Computational Statistics. Chapter 3: EM algorithm. Solution of exercises

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Exercise 1

Question 1a

We first give values to the model parameters:

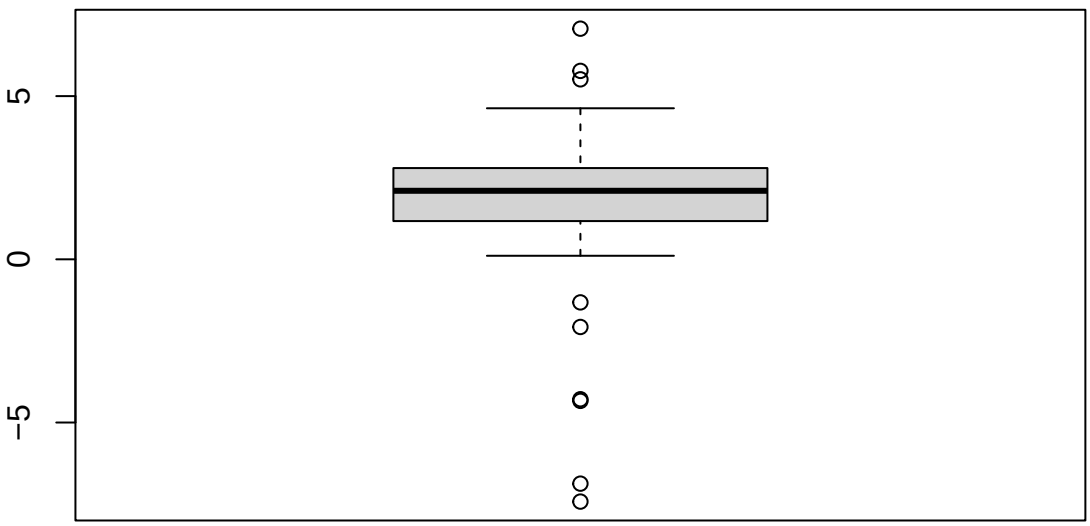
```
pi <- 0.90
mu <- 2
sig <- 1
a <- 10
c <- 1/(2*a)
n <- 100
```

We then generate the data:

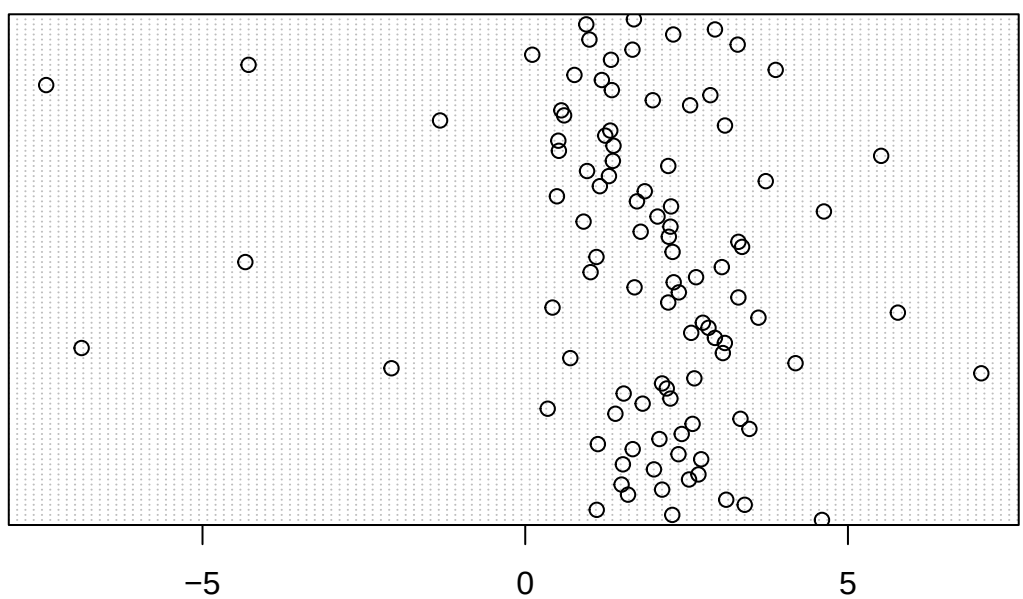
```
y<-vector("numeric",n)
z<-vector("numeric",n)
for(i in 1:n){
  z[i] <- sample(c(1,0),size=1,prob=c(pi,1-pi))
  if(z[i]==1)
    y[i] <- rnorm(1,mean=mu,sd=sig)
  else y[i] <- runif(1,min=-a,max=a)
}
```

Finally, we generate box and dot plots the data:

```
boxplot(y)
```




```r
dotchart(y)
```



Question 1b

We first write a function that computes the observed-data log-likelihood:

```r
loglik<- function(theta,y){
    phi <- sapply(y,dnorm,mean=theta[1],sd=theta[2])
    logL <- sum(log(theta[3]*phi+(1-theta[3])*c))
    return(logL)
}
```

We then write the EM algorithm for this problem. The inputs are the data `y`, the initial parameter value `theta0`, the constant `a`, the threshold `epsi` used in the stopping criterion, and a flag `disp` that controls the display of the intermediate results. The outputs are the maximum observed-data log-likelihood, the corresponding MLE of θ, and the vector $\boldsymbol{z}$ of estimated probabilities.

```r
em_outlier <- function(y,theta0,a,epsi,disp=TRUE){
    go_on<-TRUE
    logL0 <- loglik(theta0,y)
    t<-0
    c<-1/(2*a)
    n<-length(y)
    if(disp) print(c(t,logL0))
    while(go_on){
        t<-t+1
        # E-step
        phi <- sapply(y,dnorm,mean=theta0[1],sd=theta0[2])
        z<-  phi*theta0[3]/(phi*theta0[3]+c*(1-theta0[3]))
        # M-step
        S<- sum(z)
        pi<-S/n
        mu<- sum(y*z)/S
        sig<-sqrt(sum(z*(y-mu)^2)/S)
        theta<-c(mu,sig,pi)
        logL<-loglik(theta,y)
        if (logL-logL0 < epsi) go_on <- FALSE
        logL0 <- logL
        theta0<-theta
        if(disp) print(c(t,logL))
        }
    return(list(loglik=logL,theta=theta,z=z))
}
```

Question 1c

Let us now run the above function with our data. We initialize parameters μ and σ with the mean and standard deviation of the data, and we set $\pi_0 = 0.5$:

```r
mu0<-mean(y) # +rnorm(1,mean=0,sd=0.5)
sig0<-sd(y)
pi0<-0.5
theta0<-c(mu0,sig0,pi0)
```

