Computational Statistics. Chapter 3: EM algorithm. Solution of exercises 1 and 2

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Exercise 1

Question 1a

We first give values to the model parameters:

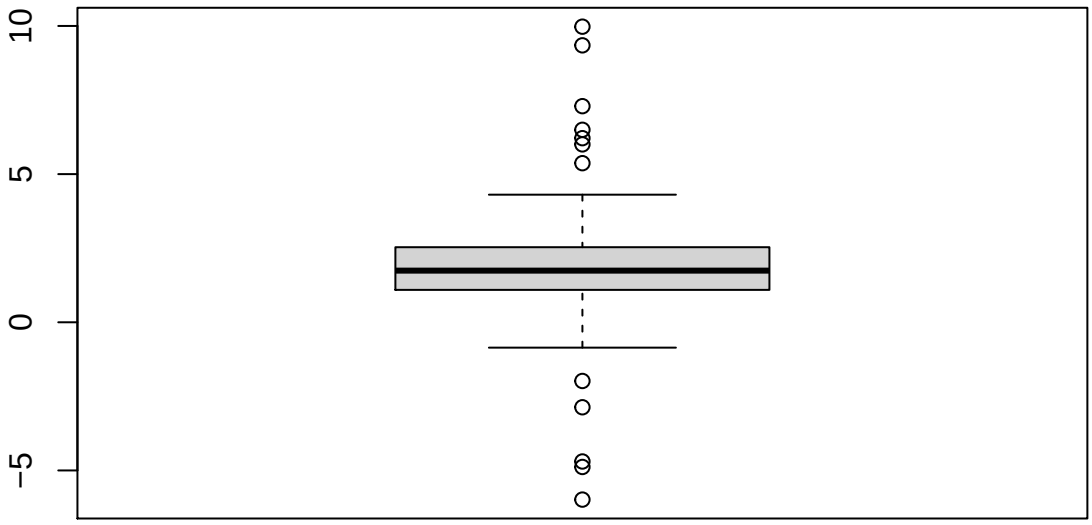
```
pi<-0.90
mu<-2
sig<- 1
a=10;
c<-1/(2*a)
n<- 100
```

We then generate the data:

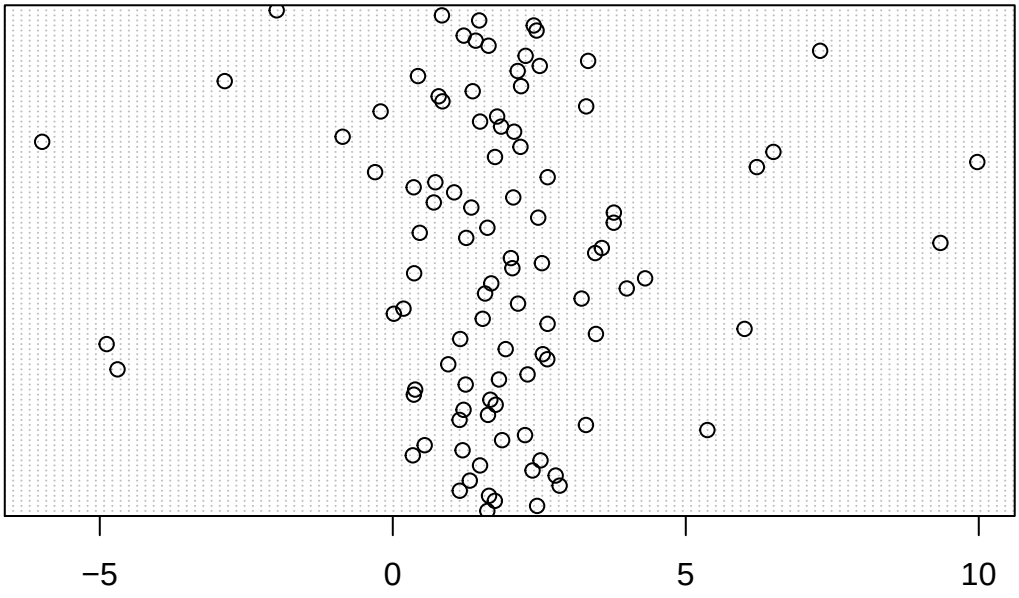
```
y<-vector("numeric",n)
z<-vector("numeric",n)
for(i in 1:n){
    z[i]=sample(c(1,0),size=1,prob=c(pi,1-pi))
    if(z[i]==1){
        y[i]<- rnorm(1,mean=mu,sd=sig)
    }
    else y[i]<-runif(1,min=-a,max=a)
}
```

