Computational Statistics. Chapter 5: MCMC. Solution of exercises

Thierry Denoeux

2023-01-24

```
set.seed(2021)
```

Exercise 1

As the density of ϵ is symmetric, the MH ratio is the ratio of the densities at x^* and $x^{(t-1)}$, i.e., we have

$$R(x^{(t-1)}, x^*) = \frac{f(x^*)}{f(x^{(t-1)})} = \exp(|x^{(t-1)}| - |x^*|).$$

The following function `MH_Laplace` implements the random walk MH algorithm for this problem:

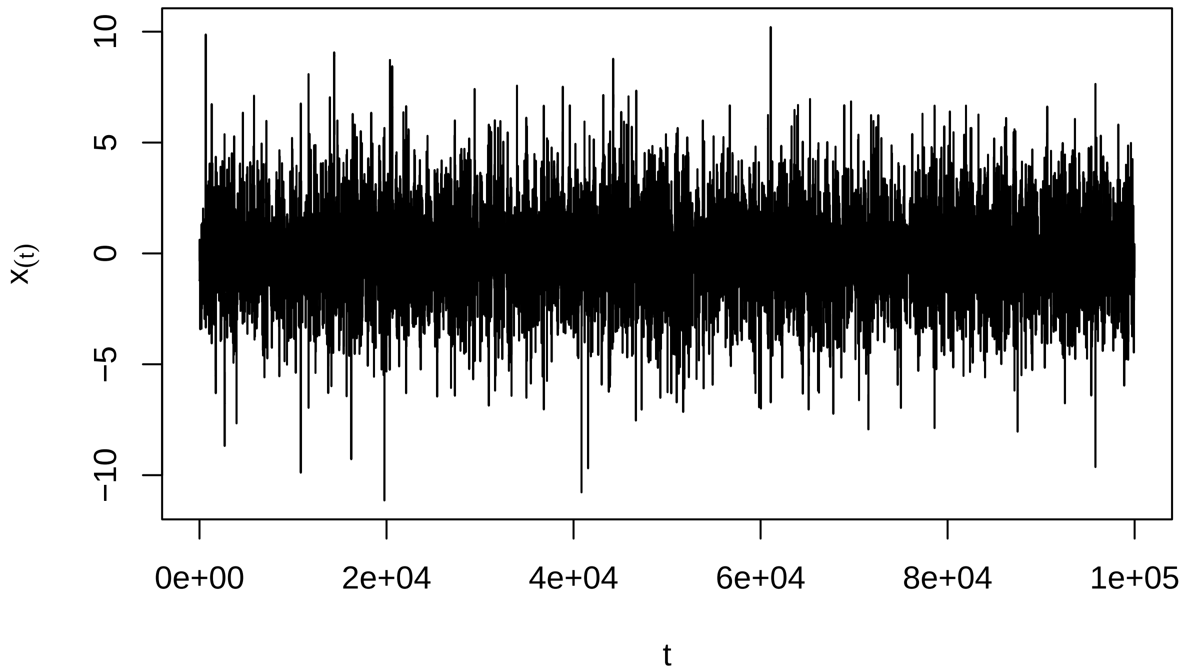
```
MH_Laplace <- function(N,sig){
  x<-vector(N,mode="numeric")
  x[1]<-rnorm(1,mean=0,sd=sig)
  for(t in (2:N)){
    epsilon<-rnorm(1,mean=0,sd=sig)
    xstar<-x[t-1]+ epsilon
    U<-runif(1)
    R<-exp(abs(x[t-1]) - abs(xstar))
    if(U <= R) x[t]<-xstar else x[t]<-x[t-1]
  }
  return(x)
}
```

Let us generate a sample of size 10^5 with $\sigma = 10$:

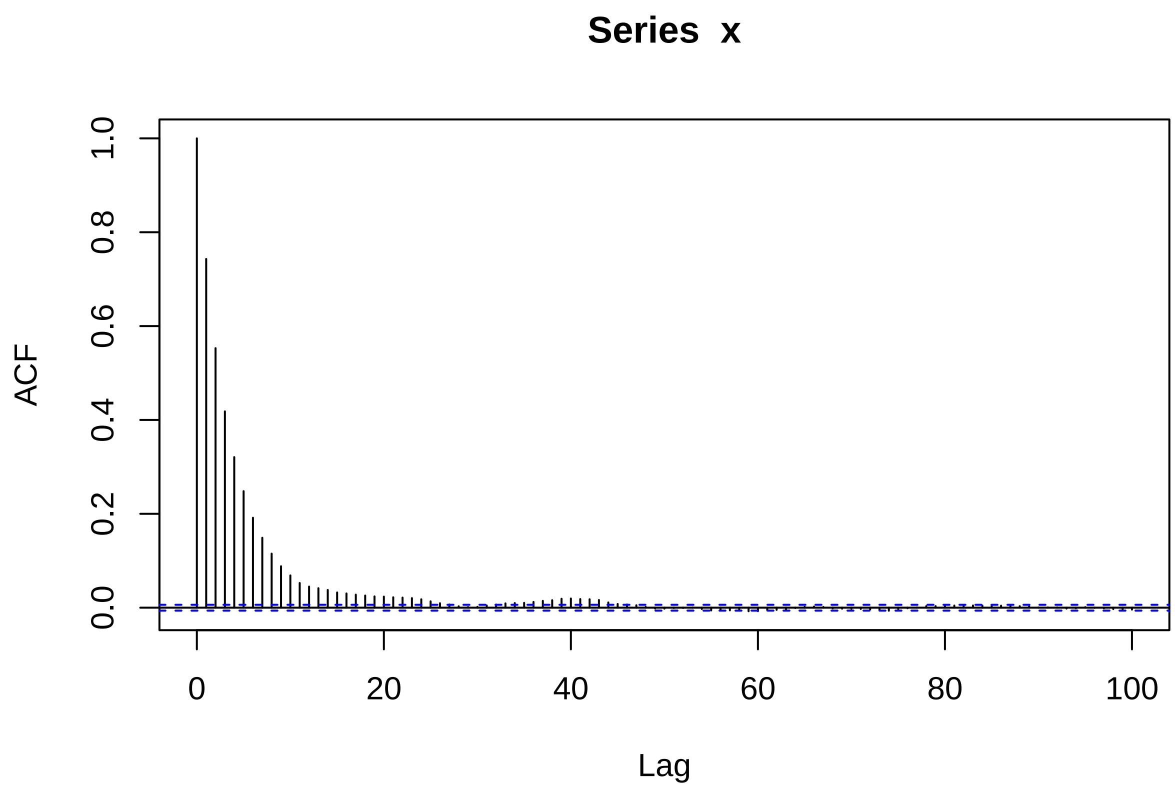
```
x<-MH_Laplace(100000,10)
```

The sample path and correlation plots show good mixing (the chain quickly moves away from its starting value, and the autocorrelation decreases quickly as the lag between iterations increases):

```
plot(x,type="l",xlab='t',ylab=expression(x[(t)]))
```