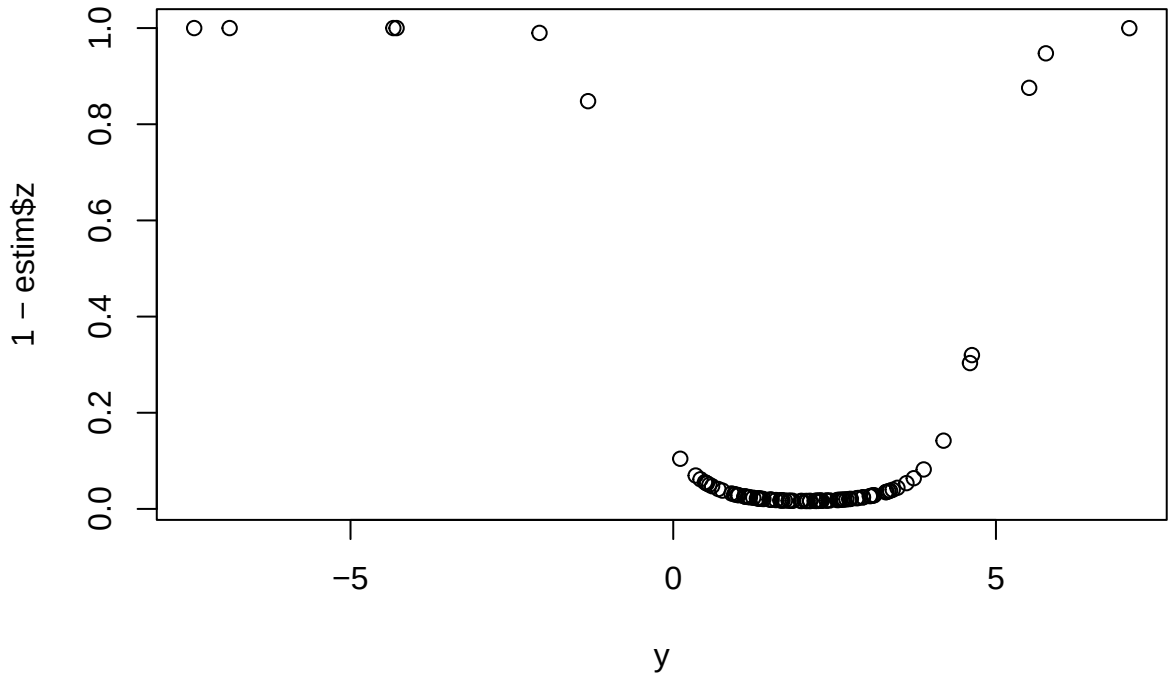
We then run function `em_outlier`:

```r
estim<-em_outlier(y,theta0,a,epsi=1e-6)
```

```
## [1]    0.0000 -230.5982
## [1]    1.000 -190.173
## [1]    2.0000 -183.3387
## [1]    3.0000 -182.9011
## [1]    4.0000 -182.8613
## [1]    5.000 -182.854
## [1]    6.0000 -182.8523
## [1]    7.0000 -182.8519
## [1]    8.0000 -182.8518
## [1]    9.0000 -182.8518
## [1]   10.0000 -182.8518
## [1]   11.0000 -182.8518
## [1]   12.0000 -182.8518
```

Finally, we plot the estimated probabilities $1 - z_i$ against the inputs y_i:

```
plot(y,1-estim$z)
```



We can see that the outliers have a high estimated probability of being drawn from the uniform distribution, as expected.

Exercise 2

Question 2a

We set the parameters:

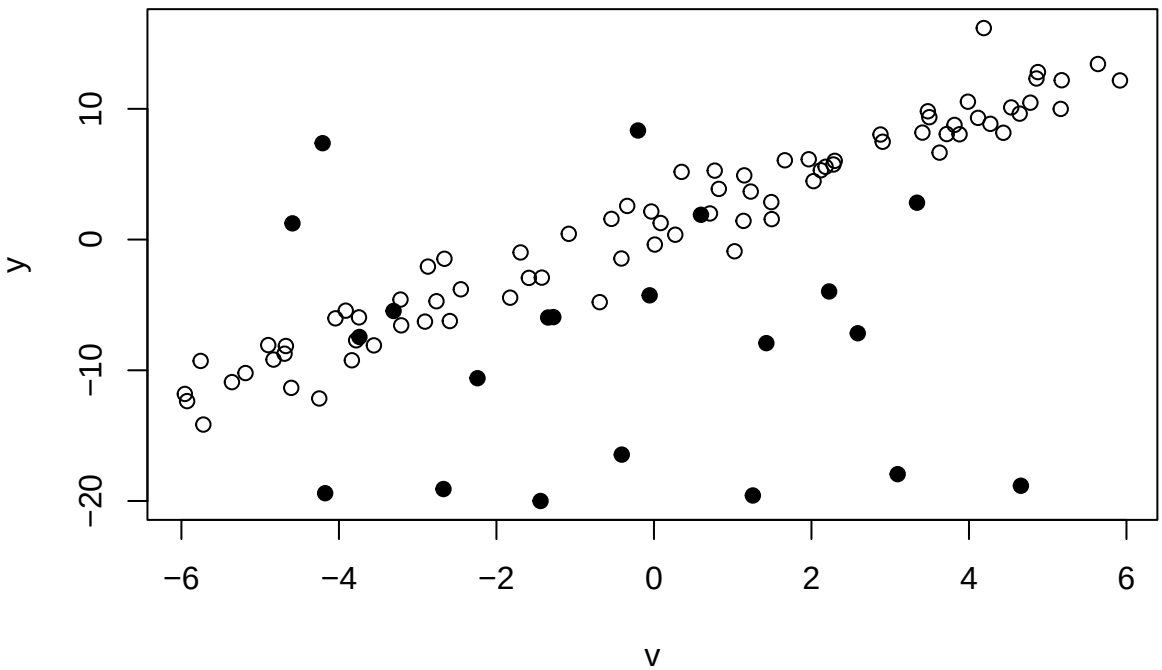
```
pi<-0.8
beta<-c(1,2)
sig<- 2
a<-20
c<-1/(2*a)
n<- 100
```

We then generate the data:

```
y<-vector("numeric",n)
v <- runif(n,min=-6,max=6)
z<-vector("numeric",n)
for(i in 1:n){
  z[i]=sample(c(1,0),size=1,prob=c(pi,1-pi))
  if(z[i]==1)
    y[i]<-rnorm(1,mean=beta[1]+v[i]*beta[2],sd=sig)
  else y[i]<-runif(1,min=-a,max=a)
}
```

