) = q_j$ $(j = 1, \ldots, J)$. The vacuous extension of these bba's on the joint space yields:

$$m^{X\uparrow XES}(\{x_i\} \times E \times S) \quad = \quad p_i \quad i = 0, \ldots, I \tag{32}$$

$$m^{E\uparrow XES}(X \times \{e_j\} \times S) \quad = \quad q_j \quad j = 1, \ldots, J. \tag{33}$$

Let m_b^{XES} denote the combination of m_a^{XES} with $m^{X\uparrow XES}$. We have:

$$m_b^{XES}(\{x_i\} \times B_i) = p_i \quad i = 0, \ldots, I. \tag{34}$$

Finally, let m^{XES} denote the combination of m_b^{XES} with $m^{E\uparrow XES}$. By construction, there is only one k such that $e_j \in A_{k,i}$, considering a fixed state i of the plant. Consequently, the resulting focal sets are all of the form:

$$(\{x_i\} \times \bigcup_{k=1}^{K} (A_{k,i} \times \{s_k\})) \cap (X \times \{e_j\} \times S) = \{(x_i, e_j, s_k)\} \tag{35}$$

with $e_j \in A_{k,i}$. Hence, m^{XES} is a Bayesian bba (a probability function). It is defined as:

$$m^{XES}(\{(x_i, e_j, s_k)\}) = \begin{cases} p_i q_j & \text{if } e_j \in A_{k,i}, \\ 0 & \text{otherwise.} \end{cases}$$

Let us now compute the degree of belief in the event $\mathbf{s} \geq N$. Denoting by k_0 the index such that $N = \sigma_{k_0-1}$, and by D_{k_0} the set of output states corresponding to noncompliant water: $\{s_{k_0}, \dots, s_K\}$, this degree of belief is equal to

$$\begin{aligned} bel^S(D_{k_0}) &= bel^{XES}(X \times E \times D_{k_0}) \\ &= \sum_{\{(i,j,k)|e_j \in A_{ki}, k \geq k_0\}} m^{XES}(\{(x_i, e_j, s_k)\}) \\ &= \sum_{i=0}^{I} \sum_{\{j|\eta_{j-1} \geq \theta_{k_0-1,i}\}} p_i q_j \\ &= \sum_{i=0}^{I} p_i bel^E(\mathbf{e} \geq \theta_{k_0-1,i}). \end{aligned}$$

By definition, the notation $\theta_{k_0-1,i}$ has the same meaning as θ_i in (2). Hence, $bel^S(D_{k_0})$ is equal to $P(\mathbf{s} \geq N)$, which completes the proof.

References

[1] R. G. Almond. *Graphical belief models*. Chapman and Hall, London, 1995.

[2] N. O. Attoh-Okine. Aggregating evidence in pavement management decision-making using belief functions and qualitative Markov trees. *IEEE Transaction on Systems, Man and Cybernetics C*, 32(3):243–251, 2002.

[3] S. Démotier, K. Odeh, W. Schön, P. Charles, F. Fotoohi, and J.-F. Allioux. Risk assessment for drinking water production process. In *Proceedings of European Conference on System Dependability and Safety*, pages 544–550, Lyon, France, 2002.

[4] A. P. Dempster. Upper and lower probabilities induced by a multivalued mapping. *Annals of Mathematical Statistics*, 38:325–339, 1967.

[5] T. Denœux. A k-nearest neighbor classification rule based on Dempster-Shafer theory. *IEEE Trans. on Systems, Man and Cybernetics*, 25(05):804–813, 1995.

[6] T. Denœux. A neural network classifier based on Dempster-Shafer theory. *IEEE Trans. on Systems, Man and Cybernetics A*, 30(2):131–150, 2000.

[7] T. Denoeux and A. Ben Yaghlane. Approximating the combination of belief functions using the fast moebius transform in a coarsened frame. *International Journal of Approximate Reasoning*, 31(1-2):77–101, 2002.

[8] D. Dubois and H. Prade. *Possibility Theory: An approach to computerized processing of uncertainty.* Plenum Press, New-York, 1988.