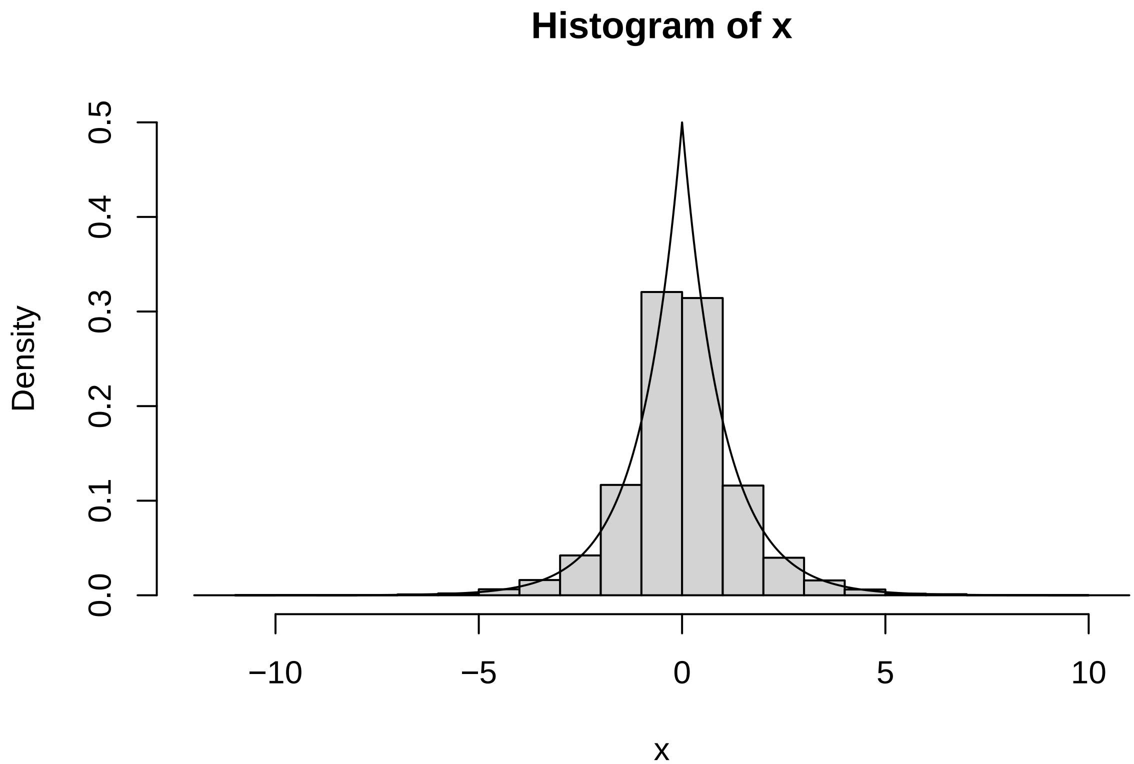



```
acf(x,lag.max=100)
```



Plot of the histogram with the Laplace density:

```
u<-seq(-10,10,0.01)
fu<-0.5*exp(-abs(u))
hist(x,freq=FALSE,ylim=range(fu))
lines(u,0.5*exp(-abs(u)))
```

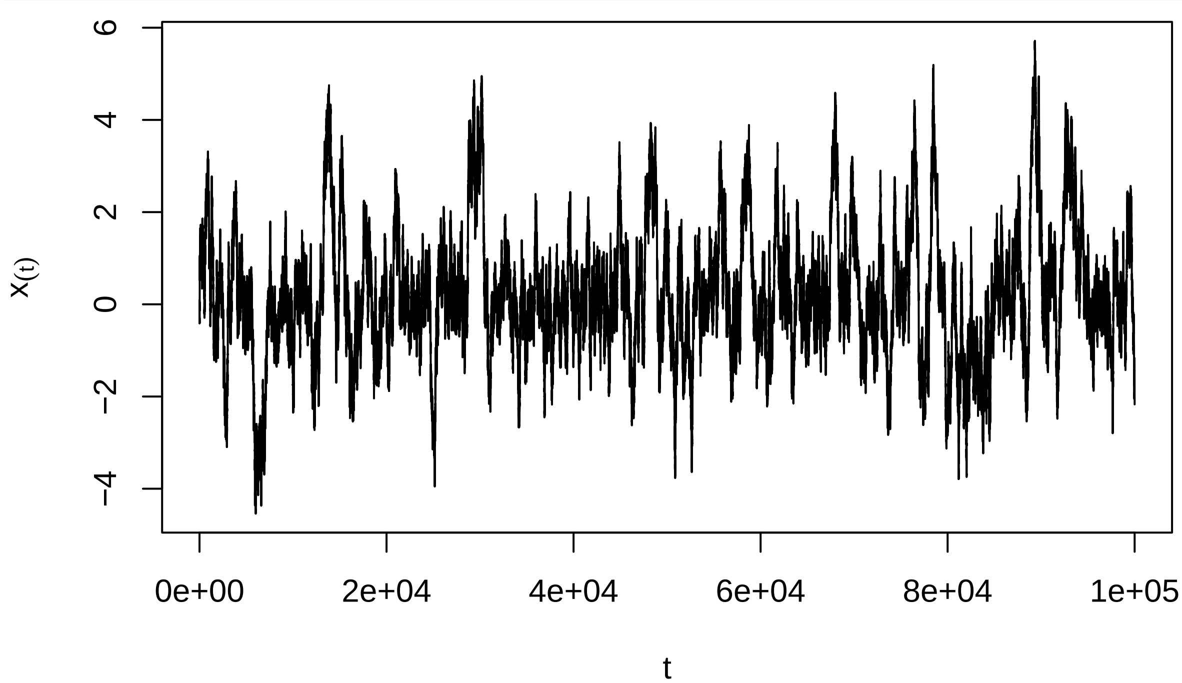


Let us now generate another sample of the same size, this time with $\sigma = 0.1$:

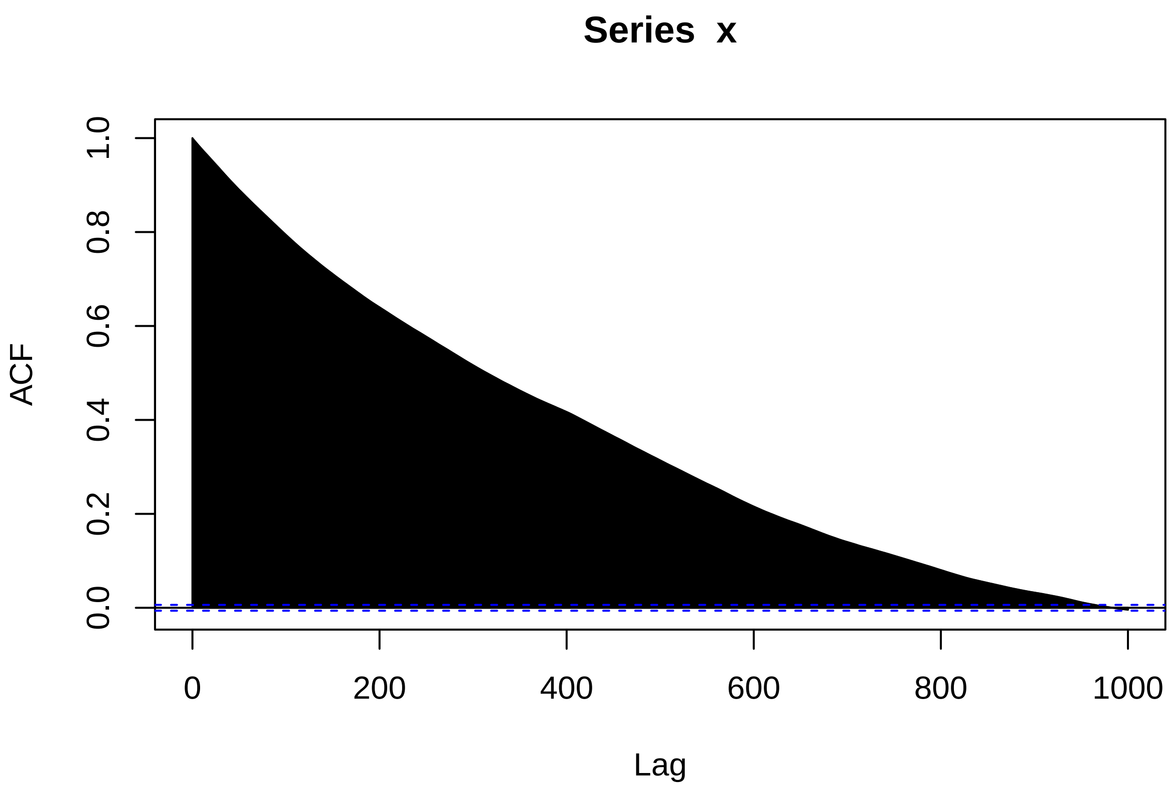
```
x<-MH_Laplace(100000,0.1)
```

This time, the sample path and correlation plots show poor mixing (the chain remains at or near the same value for many iterations, and the autocorrelation decays very slowly):

```
plot(x,type="l",xlab='t',ylab=expression(x[(t)]))
```



```r
acf(x,lag.max=1000)
```



Exercise 2

Question a

The likelihood function is

$$L(\beta; y_1, \dots, y_n) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}(y_i - \beta x_i)^2\right) = (2\pi)^{-n/2} \exp\left(-\frac{1}{2}\sum_{i=1}^{n}(y_i - \beta x_i)^2\right).$$

The density of the Gamma distribution with shape parameter a and rate b is $f(\beta) \propto \beta^{a-1}\exp(-b\beta)I(\beta > 0)$. Here $a = 2$ and $b = 1$, so $f(\beta) \propto \beta\exp(-\beta)I(\beta > 0)$. Consequently, the posterior density is

$$f(\beta \mid y_1, \dots, y_n) \propto \beta\exp(-\beta)\exp\left(-\frac{1}{2}\sum_{i=1}^{n}(y_i - \beta x_i)^2\right) I(\beta > 0).$$

Question b

We first write a function that computes the log-likelihood:

```r
loglik <- function(beta,x,y){
    n<- length(x)
    return(-0.5 * sum((y-beta*x)^2) - n/2*log(2*pi))
}
```

We then write a function that generates a MC of size N for a given data set:

```
gen_MH<-function(x,y,N){
  beta<-vector(N,mode="numeric")
  beta[1]<-rgamma(1,shape=2,rate=1)
  for(t in (2:N)){
    beta_star<-rgamma(1,shape=2,rate=1)
    u<-runif(1)
    logR <-loglik(beta_star,x,y)-loglik(beta[t-1],x,y)
    if( log(u) <= logR ) beta[t]<-beta_star else beta[t]<- beta[t-1]
  }
  return(beta)
}
```

Question c

Data generation:

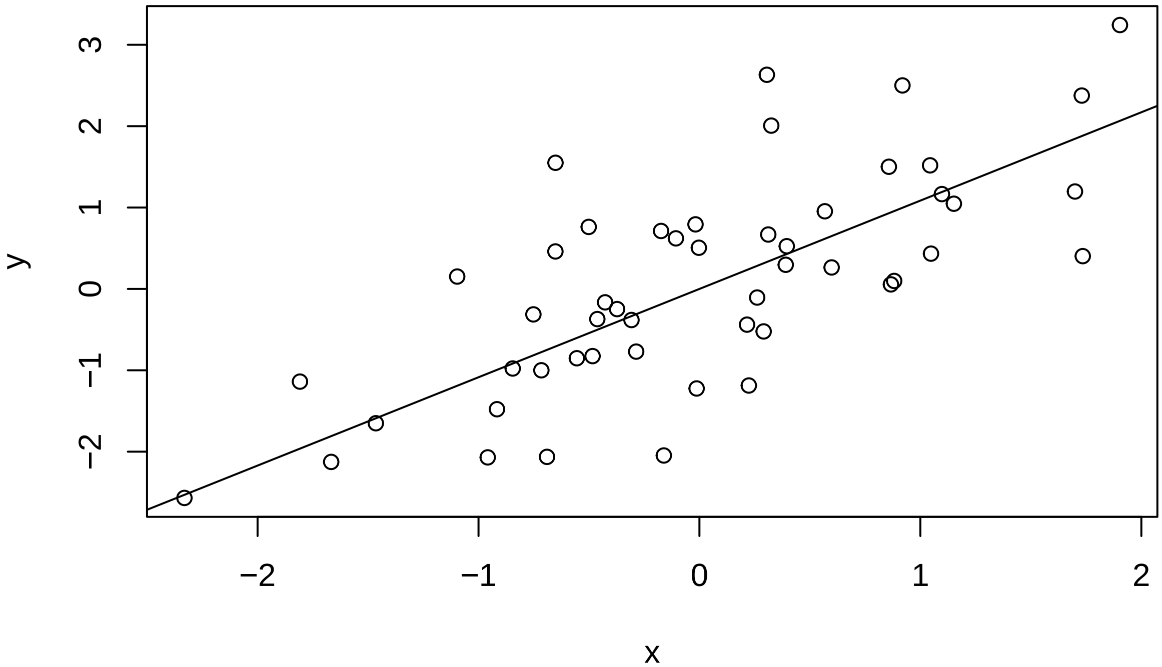
```
beta0<- rgamma(1,shape=2,rate=1) # Generation of beta
cat(beta0)
```

```
## 1.085444
```

```
n<-50
x<-rnorm(n)
y<-x*beta0+rnorm(n)
```