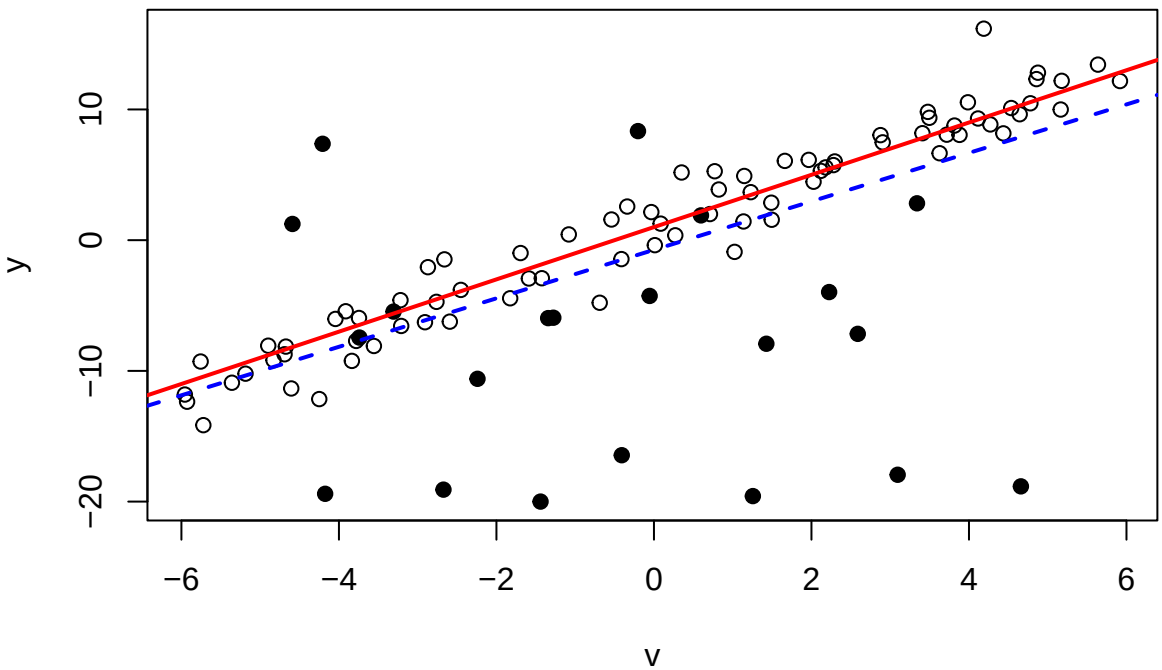
Finally, we plot the data. The data points generated from the uniform distribution (outliers) are highlighted:

```r
plot(v,y)
points(v[z==0],y[z==0],pch=16)
```



Question 2b

```r
reg<-lm(y~v)
plot(v,y)
points(v[z==0],y[z==0],pch=16)
abline(reg,lty=2,col="blue",lwd=2) # LS line
abline(1,2,lty=1,col="red",lwd=2) #true regression line
```



Question 2c

In this problem the observed data is the vector $\mathbf{y} = (y_1, \ldots, y_n)$. The observed-data likelihood is

$$L(\boldsymbol{\theta}) = \prod_{i=1}^{n} f(y_i; \boldsymbol{\theta}) = \prod_{i=1}^{n} \left[\pi\phi(y_i, v_i^T\beta, \sigma) + (1-\pi)c\right].$$

The missing data is the random vector $\mathbf{Z} = (Z_1, \ldots, Z_n)$, where $Z_i = 1$ if the observation is not an outlier, and $Z_i = 0$ otherwise. Clearly, Z_i has a Bernoulli distribution $\mathcal{B}(\pi)$. The complete-data likelihood is

$$L_c(\boldsymbol{\theta}) = \prod_{i=1}^{n} f(y_i \mid z_i; \boldsymbol{\theta}) f(z_i; \boldsymbol{\theta}) = \prod_{i=1}^{n} \left[\phi(y_i, v_i^T\beta, \sigma)^{z_i} c^{1-z_i} \pi^{z_i} (1-\pi)^{1-z_i}\right],$$

and its logarithm is

$$\ell_c(\boldsymbol{\theta}) = \sum_{i=1}^{n} \left[z_i \log\phi(y_i, v_i^T\beta, \sigma) + (1-z_i)\log c + z_i \log\pi + (1-z_i)\log(1-\pi)\right].$$

Function Q is obtained by replacing the terms z_i by their conditional expectations given $\mathbf{Y} = \mathbf{y}$, for $\boldsymbol{\theta} = \boldsymbol{\theta}^{(t)}$. In the E-step, we will compute these conditional expectations as

$$z_i^{(t)} = \mathbb{E}_{\boldsymbol{\theta}^{(t)}}[Z_i \mid y_i] = \mathbb{P}_{\boldsymbol{\theta}^{(t)}}[Z_i = 1 \mid y_i] = \frac{\phi(y_i; v_i^T\beta^{(t)}, \sigma^{(t)})\pi^{(t)}}{\phi(y_i; v_i^T\beta^{(t)}, \sigma^{(t)})\pi^{(t)} + c(1-\pi^{(t)})} \tag{1}$$

and

$$\begin{aligned} Q(\boldsymbol{\theta}, \boldsymbol{\theta}^{(t)}) &= \sum_{i=1}^{n} \left[z_i^{(t)} \log\phi(y_i, v_i^T\beta, \sigma) + (1-z_i^{(t)})\log c + z_i^{(t)}\log\pi + (1-z_i^{(t)})\log(1-\pi)\right] \\ &= \underbrace{\sum_{i=1}^{n} z_i^{(t)} \log\phi(y_i, v_i^T\beta, \sigma)}_{A} + \underbrace{\log\pi \sum_{i=1}^{n} z_i^{(t)} + \log(1-\pi)\left(n - \sum_{i=1}^{n} z_i^{(t)}\right)}_{B} + \underbrace{(1-z_i^{(t)})\log c}_{C}. \end{aligned}$$

In the M-step, we maximize $Q(\boldsymbol{\theta}, \boldsymbol{\theta}^{(t)})$ with respect to $\boldsymbol{\theta}$. In the above equation A depends only on β and σ, B depends on π, and C is a constant. We can thus maximize A and B separately. We have

$$A = -\frac{1}{2}\left(\sum_{i=1}^{n} z_i^{(t)}\right)\log(2\pi) - \left(\sum_{i=1}^{n} z_i^{(t)}\right)\log\sigma - \frac{1}{2\sigma^2}\sum_{i=1}^{n} z_i^{(t)}(y_i - v_i^T\beta)^2.$$

We can see that the vector β maximizing A (and Q) is the solution of a weighted least-squares problem (each individual error term being weighted by $z_i^{(t)}$). We thus have the following update rule for β:

$$\beta^{(t+1)} = \left[\sum_{i=1}^{n} z_i^{(t)} v_i v_i^T\right]^{-1} \sum_{i=1}^{n} z_i^{(t)} v_i y_i.$$

To maximize A w.r.t. σ, we set $\beta = \beta^{(t+1)}$ and we compute the derivative:

$$\frac{\partial A}{\partial \sigma} = -\frac{1}{\sigma}\sum_i z_i^{(t)} + \frac{1}{\sigma^3}\sum_i z_i^{(t)}(y_i - v_i^T\beta^{(t+1)})^2.$$

Setting this derivative to zero, we get the update equation for σ:

$$\sigma^{(t+1)} = \sqrt{\frac{\sum_{i=1}^{n} z_i^{(t)}(y_i - v_i^T\beta^{(t+1)})^2}{\sum_{i=1}^{n} z_i^{(t)}}}.$$

Finally, to maximize B, we compute the compute the derivative

$$\frac{\partial B}{\partial \pi} = \frac{\sum_{i=1}^{n} z_i^{(t)}}{\pi} - \frac{n - \sum_{i=1}^{n} z_i^{(t)}}{1-\pi}.$$