[9] D. Dubois, H. Prade, and Ph. Smets. Representing partial ignorance. *IEEE Transactions on Systems, Man and Cybernetics*, 26:361–377, 1996.

[10] G. J. Klir and M. J. Wierman. *Uncertainty-Based Information. Elements of Generalized Information Theory.* Springer-Verlag, New-York, 1998.

[11] J.-M. Lainé, S. Démotier, K. Odeh, W. Schön, and P. Charles. Risk assessment for drinking water production. assessing the potential risk due to the presence of cryptosporidium oocysts in water. In *Proceedings of IWA*, Berlin, Germany, 2001.

[12] N. Milisavljević and I. Bloch. Sensor fusion in anti-personnel mine detection using a two-level belief function model. *IEEE Transaction on Systems, Man and Cybernetics C*, 33(2):269–283, 2003.

[13] G. Shafer. *A mathematical theory of evidence.* Princeton University Press, Princeton, N.J., 1976.

[14] G. Shafer, P. P. Shenoy, and K. Mellouli. Propagating belief functions in qualitative Markov trees. *International Journal of Approximate Reasoning*, 1:349–400, 1987.

[15] P. P. Shenoy. A valuation-based language for expert systems. *International Journal of Approximate Reasoning*, 3:383–411, 1989.

[16] P. P. Shenoy. Binary joint trees for computing marginals in the Shenoy-Shafer architecture. *International Journal of Approximate Reasoning*, 17:239–263, 1997.

[17] P. Smets. Belief functions: the disjunctive rule of combination and the generalized Bayesian theorem. *International Journal of Approximate Reasoning*, 9:1–35, 1993.

[18] P. Smets. The Transferable Belief Model for quantified belief representation. In D. M. Gabbay and P. Smets, editors, *Handbook of Defeasible reasoning and uncertainty management systems*, volume 1, pages 267–301. Kluwer Academic Publishers, Dordrecht, 1998.

[19] P. Smets. Practical uses of belief functions. In K. B. Laskey and H. Prade, editors, *Uncertainty in Artificial Intelligence 15 (UAI99)*, pages 612–621, Stockholm, Sweden, 1999.

[20] P. Smets and R. Kennes. The Transferable Belief Model. *Artificial Intelligence*, 66:191–243, 1994.

[21] P. Walley. *Statistical Reasoning with Imprecise Probabilities.* Chapman and Hall, London, 1991.

[22] L. A. Zadeh. Fuzzy sets as a basis for a theory of possibility. *Fuzzy Sets and Systems*, 1:3–28, 1978.

Table 1: Degree of belief in the event: $\mathbf{s}_1 < N_1$ and $\mathbf{s}_2 < N_2$, for different values of N_1 and N_2.

	N_2			
N_1	3	4	5	6
2	0.2708	0.4061	0.4147	0.4203
4	0.4510	0.7262	0.7498	0.7620
6	0.4966	0.8262	0.8624	0.886
8	0.5008	0.8405	0.8824	0.9127
10	0.5032	0.8491	0.8944	0.9292

Table 2: Plausibility of the event: $\mathbf{s}_1 < N_1$ and $\mathbf{s}_2 < N_2$, for different values of N_1 and N_2.

	N_2			
N_1	3	4	5	6
2	0.6627	0.6909	0.6943	0.6944
4	0.8841	0.9454	0.9498	0.9510
6	0.9009	0.9734	0.9792	0.9826
8	0.9026	0.9765	0.9825	0.9874
10	0.9026	0.9765	0.9827	0.9885

Table 3: Probability of the event: $\mathbf{s}_1 < N_1$ and $\mathbf{s}_2 < N_2$, computed using the classical approach, for different values of N_1 and N_2.