Finally, we generate box and dot plots the data:

```
boxplot(y)
```




```
dotchart(y)
```



Question 1b

We first write a function that computes the observed-data log-likelihood:

```
loglik<- function(theta,y){
    phi <- sapply(y,dnorm,mean=theta[1],sd=theta[2])
    logL <- sum(log(theta[3]*phi+(1-theta[3])*c))
    return(logL)
}
```

We then write the EM algorithm for this problem. The inputs are the data `y`, the initial parameter value `theta0`, the constant `a`, the threshold `epsi` used in the stopping criterion, and a flag `disp` that conrols the display of the intermediate results. The outputs are the maximum observed-data log-likelihood, the corresponding MLE of θ, and the vector $\boldsymbol{z}$ of estimated probabilities.

```
em_outlier <- function(y,theta0,a,epsi,disp=TRUE){
    go_on<-TRUE
    logL0<- loglik(theta0,y)
    t<-0
    c<-1/(2*a)
    n<-length(y)
    if(disp) print(c(t,logL0))
    while(go_on){
        t<-t+1
        # E-step
        phi <- sapply(y,dnorm,mean=theta0[1],sd=theta0[2])
        z<-  phi*theta0[3]/(phi*theta0[3]+c*(1-theta0[3]))
        # M-step
        S<- sum(z)
        pi<-S/n
        mu<- sum(y*z)/S
        sig<-sqrt(sum(z*(y-mu)^2)/S)
        theta<-c(mu,sig,pi)
        logL<-loglik(theta,y)
        if (logL-logL0 < epsi) go_on <- FALSE
        logL0 <- logL
        theta0<-theta
        if(disp) print(c(t,logL))
        }
    return(list(loglik=logL,theta=theta,z=z))
}
```

Question 1c

Let us now run the above function with our data. We initialize parameters μ and σ with the mean and standard deviation of the data, and we set $\pi_0 = 0.5$:

```
mu0<-mean(y) # +rnorm(1,mean=0,sd=0.5)
sig0<-sd(y)
pi0<-0.5
theta0<-c(mu0,sig0,pi0)
```

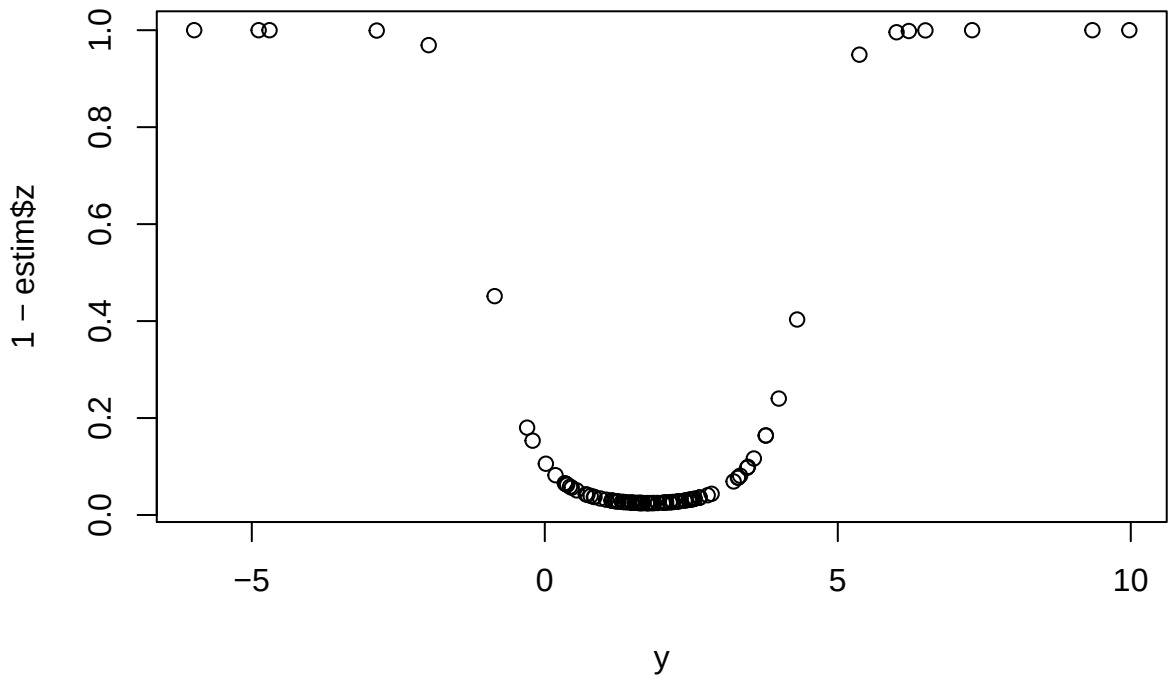
We then run function `em_outlier`:

```
estim<-em_outlier(y,theta0,a,epsi=1e-6)
```

```
## [1]    0.0000 -239.5189
## [1]    1.0000 -204.4045
## [1]    2.0000 -196.4312
## [1]    3.000 -195.986
## [1]    4.0000 -195.9742
## [1]    5.0000 -195.9738
## [1]    6.0000 -195.9737
## [1]    7.0000 -195.9737
## [1]    8.0000 -195.9737
## [1]    9.0000 -195.9737
```