Setting this derivative to zero we get the update rule for π:

$$\pi^{(t+1)} = \frac{1}{n}\sum_{i=1}^{n} z_i^{(t)}.$$

We can now write an EM algorithm for this problem. We first write a function that computes the observed-data log-likelihood:

```
loglik_reg<- function(theta,y,v){
    n<-length(y)
    phi<-vector(n,mode="numeric")
    for(i in 1:n)
      phi[i] <- dnorm(y[i],mean=theta[1]+theta[2]*v[i],sd=theta[3])
    logL <- sum(log(theta[4]*phi+(1-theta[4])*c))
    return(logL)
}
```

We then write the EM algorithm, which is similar to that of Exercise 1:

```
em_outlier_reg <- function(y,v,theta0,a,epsi){
    go_on<-TRUE
    logL0<- loglik_reg(theta0,y,v)
    t<-0
    c<-1/(2*a)
    n<-length(y)
    phi<-vector(n,mode="numeric")
    print(c(t,logL0))
```

```
    while(go_on){
        t<-t+1
        # E-step
        for(i in 1:n)
          phi[i] <- dnorm(y[i],mean=theta0[1]+theta0[2]*v[i],sd=theta0[3])
        z <- phi*theta0[4]/(phi*theta0[4]+c*(1-theta0[4]))
        # M-step
        S<- sum(z)
        pi<-S/n
        reg<-lm(y ~v,weights=z)
        beta<-reg$coefficients
        sig<-sqrt(sum(z*reg$residuals^2)/S)
        theta<-c(beta,sig,pi)
        logL<-loglik_reg(theta,y,v)
        if (logL-logL0 < epsi) go_on <- FALSE
        logL0 <- logL
        theta0<-theta
        print(c(t,logL))
        }
    return(list(loglik=logL,theta=theta,z=z))
}
```

Question 2d

We first initialize the parameters to the OLS estimates and some arbitrary value of π:

```
beta0<-reg$coefficients
sig0<-sd(reg$residuals)
pi0<-0.8
theta0<-c(beta0,sig0,pi0)
```

We then run function `em_outlier_reg`:

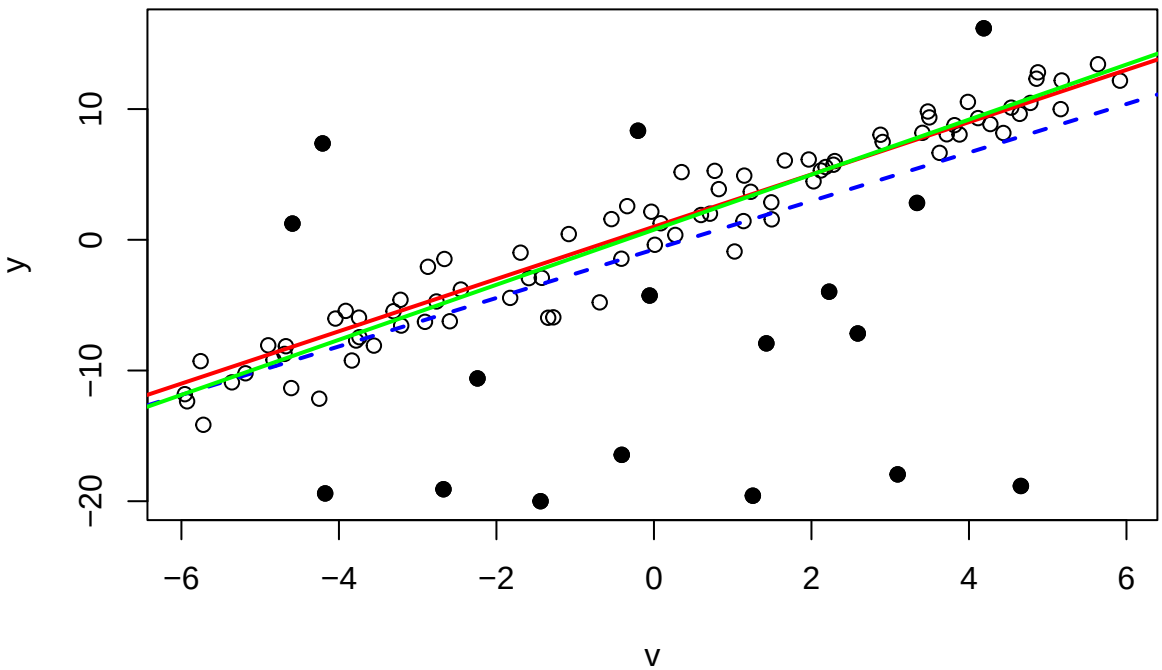
```
estim<-em_outlier_reg(y,v,theta0,a,epsi=1e-6)
```

```
## [1]    0.0000 -316.4834
## [1]    1.0000 -288.0812
## [1]    2.0000 -271.7673
## [1]    3.000 -265.862
## [1]    4.0000 -264.6194
## [1]    5.0000 -264.3309
## [1]    6.0000 -264.2446
## [1]    7.0000 -264.2149
## [1]    8.0000 -264.2039
## [1]    9.0000 -264.1997
## [1]   10.000 -264.198
## [1]   11.0000 -264.1973
## [1]   12.0000 -264.1971
## [1]   13.000 -264.197
## [1]   14.0000 -264.1969
## [1]   15.0000 -264.1969
## [1]   16.0000 -264.1969
## [1]   17.0000 -264.1969
## [1]   18.0000 -264.1969
```

```
## [1]   19.0000 -264.1969
```

Finally, we plot the line estimated by EM, together with the true regression line and the OLS line. We also plot the points identified as outliers:

```
plot(v,y)
abline(reg,lty=2,col="blue",lwd=2) # LS line
abline(1,2,lty=1,col="red",lwd=2) # regression line
abline(estim$theta[1],estim$theta[2],lty=1,col='green',lwd=2) # line estimated using EM
points(v[estim$z<0.5],y[estim$z<0.5],pch=19) # points identified as outliers
```



Exercise 3

Question 3a

We set the parameters:

```
pi <- 0.8
beta1 <- c(1,2)
sig1 <- 2
beta2 <- c(0,0)
sig2 <- 10
n <- 100
theta_true <- c(beta1,sig1,beta2,sig2,pi)
```