	N_2			
N_1	3	4	5	6
2	0.5269	0.5477	0.5556	0.5556
4	0.8164	0.8842	0.8924	0.8936
6	0.8372	0.9337	0.9448	0.9518
8	0.8398	0.9416	0.9527	0.9613
10	0.8398	0.9416	0.9527	0.9629

Figures

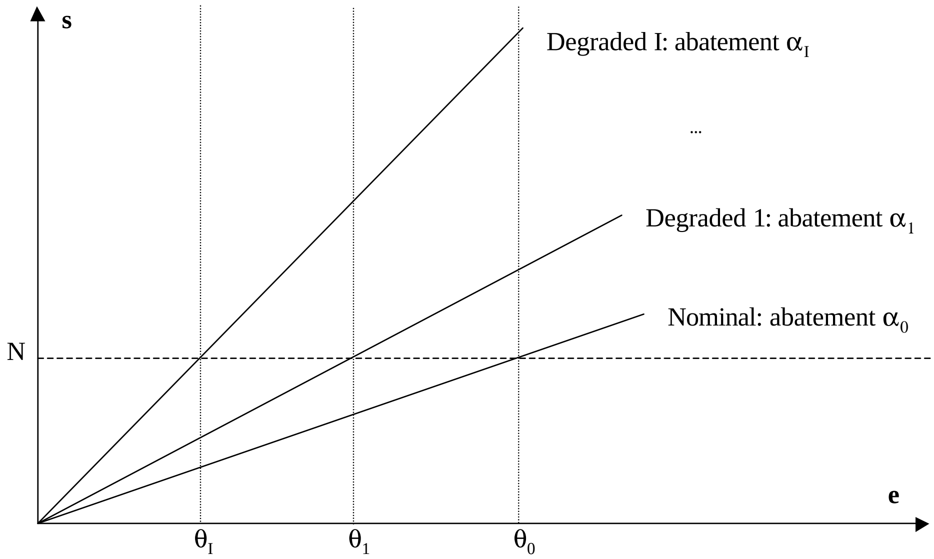

Figure 1: Transfer function (output concentration of an undesirable water characteristics, as a function of the input concentration) in nominal mode and for I degraded mode of the treatment plant.

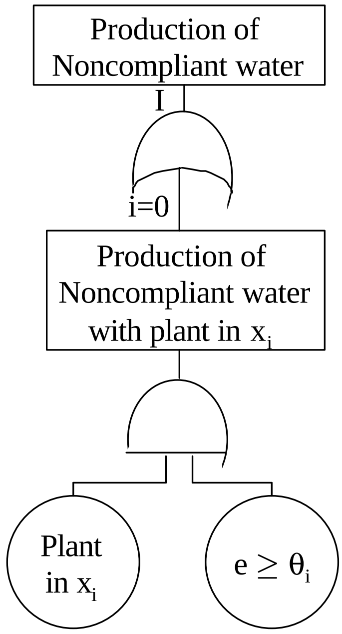

Figure 2: Fault tree analysis of event “Production of non compliant water ”.

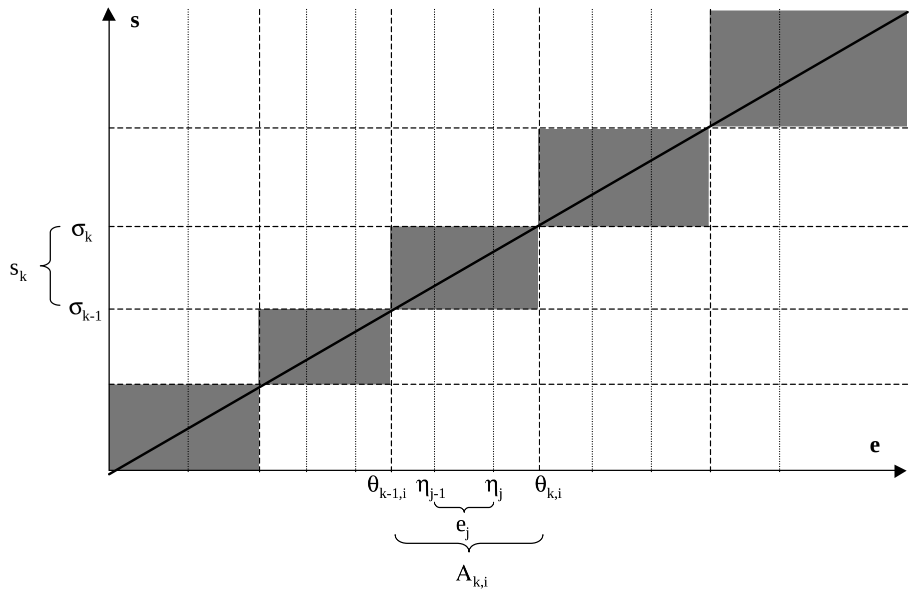

Figure 3: Discretization of input and output levels of the quality parameter. The bold diagonal line represents the transfer function of mode x_i: $\mathbf{s} = (1 - \alpha_i)\mathbf{e}$. The gray rectangles are crossed by this line along their diagonal.