Finally, we plot the esimates probabilities $1 - z_i$ against the inputs y_i:

```
plot(y,1-estim$z)
```



We can see that the outliers have a high estimated probability of being drawn from the uniform distribution, as expected.

Exercise 2

Question 2a

We set the parameters:

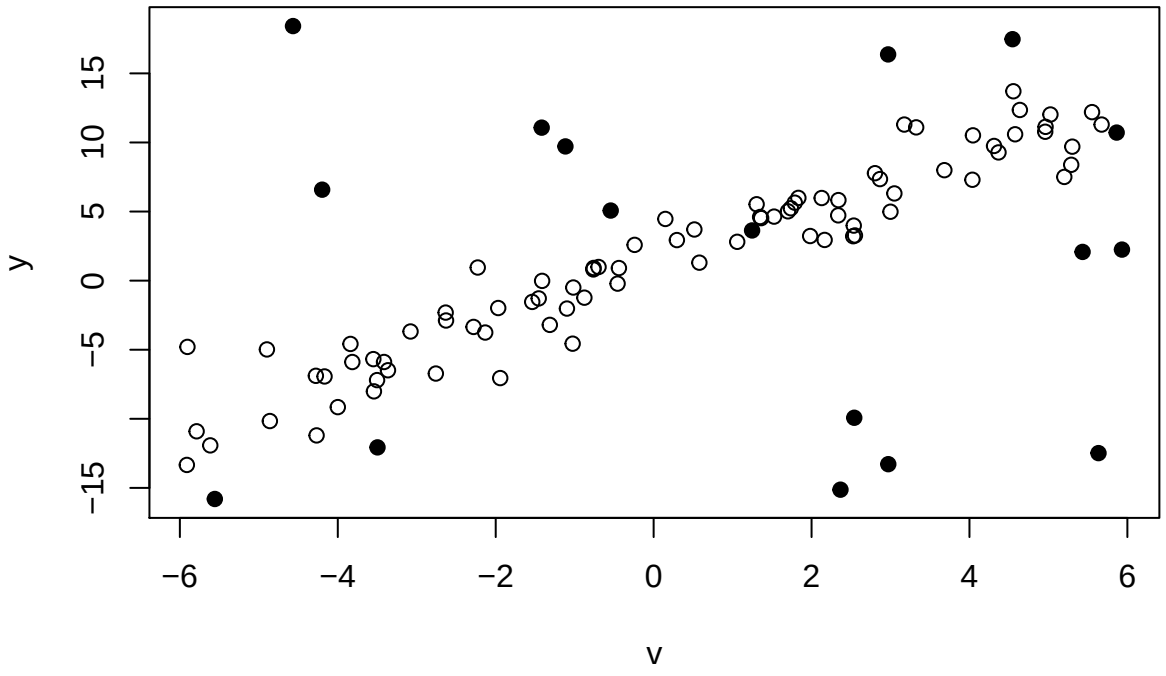
```
pi<-0.8
beta<-c(1,2)
sig<- 2
a=20
c<-1/(2*a)
n<- 100
```