Plot of the data:

```
plot(x,y)
abline(0,beta0)
```



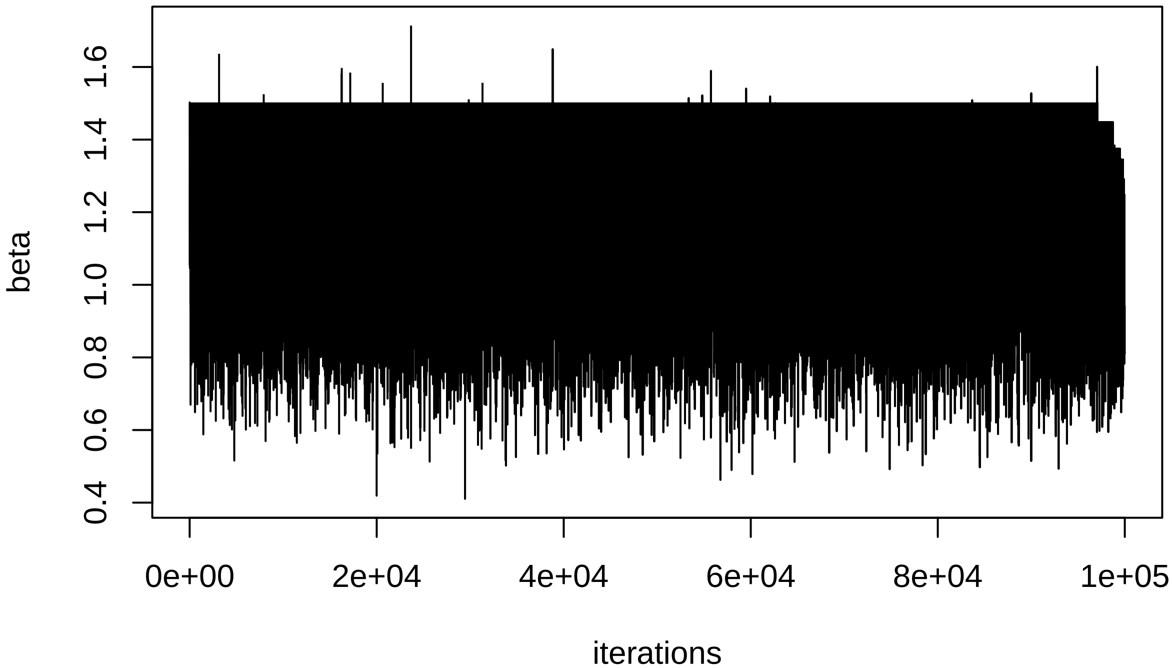
Running the MH algorithm:

```
N<-100000
beta<-gen_MH(x,y,N)
```

Question d

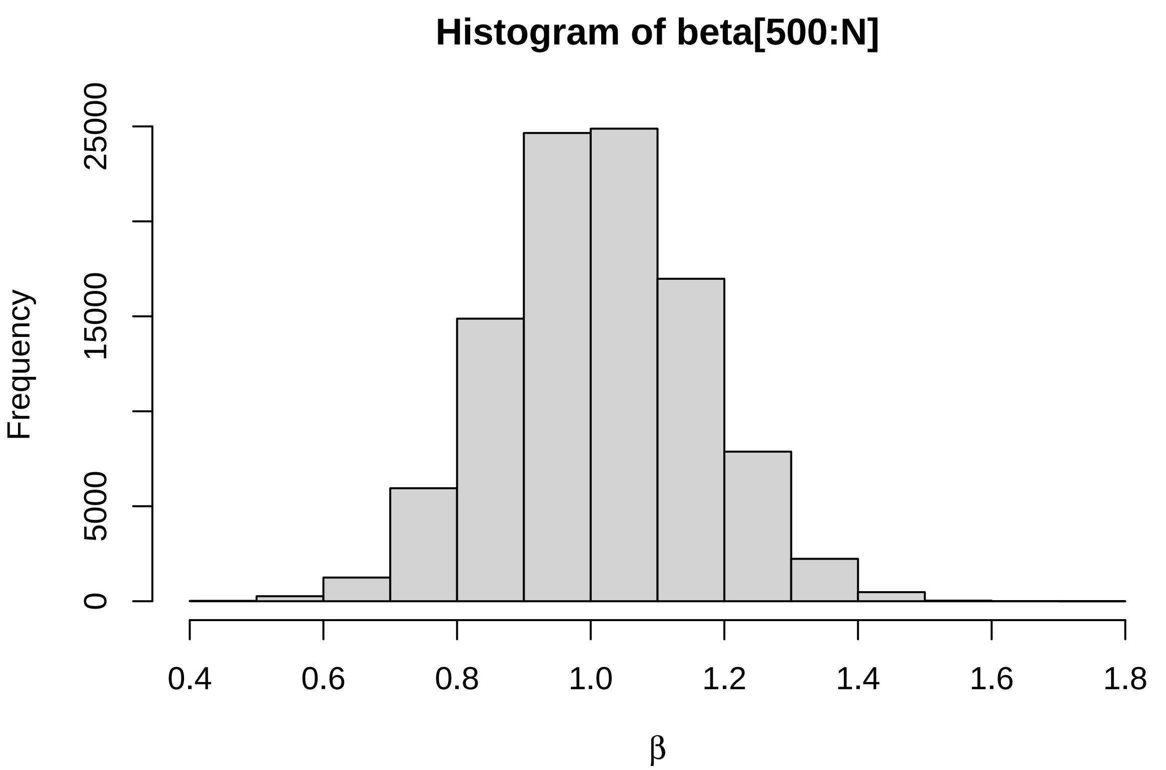
Sample path:

```
plot(beta,type="l",xlab="iterations")
```



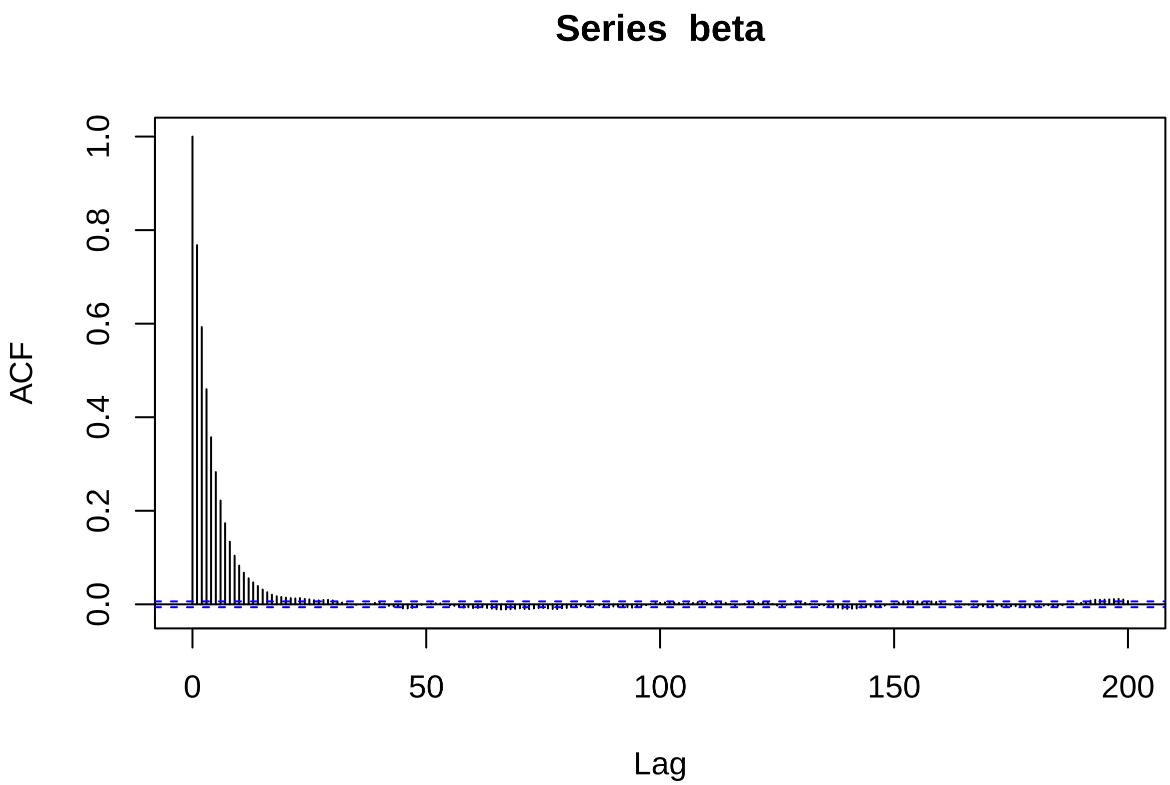
Histogram (leaving out the first 500 values):

```
hist(beta[500:N],xlab=expression(beta))
```