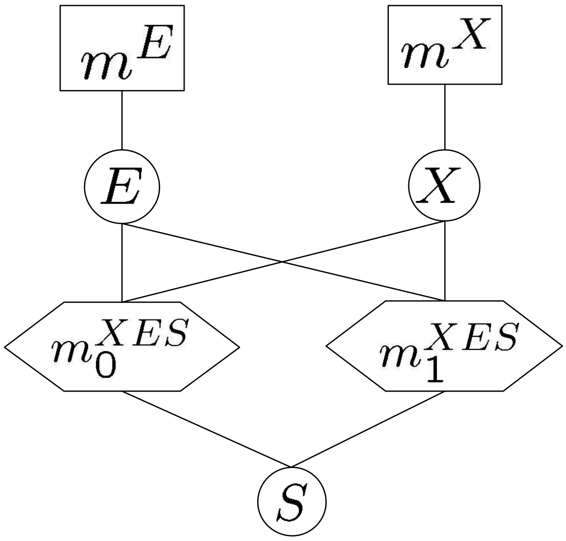

Figure 4: Belief network for the TBM model in the one-parameter case, with $I = 1$ (one failure mode). Each circle corresponds to a variable in the model. Hyperedges between sets of variables are labeled by belief functions on the corresponding product space.

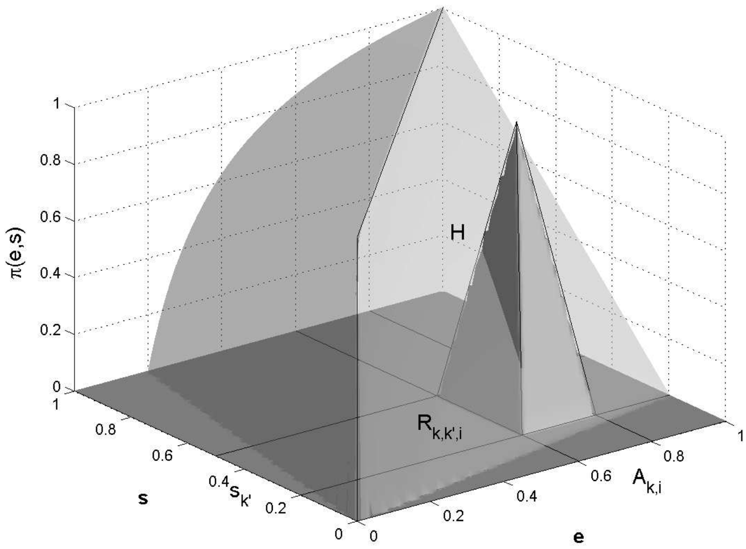

Figure 5: Possibility distribution on $E \times S$ and discretization procedure. The possibility distribution defined by (25) is represented by two pieces of hyperbolic paraboloids. The discretized possibility value given to the rectangle $R_{k,k',i} = A_{k,i} \times s_{k'}$ is defined as the height of the intersection between the surface representing the continuous possibility distribution, and a pyramid with base $R_{k,k',i}$, which corresponds to the height of point H in this figure.