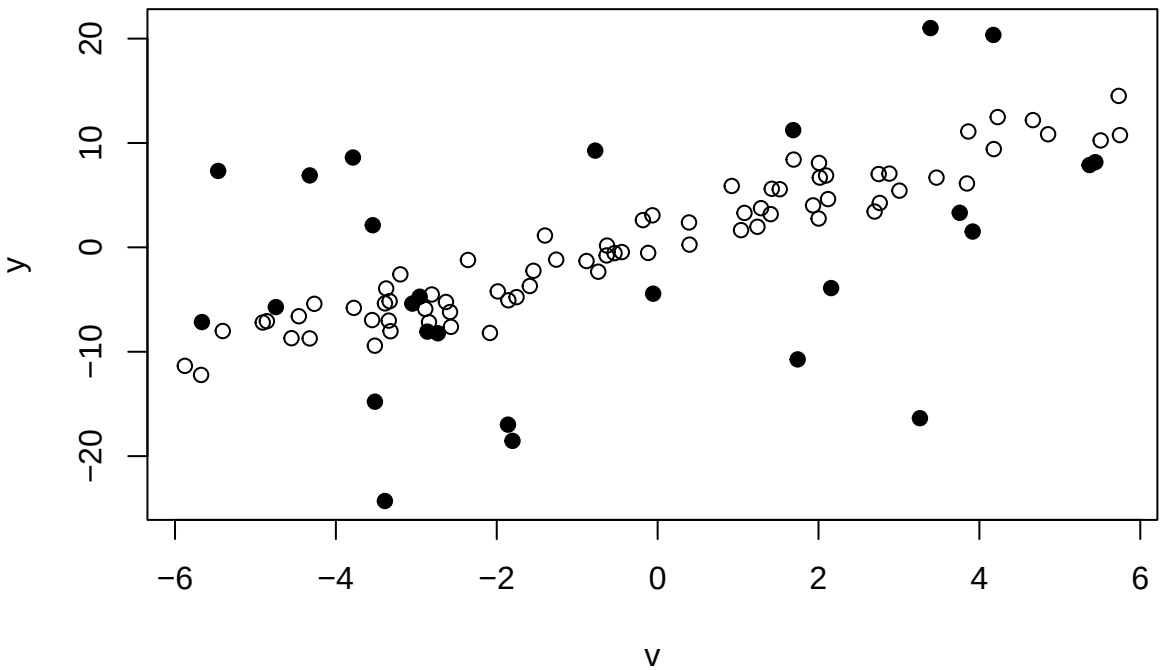
We then generate the data:

```
set.seed(2024022)
y<-vector("numeric",n)
v <- runif(n,min=-6,max=6)
z<-vector("numeric",n)
for(i in 1:n){
  z[i] <- sample(c(1,0),size=1,prob=c(pi,1-pi))
```

```
  if(z[i]==1)
    y[i]<- rnorm(1,mean=beta1[1]+v[i]*beta1[2],sd=sig1)
  else y[i]<-rnorm(1,mean=beta2[1]+v[i]*beta1[2],sd=sig2)
}
```

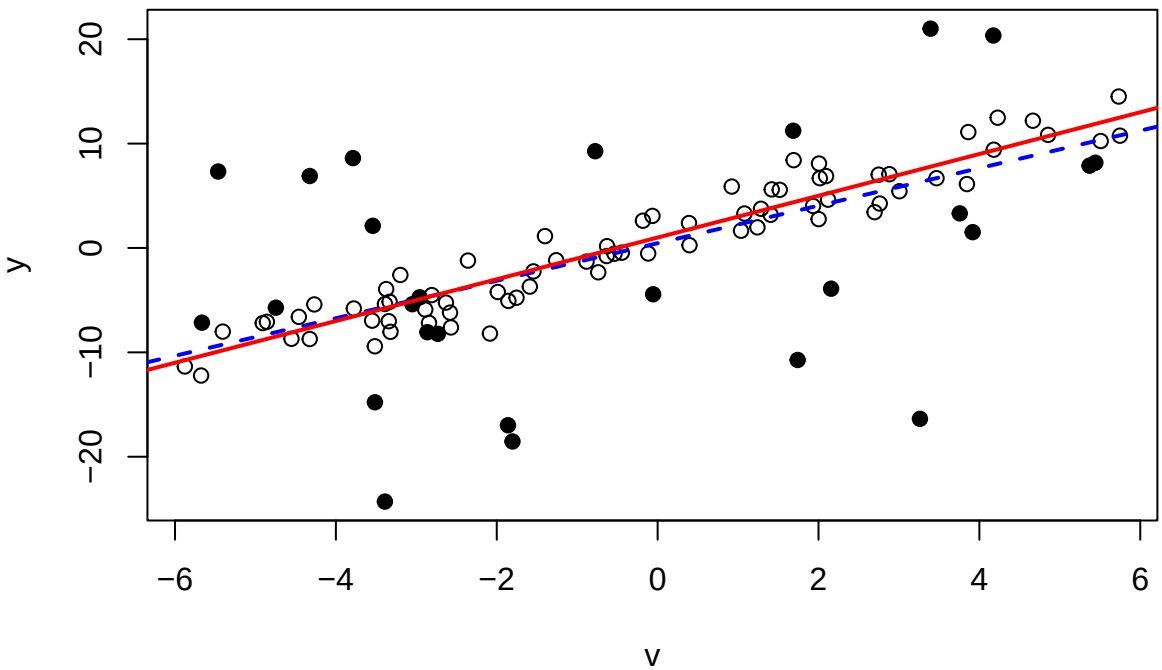
Finally, we plot the data. The data points generated from the uniform distribution (outliers) are highlighted:

```
plot(v,y)
points(v[z==0],y[z==0],pch=16)
```



Question 3b

```
reg<-lm(y~v)
plot(v,y)
points(v[z==0],y[z==0],pch=16)
abline(reg,lty=2,col="blue",lwd=2) # LS line
abline(1,2,lty=1,col="red",lwd=2) #true regression line
```




```r
print(sd(reg$residuals))
```

```
## [1] 5.850116
```

We can see that the error standard deviation in the regression model is grossly overestimated because of the outliers.

Question 3c

The observed-data likelihood is now

$$L(\boldsymbol{\theta}) = \prod_{i=1}^{n} f(y_i; \boldsymbol{\theta}) = \prod_{i=1}^{n} \left[\pi\phi(y_i, v_i^T\beta_1, \sigma_1) + (1-\pi)\phi(y_i, v_i^T\beta_2, \sigma_2)\right].$$

The complete-data likelihood is

$$L_c(\boldsymbol{\theta}) = \prod_{i=1}^{n} f(y_i \mid z_i; \boldsymbol{\theta}) f(z_i; \boldsymbol{\theta}) = \prod_{i=1}^{n} \left[\phi(y_i, v_i^T\beta_1, \sigma_1)^{z_i}\phi(y_i, v_i^T\beta_2, \sigma_2)^{1-z_i}\pi^{z_i}(1-\pi)^{1-z_i}\right],$$

and its logarithm is

$$\ell_c(\boldsymbol{\theta}) = \sum_{i=1}^{n} \left[z_i \log\phi(y_i, v_i^T\beta_1, \sigma_1) + (1-z_i)\log\phi(y_i, v_i^T\beta_2, \sigma_2) + z_i\log\pi + (1-z_i)\log(1-\pi)\right].$$