Autocorrelation plot:

```
acf(beta,lag.max=200)
```



Question e

We use the batch means method. We first determine the lag k_0 such that the autocorrelation is small enough to be neglected:

```
ACF<-acf(beta,lag.max=200,plot=FALSE)
k0<-ACF$lag[min(which(abs(ACF$acf)<0.01))]
cat(k0)
```

```
## 26
```

We fix the burn-in period and we compute the number of batches:

```
D<-1000 # burn in
B<-floor((N-D)/k0)
```

We compute the means within each block:

```
Z<-vector(B,mode="numeric")
for(b in (1:B)) Z[b]<-mean(beta[(D+(b-1)*k0+1):(D+b*k0)])
```

The estimated simulation standard error is the standard deviation of the batch means divided by the square root of the number of batches:

```
se <- sd(Z)/sqrt(B)
```

Estimated posterior expectation of β and simulation standard error:

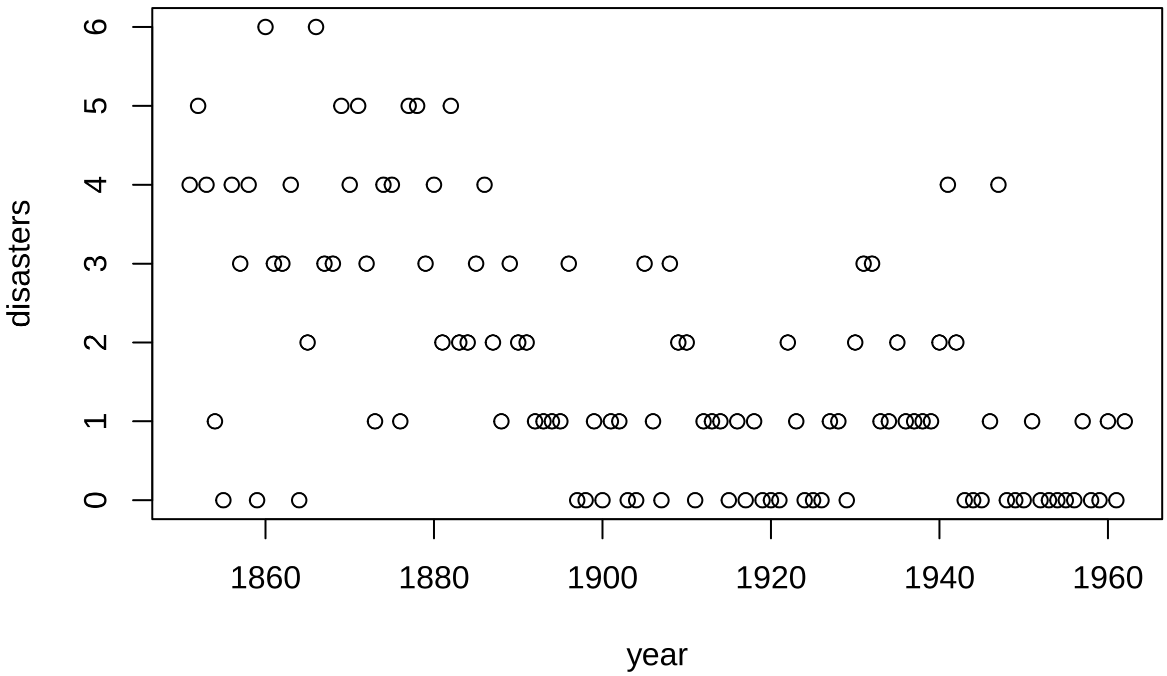
```r
print(c(mean(beta[(D+1):N]),se),3)
```

```
## [1] 1.01312 0.00123
```

Exercise 3

Question a

```r
coal <- read.table("/Users/Thierry/Documents/R/Data/Compstat/coal.dat",header=TRUE)
plot(coal)
```



Question b

The likelihood function is

$$L(\theta_1, \theta_2, k \mid \mathbf{x}) \propto \prod_{i=1}^{k} e^{-\theta_1} \theta_1^{x_i} \prod_{i=k+1}^{n} e^{-\theta_2} \theta_2^{x_i}.$$

We obtain the posterior distribution by multiplying the likelihood and the prior:

$$f(\theta_1, \theta_2, k|\mathbf{x}) \propto \underbrace{\theta_1^{\alpha_{01}-1} e^{-\beta_{01}\theta_1}}_{f(\theta_1)} \underbrace{\theta_2^{\alpha_{02}-1} e^{-\beta_{02}\theta_2}}_{f(\theta_2)} \underbrace{\prod_{i=1}^{k} e^{-\theta_1} \theta_1^{x_i} \prod_{i=k+1}^{n} e^{-\theta_2} \theta_2^{x_i}}_{L(\theta_1,\theta_2,k|\mathbf{x})}.$$

Now,

$$
\begin{aligned}
f(\theta_1 \mid \theta_2, k, \mathbf{x}) &\propto \frac{f(\theta_1, \theta_2, k \mid \mathbf{x})}{f(\theta_2, k)} \\
&\propto L(\theta_1, \theta_2, k \mid \mathbf{x}) f(\theta_1) \\
&\propto \theta_1^{\alpha_{01}-1} e^{-\beta_{01}\theta_1} \prod_{i=1}^{k} e^{-\theta_1} \theta_1^{x_i} \prod_{i=k+1}^{n} e^{-\theta_2} \theta_2^{x_i} \\
&\propto \theta_1^{\alpha_{01}+\sum_{i=1}^{k} x_i - 1} \exp\left(-(\beta_{01}+k)\theta_1\right).
\end{aligned}
$$

Consequently,

$$
f(\theta_1 \mid \theta_2, k, \mathbf{x}) = f(\theta_1 \mid k, \mathbf{x}) \sim G(\alpha_{01} + \sum_{i=1}^{k} x_i, \beta_{01} + k).
$$

Symmetrically, we obtain in the same way

$$
f(\theta_2 \mid \theta_1, k, \mathbf{x}) = f(\theta_2 \mid k, \mathbf{x}) \sim G(\alpha_{02} + \sum_{i=k+1}^{n} x_i, \beta_{02} + k).
$$

We can observe that θ_1 and θ_2 are conditionally independent given k and $\mathbf{x}$.