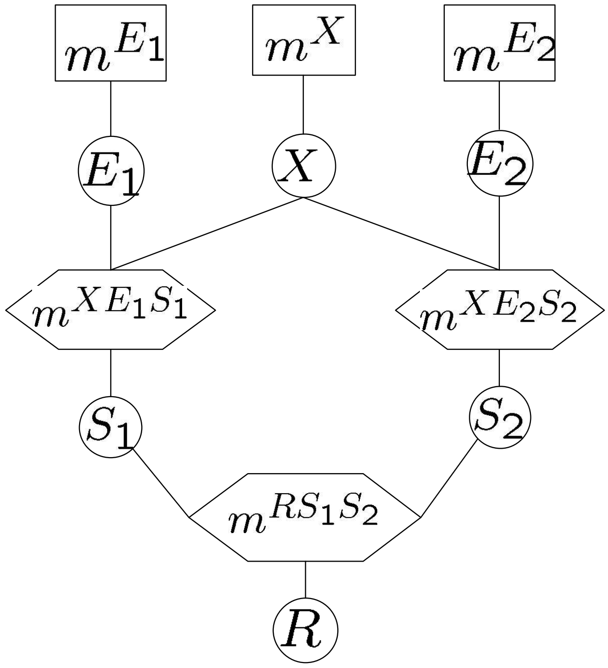

Figure 6: Belief network for the TBM model in the case of two independent quality parameters.

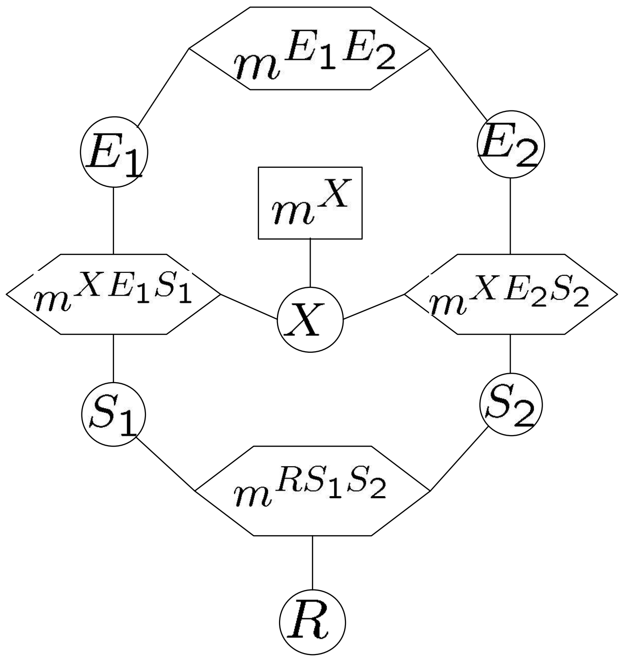

Figure 7: Belief network for the TBM model in the case of two dependent quality parameters.

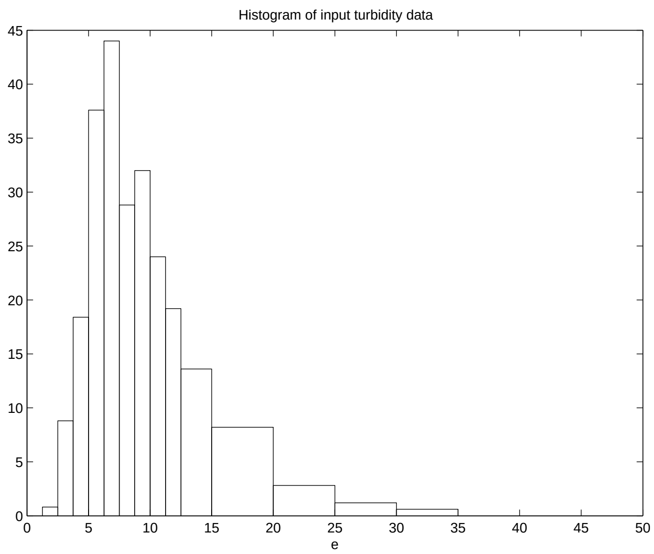

Figure 8: Histogram of 366 turbidity measurements in raw water.