We then generate the data

```
y<-vector("numeric",n)
v <- runif(n,min=-6,max=6)
z<-vector("numeric",n)
for(i in 1:n){
  z[i]=sample(c(1,0),size=1,prob=c(pi,1-pi))
  if(z[i]==1){
    y[i]<-rnorm(1,mean=beta[1]+v[i]*beta[2],sd=sig)
    } else y[i]<-runif(1,min=-a,max=a) }
```

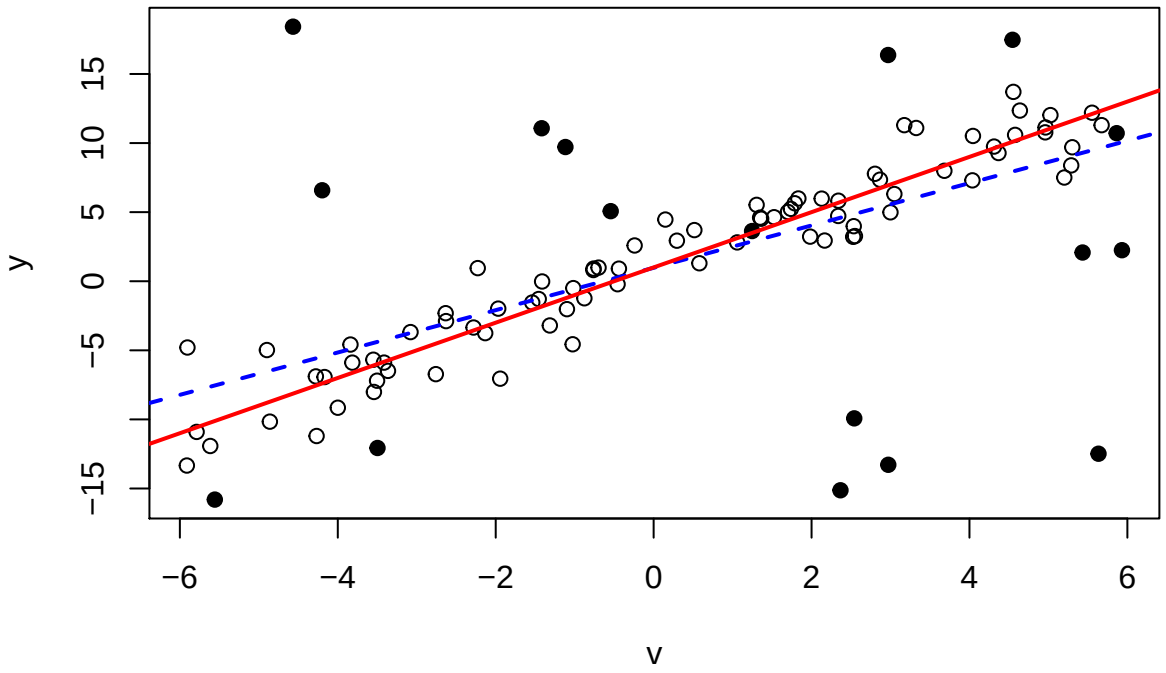
Finally, we plot the data. The data points generated from the uniform distribution (outliers) are highlighted:

```r
plot(v,y)
points(v[z==0],y[z==0],pch=16)
```



Question 2b

```r
reg<-lm(y~v)
plot(v,y)
points(v[z==0],y[z==0],pch=16)
abline(reg,lty=2,col="blue",lwd=2) # LS line
abline(1,2,lty=1,col="red",lwd=2) #true regression line
```