In the E-step, we can compute the conditional expectations $z_i^{(t)}$ as

$$z_i^{(t)} = \mathbb{E}_{\boldsymbol{\theta}^{(t)}}[Z_i \mid y_i] = \mathbb{P}_{\boldsymbol{\theta}^{(t)}}[Z_i = 1 \mid y_i] = \frac{\phi(y_i; v_i^T\beta_1^{(t)}, \sigma_1^{(t)})\pi^{(t)}}{\phi(y_i; v_i^T\beta_1^{(t)}, \sigma_1^{(t)})\pi^{(t)} + \phi(y_i; v_i^T\beta_2^{(t)}, \sigma_2^{(t)})(1-\pi^{(t)})} \tag{2}$$

and

$$Q(\boldsymbol{\theta}, \boldsymbol{\theta}^{(t)}) = \underbrace{\sum_{i=1}^{n} z_i^{(t)} \log \phi(y_i, v_i^T \beta_1, \sigma_1)}_{A} + \underbrace{\sum_{i=1}^{n} (1 - z_i^{(t)}) \log \phi(y_i, v_i^T \beta_2, \sigma_2)}_{B} + \underbrace{\log \pi \sum_{i=1}^{n} z_i^{(t)} + \log(1-\pi) \left(n - \sum_{i=1}^{n} z_i^{(t)} \right)}_{C}.$$

In the M-step, we maximize $Q(\boldsymbol{\theta}, \boldsymbol{\theta}^{(t)})$ with respect to $\boldsymbol{\theta}$. In the above equation, A depends only on β_1 and σ_1, B depends only on β_2 and σ_2 and C depends on π. We can thus maximize these three terms separately. Reasoning as in Question 2c, we get the following update equations:

$$\beta_1^{(t+1)} = \left[\sum_{i=1}^{n} z_i^{(t)} v_i v_i^T \right]^{-1} \sum_{i=1}^{n} z_i^{(t)} v_i y_i,$$

$$\sigma_1^{(t+1)} = \sqrt{\frac{\sum_{i=1}^{n} z_i^{(t)} (y_i - v_i^T \beta_1^{(t+1)})^2}{\sum_{i=1}^{n} z_i^{(t)}}},$$

$$\beta_2^{(t+1)} = \left[\sum_{i=1}^{n} (1 - z_i^{(t)}) v_i v_i^T \right]^{-1} \sum_{i=1}^{n} (1 - z_i^{(t)}) v_i y_i,$$

$$\sigma_2^{(t+1)} = \sqrt{\frac{\sum_{i=1}^{n} (1 - z_i^{(t)}) (y_i - v_i^T \beta_2^{(t+1)})^2}{n - \sum_{i=1}^{n} z_i^{(t)}}},$$

$$\pi^{(t+1)} = \frac{1}{n} \sum_{i=1}^{n} z_i^{(t)}.$$

We can now a function that computes the observed-data log-likelihood:

```
loglik_reg1<- function(theta,y,v){
  n<-length(y)
  phi1<-vector(n,mode="numeric")
  phi2<-vector(n,mode="numeric")
  for(i in 1:n){
    phi1[i] <- dnorm(y[i],mean=theta[1]+theta[2]*v[i],sd=theta[3])
    phi2[i] <- dnorm(y[i],mean=theta[4]+theta[5]*v[i],sd=theta[6])
  }
  logL <- sum(log(theta[7]*phi1+(1-theta[4])*phi2))
  return(logL)
}
```

We then write the EM algorithm, which is similar to those of Exercises 1 an 2:

```
em_outlier_reg1 <- function(y,v,theta0,epsi){
  go_on<-TRUE
  logL0<- loglik_reg1(theta0,y,v)
  t<-0
  n<-length(y)
  phi1<-vector(n,mode="numeric")
  phi2<-vector(n,mode="numeric")
  print(c(t,logL0))
```

```r
  while(go_on){
    t<-t+1
    # E-step
    for(i in 1:n){
      phi1[i] <- dnorm(y[i],mean=theta0[1]+theta0[2]*v[i],sd=theta0[3])
      phi2[i] <- dnorm(y[i],mean=theta0[4]+theta0[5]*v[i],sd=theta0[6])
    }
    z<-  phi1*theta0[7]/(phi1*theta0[7]+phi2*(1-theta0[7]))
    # M-step
    S<- sum(z)
    pi<-S/n
    reg1<-lm(y ~v,weights=z)
    beta1<-reg1$coefficients
    reg2<-lm(y ~v,weights=1-z)
    beta2<-reg2$coefficients
    sig1<-sqrt(sum(z*reg1$residuals^2)/S)
    sig2<-sqrt(sum((1-z)*reg2$residuals^2)/(n-S))
    theta<-c(beta1,sig1,beta2,sig2,pi)
    logL<-loglik_reg1(theta,y,v)
    if (logL-logL0 < epsi){
      go_on <- FALSE
    }
    logL0 <- logL
    theta0<-theta
    print(c(t,logL))
  }
  return(list(loglik=logL,theta=theta,z=z))
}
```

Question 3d

We first initialize the coeffcients β_1 to the OLS estimates and β_2 randomly:

```r
beta10 <- reg$coefficients
sig10 <- sd(reg$residuals)
beta20 <- 0.1*rnorm(2)
sig20 <- sig10
pi0 <- 0.8
theta0<-c(beta10,sig10,beta20,sig20,pi0)
```

We then run function `em_outlier_reg1`:

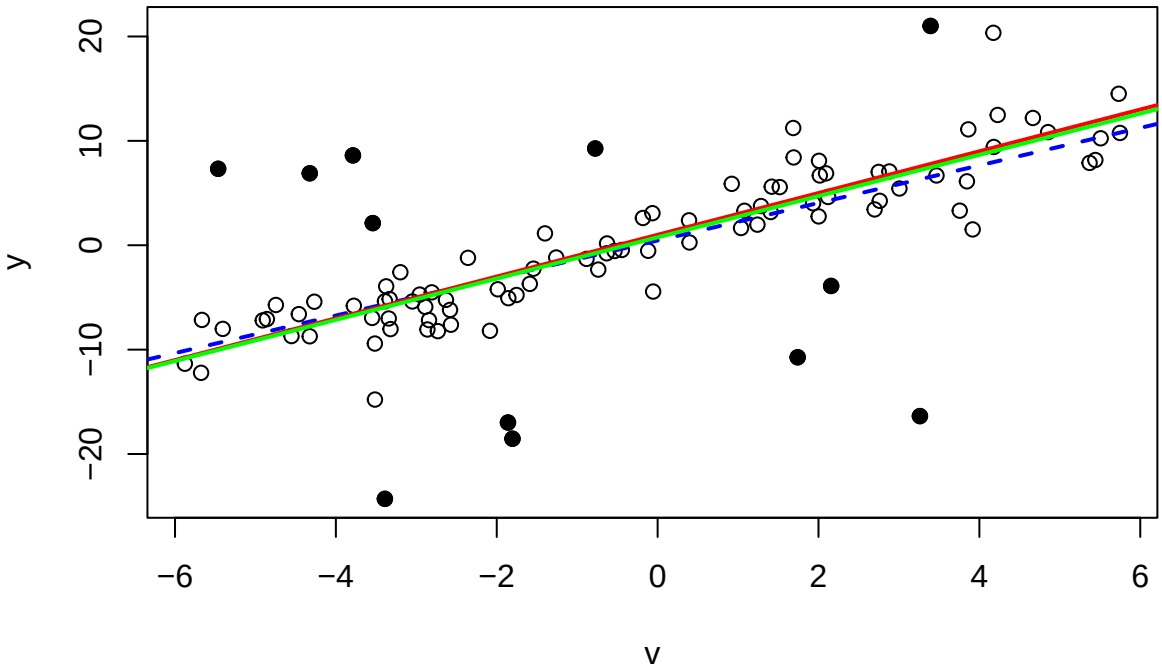
```r
estim <- em_outlier_reg1(y,v,theta0,epsi=1e-6)
```

```
## [1]    0.0000 -271.8457
## [1]    1.0000 -244.8924
## [1]    2.0000 -207.9936
## [1]    3.0000 -173.5814
## [1]    4.0000 -196.9353
```