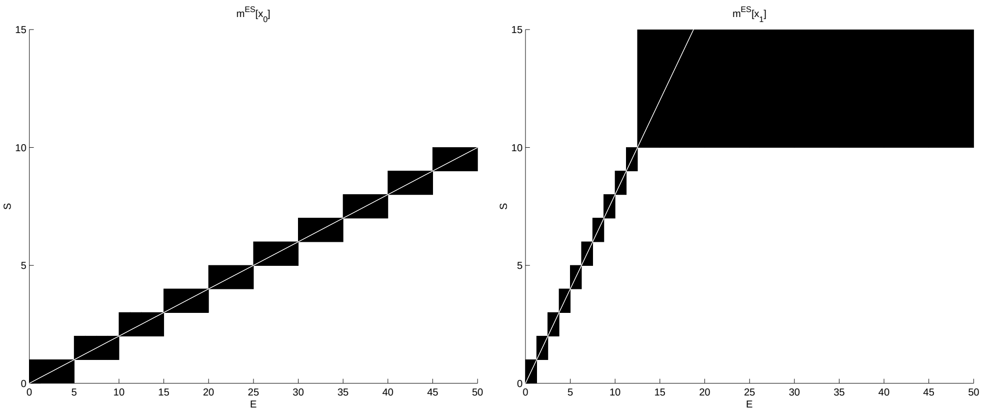

Figure 9: Conditional possibility distribution on $E \times S$ given the nominal state x_0 (left) and the degraded state x_1 (right), in the case of precise data.

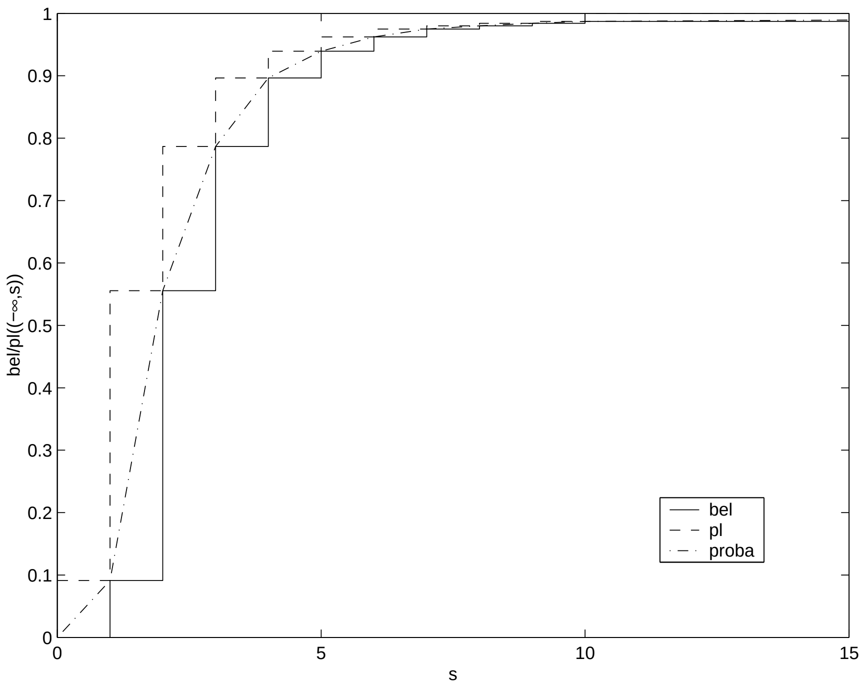

Figure 10: Cumulative belief and plausibility functions (precise data), with linearly interpolated cumulative probability function obtained using the probabilistic approach.

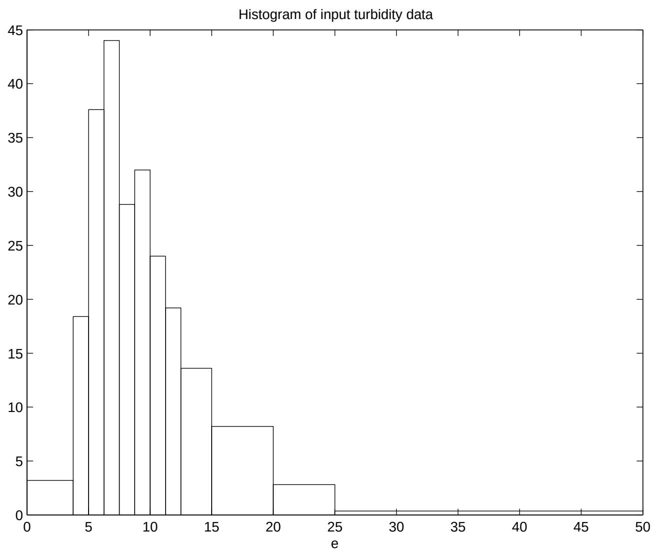

Figure 11: Histogram of 366 turbidity measurements in raw water.The first two states have been grouped, as well as the last three ones.

