SCI 22 – Refinement and coarsening of a frame of discernment

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We have already seen that the definition of a frame of discernment is, to some extent, a matter of convention, as its granularity is often a matter of choice. For instance, the income of a household can be expressed in a real interval $[0, r_{max}]$, or in a discrete frame defined, e.g., by the quartiles or the deciles. Different sources of information may provide evidence represented in frames of different granularities. Also, when several variables are defined, we may receive evidence about different subsets of variables, and we may wish to express the result of the analysis according to some specific subset containing the variables of interest. In evidential reasoning with uncertain information, we thus have to express belief functions in different frames with varying granularities.

1 Varying the granularity of the frame of discernment

Let Ω and Θ be two frames of discernment. We say that Ω is a *refinement* of a Θ (or, equivalently, Θ is a coarsening of Ω) if elements of Ω can be obtained by splitting some or all of the elements of Θ (Figure 1). Formally, Ω is a refinement of a frame Θ iff there is a mapping $\rho : 2^\Theta \to 2^\Omega$ such that:

- $\{\rho(\{\theta\}), \theta \in \Theta\} \subseteq 2^\Omega$ is a partition of Ω, and
- For all $A \subseteq \Omega$, $\rho(A) = \bigcup_{\theta \in A} \rho(\{\theta\})$.

Let m^Θ be a mass function representing some piece of evidence, and let Ω be a refinement of Θ. We can carry m^Θ from Θ to Ω by transferring each mass $m^\Theta(A)$ to $\rho(A)$. The resulting mass function is denoted by $m^{\Theta\uparrow\Omega}$ and is called the vacuous extension of m^Θ in Ω: for all $B \subseteq \Omega$,

$$m^{\Theta\uparrow\Omega}(B) = \begin{cases} m^\Theta(A) & \text{if } B = \rho(A), A \subseteq \Omega, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Conversely, given a mass function m^Ω on Ω, how to express it in the coarser frame Θ? Here, the solution is not so obvious because the mapping

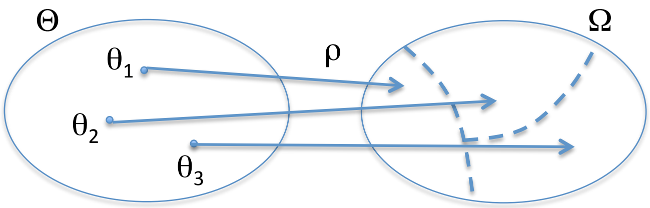

Figure 1: Refinement of a frame of discernment.

ρ is not invertible. However, we can define two generalized inverses of ρ as

$$\underline{\rho}^{-1}(B) = \{\theta \in \Theta | \rho(\{\theta\}) \subseteq B\} \tag{2}$$

$$\overline{\rho}^{-1}(B) = \{\theta \in \Theta | \rho(\{\theta\}) \cap B \neq \emptyset\}, \tag{3}$$

for all subset B of Ω. The subsets $\underline{\rho}^{-1}(B)$ and $\overline{\rho}^{-1}(B)$ are called, respectively, the *inner and outer reductions* of B. A mass function m^Ω can then be expressed in Θ by transferring each mass $m^\Omega(B)$ to the outer reduction of B. The resulting mass function is denoted by $m^{\Omega\downarrow\Theta}$ and is called the *restriction* of m^Ω in Θ: for all subset A of Θ,

$$m^{\Omega\downarrow\Theta}(A) = \sum_{\overline{\rho}^{-1}(B)=A} m^\Omega(B). \tag{4}$$

We may observe that, in the process of carrying m^Ω from Ω to the coarser frame Θ, some information may be lost. In particular, if $m^{\Omega\downarrow\Theta}$ is carried back to Ω, we will not recover m^Ω in general. The resulting mass function will usually be less informative than m^Ω, because

$$\rho[\overline{\rho}^{-1}(B)] \supseteq B \tag{5}$$

for any subset B of Ω, and the inclusion may be strict.