Finally, the conditional probability mass function of k is

$$
\begin{aligned}
f(k \mid \theta_1, \theta_2, \mathbf{x}) &\propto \frac{f(\theta_1, \theta_2, k \mid \mathbf{x})}{f(\theta_1) f(\theta_2)} \\
&\propto L(\theta_1, \theta_2, k \mid \mathbf{x}) \\
&\propto \exp\left[k(\theta_2 - \theta_1)\right] \left(\frac{\theta_1}{\theta_2}\right)^{\sum_{i=1}^{k} x_i}.
\end{aligned}
$$

Question c

The following function implements the Gibbs algorithm for this problem:

```
gibbs<-function(x,N,alpha10,beta10,alpha20,beta20){
  n<-length(x)
  # Initialization
  theta1 <- vector(length=N,mode="numeric")
  theta2 <- vector(length=N,mode="numeric")
  k <- vector(length=N,mode="numeric")
  p<-vector(length=n,mode="numeric")
  # First cycle
  # Sampling of k[1] from a uniform distribution
  k[1]<-sample(n,size=1)
  theta1[1]<-rgamma(1,shape=alpha10+sum(x[1:k[1]]),rate=beta10+k[1])
  theta2[1]<-rgamma(1,shape=alpha20+sum(x[(k[1]+1):n]),rate=beta20+n-k[1])
  for(t in (2:N)){
    # Conditional pmf of k
    for (j in (1:n)){
      p[j]<- (theta1[t-1]/theta2[t-1])^sum(x[1:j]) * exp(j*(theta2[t-1]-theta1[t-1]))
    }
    p<-p/sum(p)
    k[t]<- sample(n,size=1,prob=p)
    theta1[t]<-rgamma(1,shape=alpha10+sum(x[1:k[t]]),rate=beta10+k[t])
```

```
    theta2[t]<-rgamma(1,shape=alpha20+sum(x[(k[t]+1):n]),rate=beta20+n-k[t])
  }
  return(list(k=k,theta1=theta1,theta2=theta2))
}
```

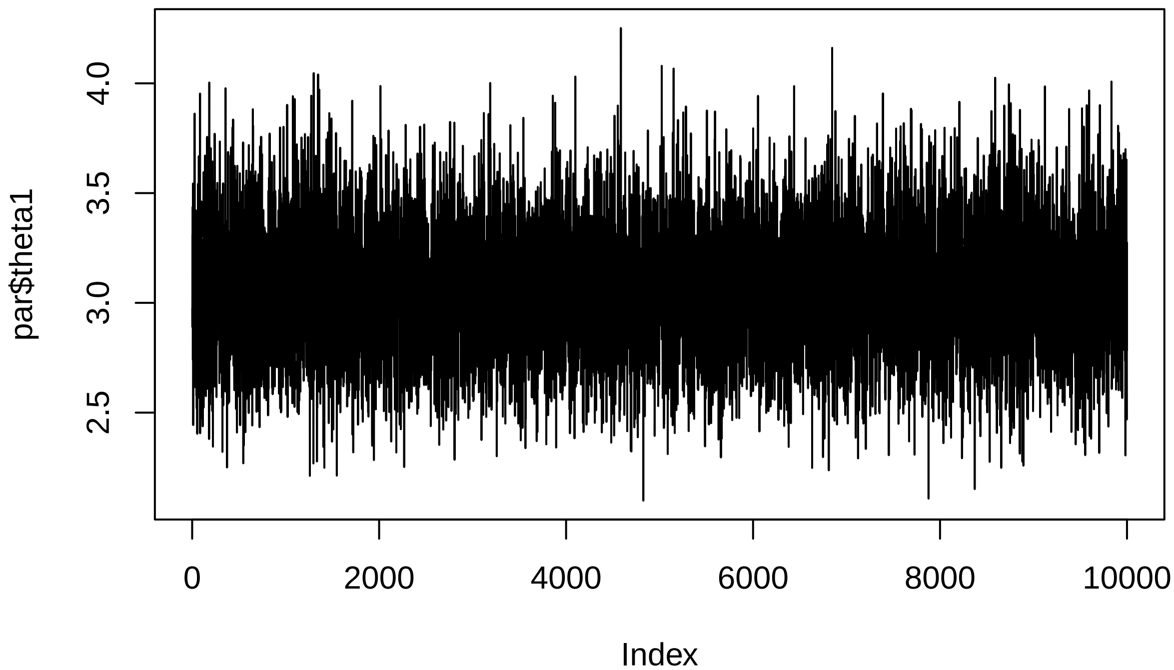
We can run this algorithm on the data:

```
N<-10000
alpha10<-0.5
alpha20<-0.5
beta10<-1
beta20<-1
par<-gibbs(x=coal$disasters,N,alpha10,beta10,alpha20,beta20)
```

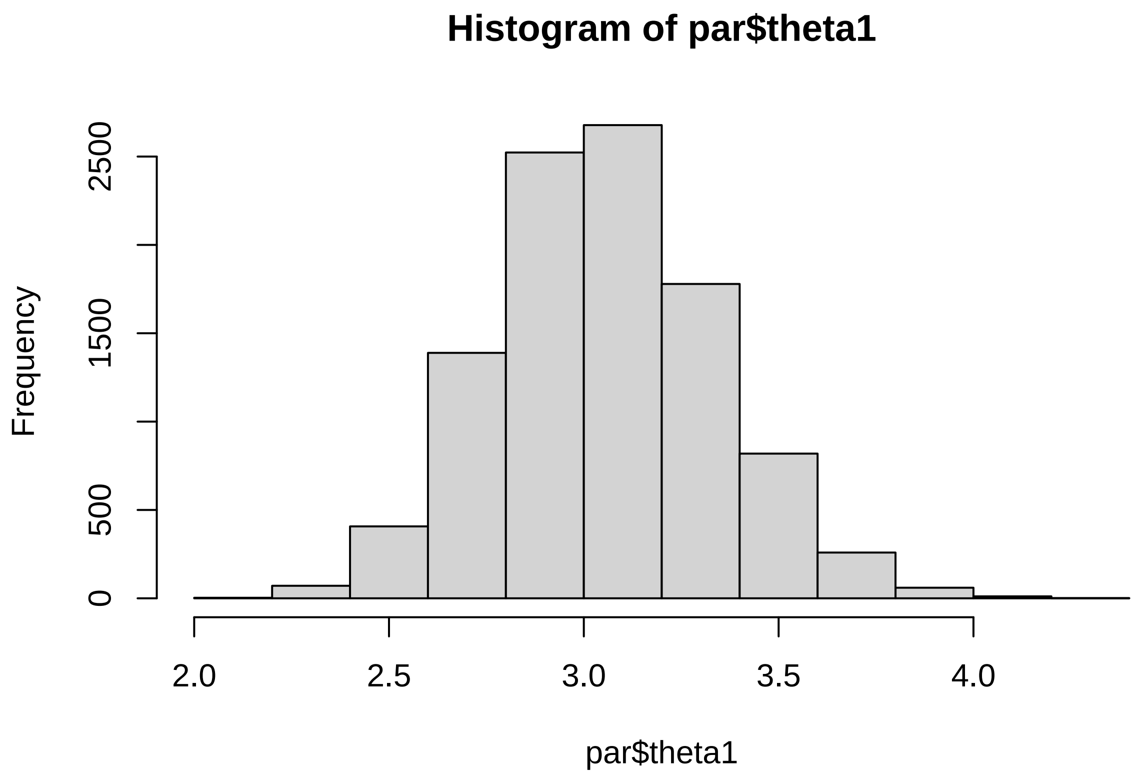
Question d

Plots for θ_1:

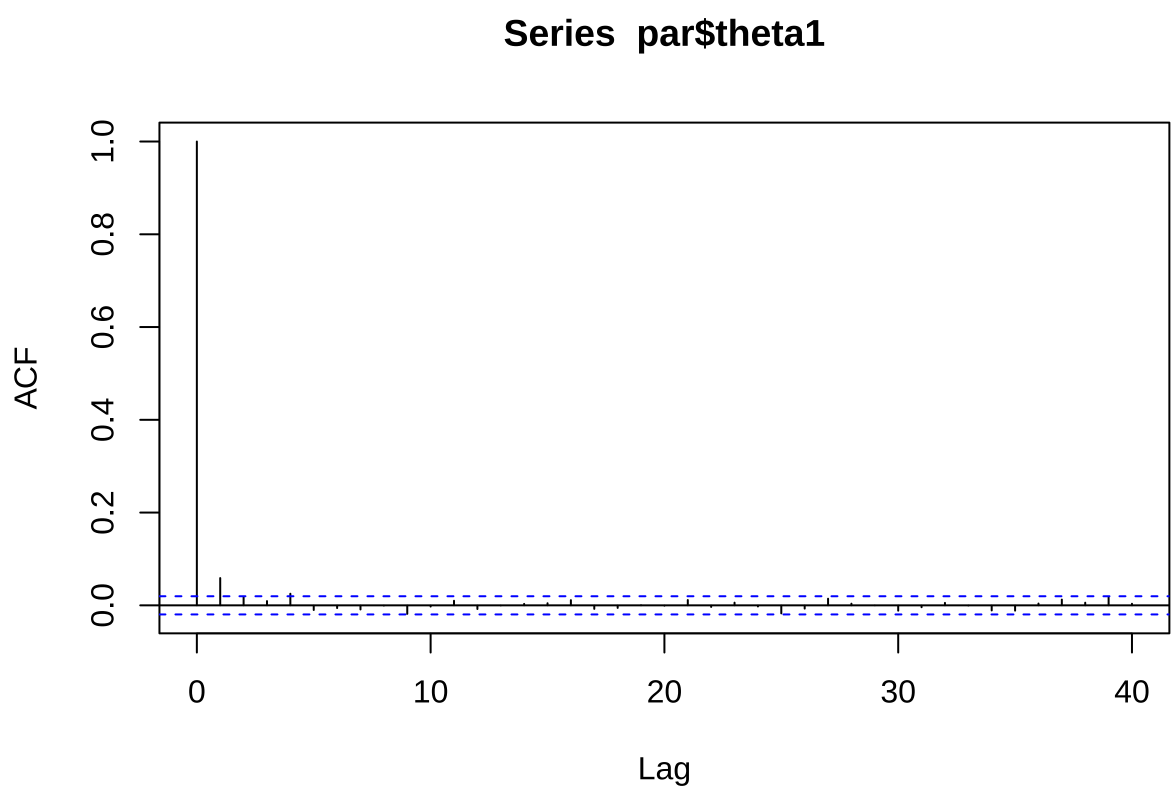
```
plot(par$theta1,type="l")
```




```
hist(par$theta1)
```