Question 2c

In this problem the observed data is the vector $\mathbf{y} = (y_1, \ldots, y_n)$ The observed-data likelihood is

$$L(\boldsymbol{\theta}) = \prod_{i=1}^{n} f(y_i; \boldsymbol{\theta}) = \prod_{i=1}^{n} \left[\pi \phi(y_i, v_i^T \beta, \sigma) + (1-\pi)c\right].$$

The missing data is the random vector $\mathbf{Z} = (Z_1, \ldots, Z_n)$, where $Z_i = 1$ if the observation is an outlier, and $Z_i = 0$ otherwise. Clearly, Z_i has a Bernoulli distribution $\mathcal{B}(1-\pi)$. The complete-data likelihood is

$$L_c(\boldsymbol{\theta}) = \prod_{i=1}^{n} f(y_i \mid z_i; \boldsymbol{\theta}) f(z_i; \boldsymbol{\theta}) = \prod_{i=1}^{n} \left[\phi(y_i, v_i^T \beta, \sigma)^{z_i} c^{1-z_i} \pi^{z_i} (1-\pi)^{1-z_i}\right],$$

and its logarithm is

$$\ell_c(\boldsymbol{\theta}) = \sum_{i=1}^{n} \left[z_i \log \phi(y_i, v_i^T \beta, \sigma) + (1-z_i) \log c + z_i \log \pi + (1-z_i) \log(1-\pi)\right].$$

Function Q is obtained by replacing the terms z_i by their conditional expectations given $\mathbf{Y} = \mathbf{y}$, for $\boldsymbol{\theta} = \boldsymbol{\theta}^{(t)}$. In the E-step, we will compute these conditional expectations as

$$z_i^{(t)} = \mathbb{E}_{\boldsymbol{\theta}^{(t)}}[Z_i \mid y_i] = \mathbb{P}_{\boldsymbol{\theta}^{(t)}}[Z_i = 1 \mid y_i] = \frac{\phi(y_i; v_i^T \beta^{(t)}, \sigma^{(t)}) \pi^{(t)}}{\phi(y_i; v_i^T \beta^{(t)}, \sigma^{(t)}) \pi^{(t)} + c(1-\pi^{(t)})} \tag{1}$$

and

$$\begin{aligned} Q(\boldsymbol{\theta}, \boldsymbol{\theta}^{(t)}) &= \sum_{i=1}^{n} \left[z_i^{(t)} \log \phi(y_i, v_i^T \beta, \sigma) + (1-z_i^{(t)}) \log c + z_i^{(t)} \log \pi + (1-z_i^{(t)}) \log(1-\pi)\right] \\ &= \underbrace{\sum_{i=1}^{n} z_i^{(t)} \log \phi(y_i, v_i^T \beta, \sigma)}_{A} + \underbrace{\log \pi \sum_{i=1}^{n} z_i^{(t)} + \log(1-\pi) \left(n - \sum_{i=1}^{n} z_i^{(t)}\right)}_{B} + \underbrace{(1-z_i^{(t)}) \log c}_{C}. \end{aligned}$$

In the M-step, we maximize $Q(\boldsymbol{\theta}, \boldsymbol{\theta}^{(t)})$ with respect to $\boldsymbol{\theta}$. In the above equation A depends only on β and σ, B depends on π, and C is a constant. We can thus maximize A and B separately. We have

$$A = -\frac{1}{2}\left(\sum_{i=1}^{n} z_i^{(t)}\right)\log(2\pi) - \left(\sum_{i=1}^{n} z_i^{(t)}\right)\log\sigma - \frac{1}{2\sigma^2}\sum_{i=1}^{n} z_i^{(t)}(y_i - v_i^T\beta)^2.$$

We can see that the vector β maximizing A (and Q) is the solution of a weighted least-squares problem (each individual error term being weighted by $z_i^{(t)}$). We thus have the following update rule for β:

$$\beta^{(t+1)} = \left[\sum_{i=1}^{n} z_i^{(t)} v_i v_i^T\right]^{-1} \sum_{i=1}^{n} z_i^{(t)} v_i y_i.$$

To maximize A w.r.t. σ, we set $\beta = \beta^{(t+1)}$ and we compute the derivative:

$$\frac{\partial A}{\partial \sigma} = -\frac{1}{\sigma}\sum_i z_i^{(t)} + \frac{1}{\sigma^3}\sum_i z_i^{(t)}(y_i - v_i^T\beta^{(t+1)})^2.$$