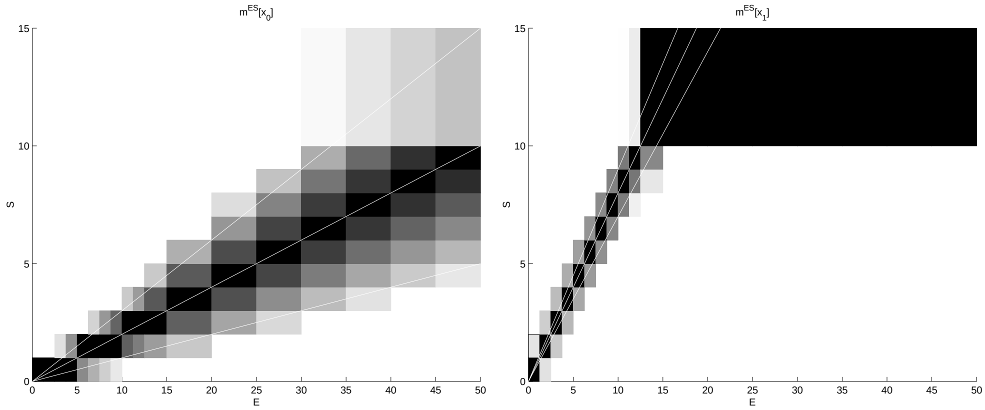

Figure 12: Conditional possibility distribution on $E \times S$ given the nominal state x_0 (left) and the degraded state x_1 (right), in the case of imprecise data.

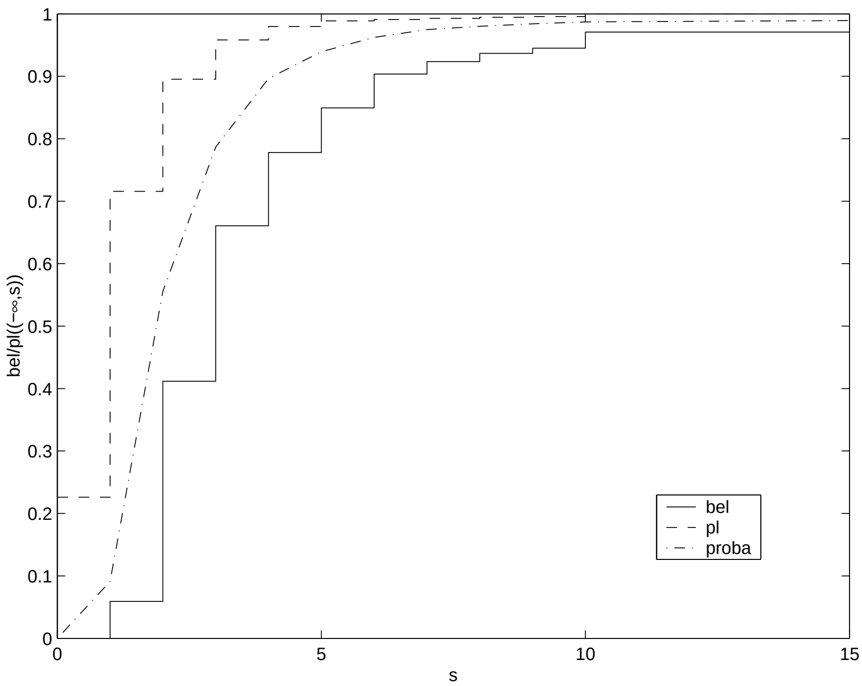

Figure 13: Cumulative belief and plausibility functions (imprecise data), and linearly interpolated cumulative probability function obtained using the probabilistic approach with the precise data of Section 6.1.