Finally, we plot the line estimated by EM, together with the true regression line and the OLS line. We also plot the points identified as outliers:

```r
plot(v,y)
abline(reg,lty=2,col="blue",lwd=2) # LS line
```

```
abline(1,2,lty=1,col="red",lwd=2) # regression line
abline(estim$theta[1],estim$theta[2],lty=1,col='green',lwd=2) # line estimated using EM
points(v[estim$z<0.5],y[estim$z<0.5],pch=19)
```



Exercise 4

Question 3a

We write a function `gen_data` that generates a sample:

```
gen_sample<- function(mu,sig,pi,a,n){
  y<-rep(0,n)
    for(i in 1:n){
        z<-sample(c(1,0),size=1,prob=c(pi,1-pi))
        if(z==1) y[i]<- rnorm(1,mean=mu,sd=sig) else y[i]<-runif(1,min=-a,max=a)
        }
    return(y)
}
```

We then define true values of the parameters:

```
pi<-0.90
mu<-0
sig<- 1
theta_true<-c(mu,sig,pi)
```

We then generate $N = 1000$ samples and estimate $\boldsymbol{\theta}$ from each sample using function `em_outlier`; the estimates $\widehat{\boldsymbol{\theta}}$ are stored in a matrix `Theta`:

```
a=5
c<-1/(2*a)
n<- 100
```

```r
N<-1000
Theta<-matrix(0,N,3)
pi0<-0.5
for(j in 1:N){
  y<- gen_sample(mu,sig,pi,a,n)
  mu0<-mean(y)
  sig0<-sd(y)
  theta0<-c(mu0,sig0,pi0)
  em<-em_outlier(y,theta0,a,epsi=1e-5,disp=FALSE)
  Theta[j,]<-em$theta }
```

Finally, we can estimate the covariance matrix of $\widehat{\boldsymbol{\theta}}$:

```r
V<-var(Theta)
print(V,2)
```

```
##          [,1]    [,2]     [,3]
## [1,]  1.4e-02 -0.0008 -9.8e-05
## [2,] -8.0e-04  0.0111  2.8e-03
## [3,] -9.8e-05  0.0028  3.0e-03
```

Question 3b

We first need to compute the complete-data information matrix

$$\widehat{i}_{\mathbf{x}}(\widehat{\boldsymbol{\theta}}) = \mathbb{E}_{\widehat{\boldsymbol{\theta}}}(-\ell_c''(\widehat{\boldsymbol{\theta}}) \mid \mathbf{y}).$$

The complete-data likelihood is

$$\begin{aligned} L_c(\boldsymbol{\theta}) &= \prod_{i=1}^n f(y_i, z_i; \boldsymbol{\theta}) = \prod_{i=1}^n f(y_i \mid z_i; \mu, \sigma) f(z_i; \pi) \\ &= \prod_{i=1}^n [\phi(y_i; \mu, \sigma)^{z_i} c^{1-z_i} \pi^{z_i} (1-\pi)^{1-z_i}] \end{aligned}$$

and the corresponding log-likelihood is:

$$\ell_c(\boldsymbol{\theta}) = \sum_{i=1}^n z_i \log \phi(y_i; \mu, \sigma) + \left(n - \sum_{i=1}^n z_i\right) \log c + \sum_{i=1}^n (z_i \log \pi + (1-z_i) \log(1-\pi))$$

We need to compute the first and then the second derivative. We have

$$\frac{\partial \ell_c}{\partial \mu} = \frac{1}{\sigma^2} \sum_i z_i (y_i - \mu)$$

$$\frac{\partial \ell_c}{\partial \sigma} = -\frac{1}{\sigma} \sum_i z_i + \frac{1}{\sigma^3} \sum_i z_i (y_i - \mu)^2$$

$$\frac{\partial \ell_c}{\partial \pi} = \sum_i \left(\frac{z_i}{\pi} - \frac{1-z_i}{1-\pi}\right)$$

And then the second derivatives:

$$\frac{\partial^2 \ell_c}{\partial \mu^2} = -\frac{1}{\sigma^2} \sum_i z_i$$

$$\frac{\partial^2 \ell_c}{\partial \mu \partial \sigma} = -\frac{2}{\sigma^3} \sum_i z_i (y_i - \mu)$$

$$\frac{\partial^2 \ell_c}{\partial \sigma^2} = \frac{1}{\sigma^2} \sum_i z_i - \frac{3}{\sigma^4} \sum_i z_i (y_i - \mu)^2$$

$$\frac{\partial^2 \ell_c}{\partial \pi^2} = \sum_i \left(\frac{-z_i}{\pi^2} - \frac{1 - z_i}{(1-\pi)^2} \right) = -\frac{\sum_i z_i}{\pi^2} - \frac{n - \sum_i z_i}{(1-\pi)^2}$$

$$\frac{\partial^2 \ell_c}{\partial \mu \partial \pi} = \frac{\partial^2 \ell_c}{\partial \sigma \partial \pi} = 0.$$

To compute the conditional expectation of $\ell_c(\boldsymbol{\theta}^{(t)})$ for $\boldsymbol{\theta} = \boldsymbol{\theta}^{(t)}$, we simply replace z_i by $z_i^{(t)}$ in the above expression. (When the EM algorithm has converged, $\boldsymbol{\theta}^{(t)} \approx \widehat{\boldsymbol{\theta}}$). The complete information matrix is then

$$\widehat{i}_{\mathbf{x}}(\widehat{\boldsymbol{\theta}}) = \begin{pmatrix} \frac{1}{\widehat{\sigma}^2} \sum_i z_i^{(t)} & \frac{2}{\widehat{\sigma}^3} \sum_i z_i^{(t)} (y_i - \widehat{\mu}) & 0 \\ \frac{2}{\widehat{\sigma}^3} \sum_i z_i^{(t)} (y_i - \widehat{\mu}) & -\frac{1}{\widehat{\sigma}^2} \sum_i z_i^{(t)} + \frac{3}{\widehat{\sigma}^4} \sum_i z_i^{(t)} (y_i - \widehat{\mu})^2 & 0 \\ 0 & 0 & \frac{\sum_i z_i^{(t)}}{\widehat{\pi}^2} + \frac{n - \sum_i z_i^{(t)}}{(1-\widehat{\pi})^2} \end{pmatrix}.$$