Setting this derivative to zero, we get the update equation for σ:

$$\sigma^{(t+1)} = \sqrt{\frac{\sum_{i=1}^n z_i^{(t)}(y_i - v_i^T\beta^{(t+1)})^2}{\sum_{i=1}^n z_i^{(t)}}}.$$

Finally, to maximize B, we compute the compute the derivative

$$\frac{\partial B}{\partial \pi} = \frac{\sum_{i=1}^n z_i^{(t)}}{\pi} - \frac{n - \sum_{i=1}^n z_i^{(t)}}{1-\pi}.$$

Setting this derivative to zero we get the update rule for π:

$$\pi^{(t+1)} = \frac{1}{n}\sum_{i=1}^{n} z_i^{(t)}.$$

We can now write an EM algorithm for this problem. We first write a function that computes the observed-data log-likelihood:

```
loglik_reg<- function(theta,y,v){
    n<-length(y)
    phi<-vector(n,mode="numeric")
    for(i in 1:n) phi[i] <- dnorm(y[i],mean=theta[1]+theta[2]*v[i],sd=theta[3])
    logL <- sum(log(theta[4]*phi+(1-theta[4])*c))
    return(logL)
}
```

We then write the EM algorithm, which is similar to that of Exercise 1:

```
em_outlier_reg <- function(y,v,theta0,a,epsi){
    go_on<-TRUE
    logL0<- loglik_reg(theta0,y,v)
    t<-0
    c<-1/(2*a)
    n<-length(y)
    phi<-vector(n,mode="numeric")
    print(c(t,logL0))
    while(go_on){
```

```
        t<-t+1
        # E-step
        for(i in 1:n) phi[i] <- dnorm(y[i],mean=theta0[1]+theta0[2]*v[i],sd=theta0[3])
        z<-  phi*theta0[4]/(phi*theta0[4]+c*(1-theta0[4]))
        # M-step
        S<- sum(z)
        pi<-S/n
        reg<-lm(y ~v,weights=z)
        beta<-reg$coefficients
        sig<-sqrt(sum(z*reg$residuals^2)/S)
        theta<-c(beta,sig,pi)
        logL<-loglik_reg(theta,y,v)
        if (logL-logL0 < epsi) go_on <- FALSE
        logL0 <- logL
        theta0<-theta
        print(c(t,logL))
        }
    return(list(loglik=logL,theta=theta,z=z))
}
```

Question 2d

We first initialize the parameters to the OLS estimates and some arbitrary value of π:

```
beta0<-reg$coefficients
sig0<-sd(reg$residuals)
pi0<-0.8
theta0<-c(beta0,sig0,pi0)
```

We then run function `em_outlier_reg`:

```
estim<-em_outlier_reg(y,v,theta0,a,epsi=1e-6)
```

```
## [1]    0.0000 -311.1579
## [1]    1.0000 -282.9658
## [1]    2.0000 -271.7421
## [1]    3.0000 -268.5889
## [1]    4.0000 -267.9363
## [1]    5.0000 -267.7326
## [1]    6.0000 -267.6643
## [1]    7.000 -267.641
## [1]    8.0000 -267.6331
## [1]    9.0000 -267.6305
## [1]   10.0000 -267.6296
## [1]   11.0000 -267.6293
## [1]   12.0000 -267.6292
## [1]   13.0000 -267.6291
## [1]   14.0000 -267.6291
## [1]   15.0000 -267.6291
## [1]   16.0000 -267.6291
## [1]   17.0000 -267.6291
```

Finally, we plot the the line estimated by EM, together with the true regression line and the OLS line. We also plot the points identified as outliers:

```r
plot(v,y)
abline(reg,lty=2,col="blue",lwd=2) # LS line
abline(1,2,lty=1,col="red",lwd=2) # regression line
abline(estim$theta[1],estim$theta[2],lty=1,col='green',lwd=2) # line estimated using EM
points(v[estim$z<0.5],y[estim$z<0.5],pch=19) # points identified as outliers
```