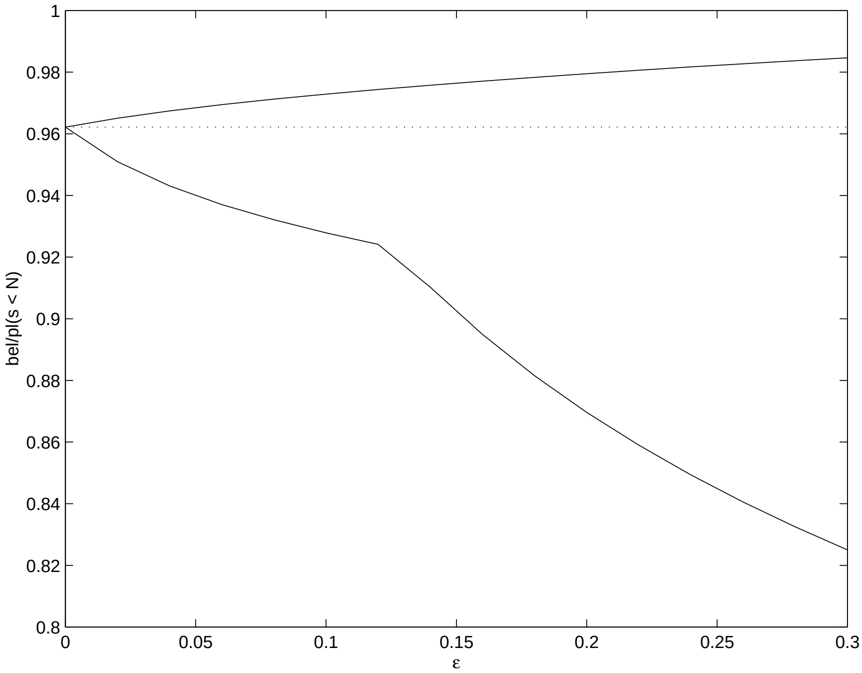

Figure 14: Belief and plausibility of not exceeding the standard level ($N = 6$), as a function of the imprecision on input data (α_0, α_1, T and λ), the input turbidity level frequency distribution being estimated with 9 histogram classes. The horizontal dotted line corresponds to the probability computed by the classical procedure.

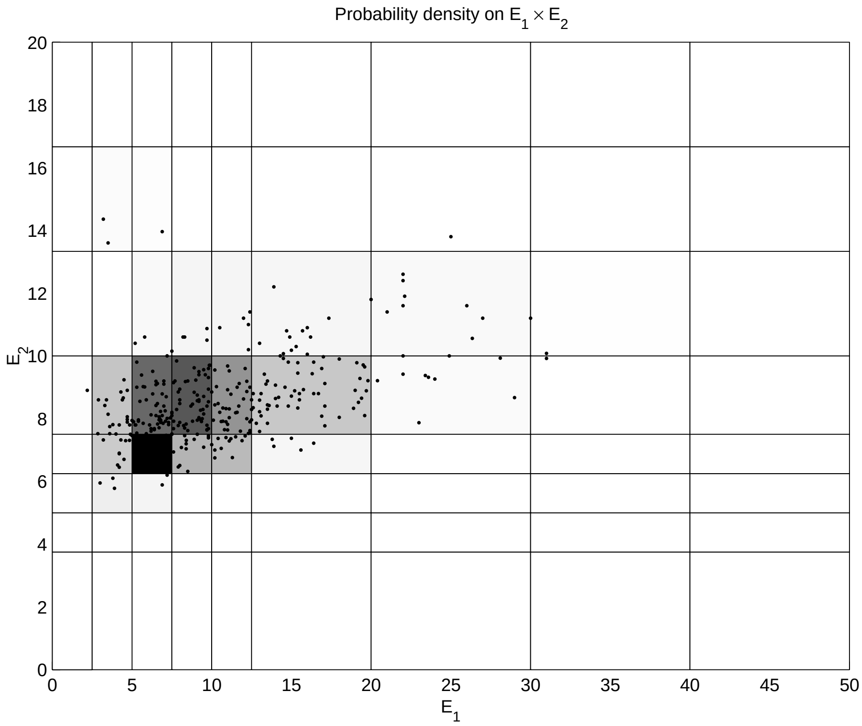

Figure 15: Input data for the two parameter case (turbidity and oxydability), and two-dimensional histogram (estimated probability density values are represented as gray levels).