The following R function computes the complete information matrix:

```
complete_information<- function(theta,y,z){
    I<-matrix(0,3,3)
    mu<-theta[1]
    sig<-theta[2]
    pi<-theta[3]
    S<-sum(z)
    I[1,1]<- S/sig^2
    I[1,2]<- 2/sig^3*sum(z*(y-mu))
    I[2,1]<-I[1,2]
    I[2,2]<- -S/sig^2 + 3/sig^4*sum(z*(y-mu)^2)
    I[3,3]<- S/pi^2 + (n-S)/(1-pi)^2
    return(I)
}
```

Next, we need to compute the missing information as the variance of the conditional score:

$$\widehat{i}_{\mathbf{Z}|\mathbf{Y}}(\boldsymbol{\theta}) = \mathsf{Var}\left[\frac{\partial \log f(\mathbf{z} \mid \mathbf{y}; \boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \mid \mathbf{y} \right].$$

For this, we first need to compute $f(\mathbf{z} \mid \mathbf{y}; \boldsymbol{\theta})$. We recall that

$$P(Z_i = 1 \mid y_i) = \frac{\phi_i \pi}{\phi_i \pi + c(1-\pi)} = p_i(\boldsymbol{\theta}).$$

with

$$\phi_i = \phi(y_i; \mu, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2\sigma^2} (y_i - \mu)^2 \right].$$

So,

$$f(\mathbf{z} \mid \mathbf{y}; \boldsymbol{\theta}) = \prod_{i=1}^n f(z_i \mid y_i; \boldsymbol{\theta}) = \prod_{i=1}^n p_i(\boldsymbol{\theta})^{z_i} (1 - p_i(\boldsymbol{\theta}))^{1-z_i},$$

and

$$\begin{aligned} \log f(\mathbf{z} \mid \mathbf{y}; \boldsymbol{\theta}) &= \sum_{i=1}^n \left(z_i (\log \phi_i + \log \pi - \log[\phi_i \pi + c(1-\pi)]) + (1 - z_i)(\log c(1-\pi) - \log[\phi_i \pi + c(1-\pi)]) \right) \\ &= \sum_{i=1}^n \left(z_i \log \phi_i + z_i \log \pi + (1 - z_i) \log c(1-\pi) - \log[\phi_i \pi + c(1-\pi)] \right). \end{aligned}$$

To compute the score, we first compute:

$$\frac{\partial \phi_i}{\partial \mu} = \phi_i \frac{y_i - \mu}{\sigma^2},$$

and

$$\frac{\partial \phi_i}{\partial \sigma} = \frac{\phi_i}{\sigma}\left(\frac{(y_i-\mu)^2}{\sigma^2} - 1\right).$$

Using elementary calculus, we get

$$\begin{aligned}
\frac{\partial \log f(\mathbf{z} \mid \mathbf{y}; \boldsymbol{\theta})}{\partial \mu} &= \frac{1}{\sigma^2}\sum_{i=1}^n z_i(y_i-\mu) - \frac{\pi}{\sigma^2}\sum_{i=1}^n \frac{(y_i-\mu)\phi_i}{\phi_i\pi + c(1-\pi)} \\
\frac{\partial \log f(\mathbf{z} \mid \mathbf{y}; \boldsymbol{\theta})}{\partial \sigma} &= \frac{1}{\sigma}\sum_{i=1}^n z_i\left(\frac{(y_i-\mu)^2}{\sigma^2} - 1\right) - \frac{\pi}{\sigma}\sum_{i=1}^n \frac{\phi_i\left(\frac{(y_i-\mu)^2}{\sigma^2} - 1\right)}{\phi_i\pi + c(1-\pi)} \\
\frac{\partial \log f(\mathbf{z} \mid \mathbf{y}; \boldsymbol{\theta})}{\partial \pi} &= \frac{\sum_{i=1}^n z_i - n\pi}{\pi(1-\pi)} - \sum_{i=1}^n \frac{\phi_i - c}{\phi_i\pi + c(1-\pi)}.
\end{aligned}$$

The calculation of the score is performed by the following R function:

```
score<- function(theta,y,z,c){
    S<-c(0,0,0)
    mu<-theta[1]
    sig<-theta[2]
    pi<-theta[3]
    SZ<-sum(z)
    phi<- sapply(y,dnorm,mean=mu,sd=sig)
    S[1]<- 1/sig^2*sum(z*(y-mu)) - pi/sig^2*sum((y-mu)*phi/(phi*pi+c*(1-pi)))
    S[2]<-  -SZ/sig + 1/sig^3*sum(z*(y-mu)^2) -
             pi/sig*sum( phi*((y-mu)^2/sig^2-1)/(phi*pi+c*(1-pi)) )
    S[3]<- (SZ-n*pi)/(pi*(1-pi)) - sum((phi-c)/(phi*pi+c*(1-pi)))
    return(S)
}
```

We can now compute the observed information for an observed sample. We first generate the data and compute the MLE:

```
set.seed(42)
y<- gen_sample(mu,sig,pi,a,n)
mu0<-mean(y)
sig0<-sd(y)
pi0<-0.5
theta0<-c(mu0,sig0,pi0)
em<-em_outlier(y,theta0,a,epsi=1e-5)
```

```
## [1]    0.0000 -191.3151
## [1]    1.0000 -173.8608
## [1]    2.0000 -169.5465
## [1]    3.0000 -168.7118
## [1]    4.0000 -168.4584
## [1]    5.0000 -168.3611
## [1]    6.0000 -168.3219
## [1]    7.0000 -168.3059
## [1]    8.0000 -168.2994
## [1]    9.0000 -168.2968
## [1]   10.0000 -168.2957
```

```
## [1]   11.0000 -168.2953
## [1]   12.0000 -168.2951
## [1]   13.000 -168.295
## [1]   14.000 -168.295
## [1]   15.000 -168.295
## [1]   16.000 -168.295
```

We then compute by simulation the missing information $\widehat{i}_{\mathbf{Z}|\mathbf{Y}}(\boldsymbol{\theta})$ as the variance of the score:

```
N<-100000
SS<-matrix(0,N,3)
for(i in (1:N)){
    u<-runif(n)
    Z<-as.numeric(u<=em$z)
    SS[i,]<-score(em$theta,y,Z,c)
}
izy<-var(SS)
```

Finally, we compute the observed information as the difference between the complete information and missing information; and we estimate the variance of $\widehat{\boldsymbol{\theta}}$ as the inverse of the observed information:

```
ix<-complete_information(em$theta,y,em$z)
iy<-ix-izy
varh<-solve(iy)
print(varh,2)
```

```
##          [,1]    [,2]     [,3]
## [1,]  0.01254 -0.0011 -0.00047
## [2,] -0.00111  0.0086  0.00175
## [3,] -0.00047  0.0018  0.00315
```

To check our calculations, we can compare our result to that using numerical differentiation:

```
library(numDeriv)
varh1<-solve(-hessian(loglik,em$theta,y=y))
print(varh1,2)
```

```
##          [,1]    [,2]     [,3]
## [1,]  0.01254 -0.0011 -0.00046
## [2,] -0.00112  0.0087  0.00179
## [3,] -0.00046  0.0018  0.00318
```