2 Special case of product spaces

2.1 Marginalization and vacuous extension

Let us now assume that we have two frames Ω_X and Ω_Y related to two different questions about, e.g., the values of two unknown variables X and Y. Let $\Omega_X \times \Omega_Y$ be the product space. It is a refinement of both Ω_X and Ω_Y. For instance, we can define the following mapping ρ from 2^{Ω_X} to $2^{\Omega_X \times \Omega_Y}$:

$$\rho(B) = B \times \Theta, \tag{6}$$

for all $B \subseteq \Omega_X$. The set $\rho(B)$ is called the *cylindrical extension* of B in $\Omega_X \times \Omega_Y$ and is denoted by $B \uparrow \Omega_X \times \Omega_Y$.

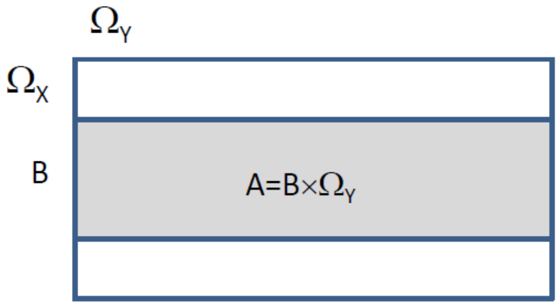

Figure 2: Vacuous extension.

The *vacuous extension* of a mass function m^X from Ω_X to $\Omega_X \times \Omega_Y$ is obtained by transferring each mass $m^X(B)$ for any subset B of Ω_X to the cylindrical extension of B (Figure 2):

$$m^{X\uparrow XY}(A) = \begin{cases} m^X(B) & \text{if } A = B \times \Omega_Y \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

Conversely, let m^{XY} be a joint mass function on the product space $\Omega_X \times \Omega_Y$. Typically, such a mass function represents partial knowledge about the relation between variables X and Y. Now, assume that we are only interested in evidence about Ω_X. We then have to compute the restriction of m^{XY} to the coarser frame Ω_X:

$$m^{XY\downarrow X}(B) = \sum_{A\downarrow\Omega_X = B} m^{XY}(A), \tag{8}$$

where $A \downarrow \Omega_X$ denotes the projection of B on Ω_X:

$$A \downarrow \Omega_X = \{x \in \Omega_X | \exists y \in \Omega_Y, (x, y) \in A\}. \tag{9}$$

The mass functions $m^{XY\downarrow X}$ and $m^{XY\downarrow Y}$ are called the *marginals* of m^{XY} and the operation that computes the marginals from a joint mass functions is called *marginalization* (Figure 3). We can observe that this operation extends both set projection and marginalization of joint probability distributions.

2.2 Application to evidential reasoning

Most problems in engineering or economics can be modeled by defining variables and relations between variables. Based on partial information about some variables, the problem is then to infer the values of variables of interest. This problem can be cast in the belief function framework, as relations are

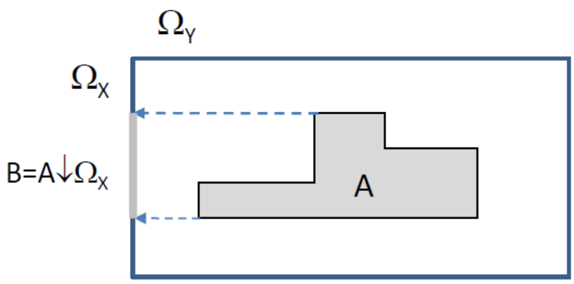

Figure 3: Marginalization.

sets and can thus be represented by joint mass functions. The three fundamental operations in evidential reasoning are Dempster's rule, marginalization and vacuous extension. For instance, assume for simplicity that we have only two variables X and Y and we have:

- Partial knowledge of X formalized as a mass function m^X, and
- A joint mass function m^{XY} representing an *uncertain relation* between X and Y.

These two pieces of evidence can be combined by vacuously extending m^X to Ω_{XY} and combining $m^{X\uparrow XY}$ with m^{XY}. The combined joint mass function can then be marginalized on Ω_Y. Formally,

$$m^Y = \left(m^{X\uparrow XY} \oplus m^{XY}\right)^{\downarrow Y}. \tag{10}$$

We can remark that these operations become infeasible with many variables and large frames of discernment, but efficient algorithms exist to carry out the operations in frames of minimal dimensions [1].

References

[1] R. G. Almond. *Graphical belief models.* Chapman and Hall, London, 1995.