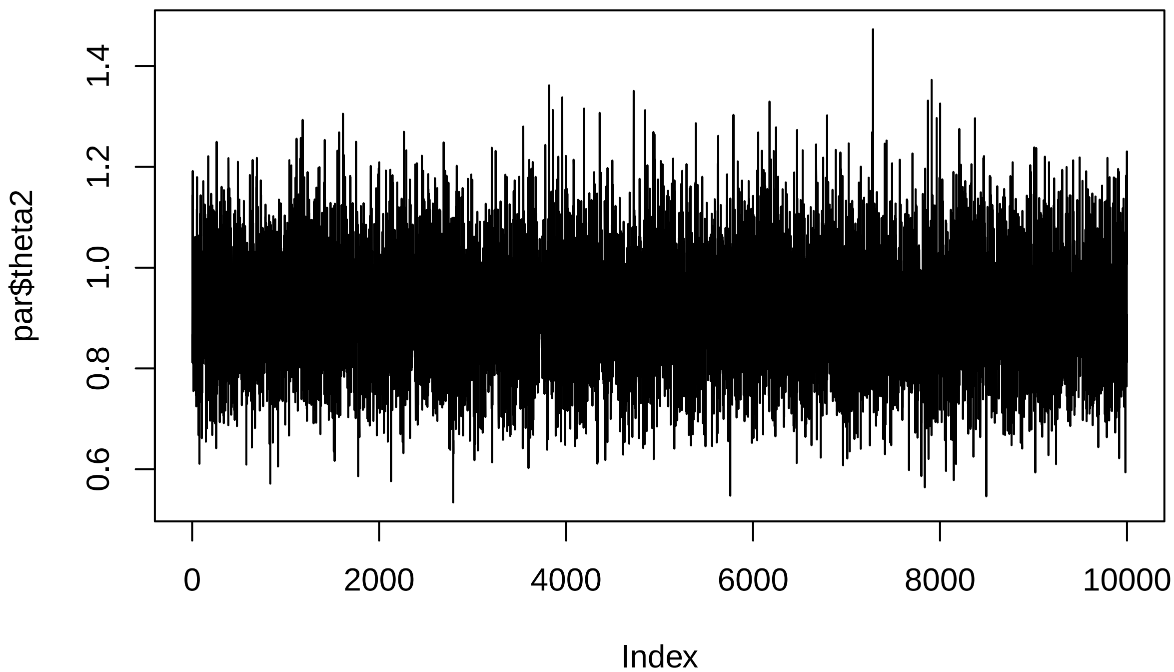



```
acf(par$theta1)
```

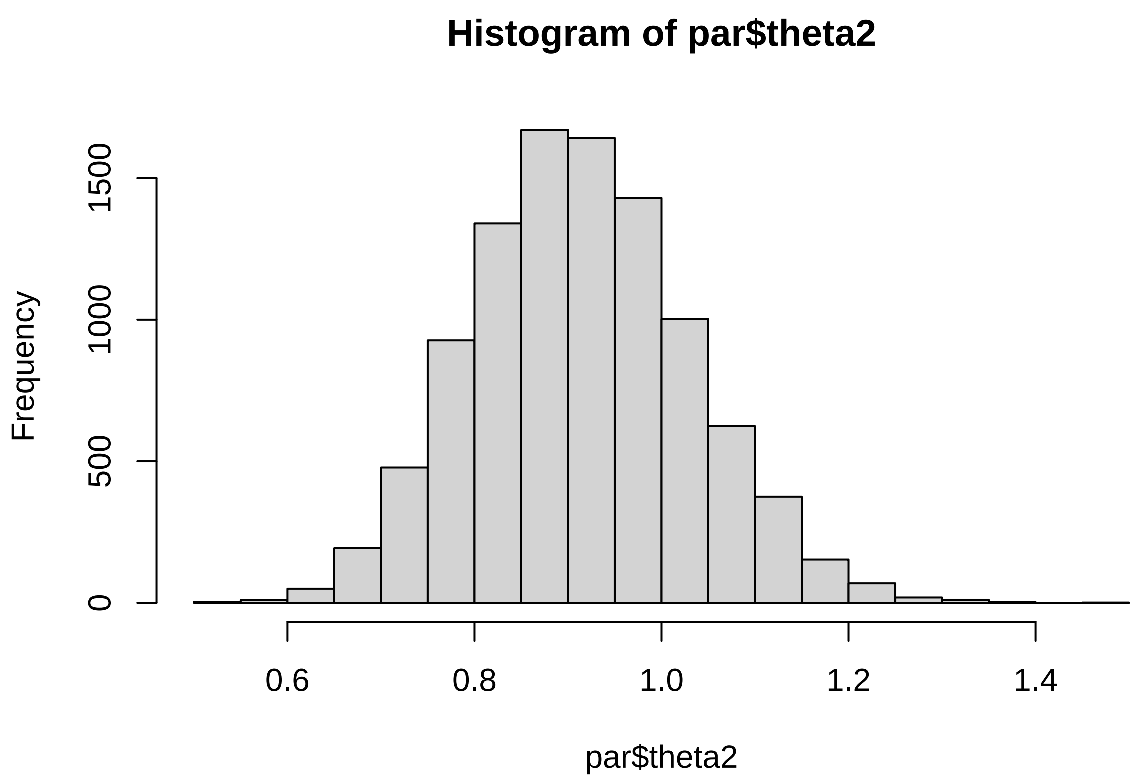


Plots for θ_2:

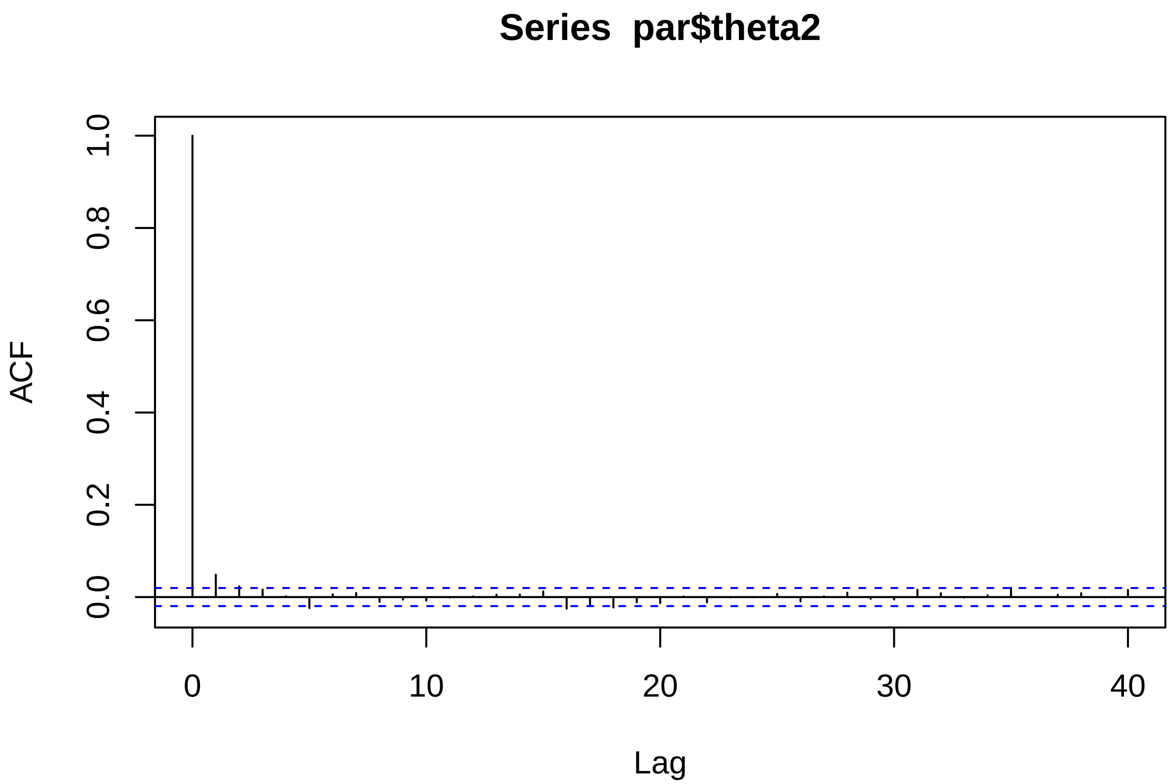
```
plot(par$theta2,type="l")
```




```
hist(par$theta2)
```

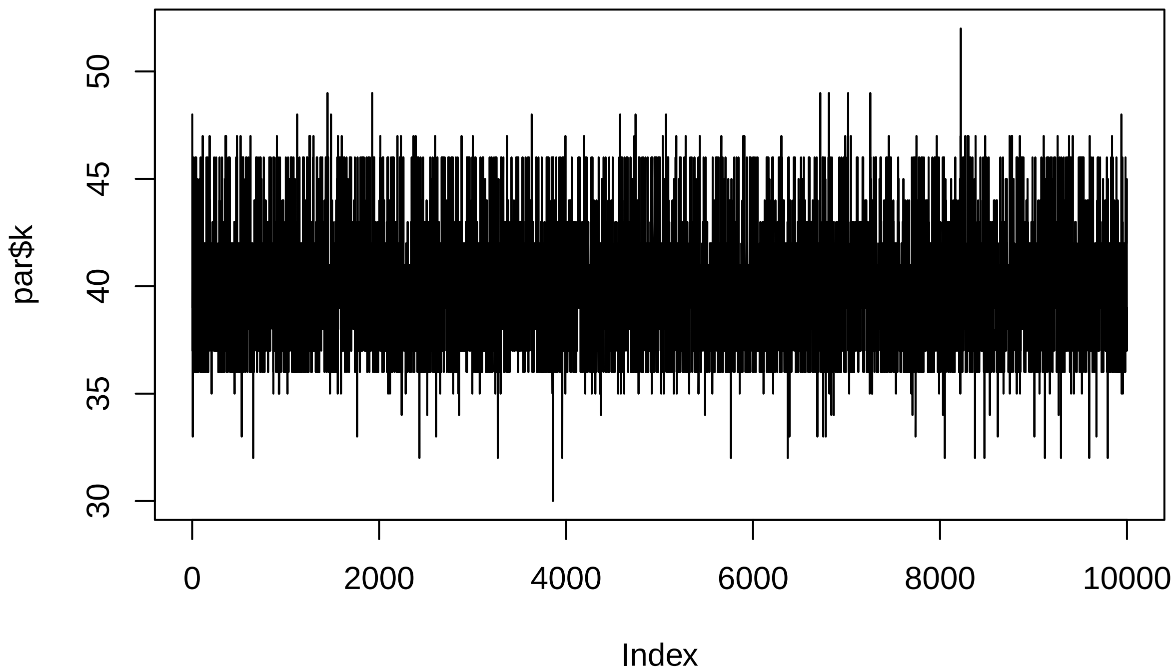



```
acf(par$theta2)
```

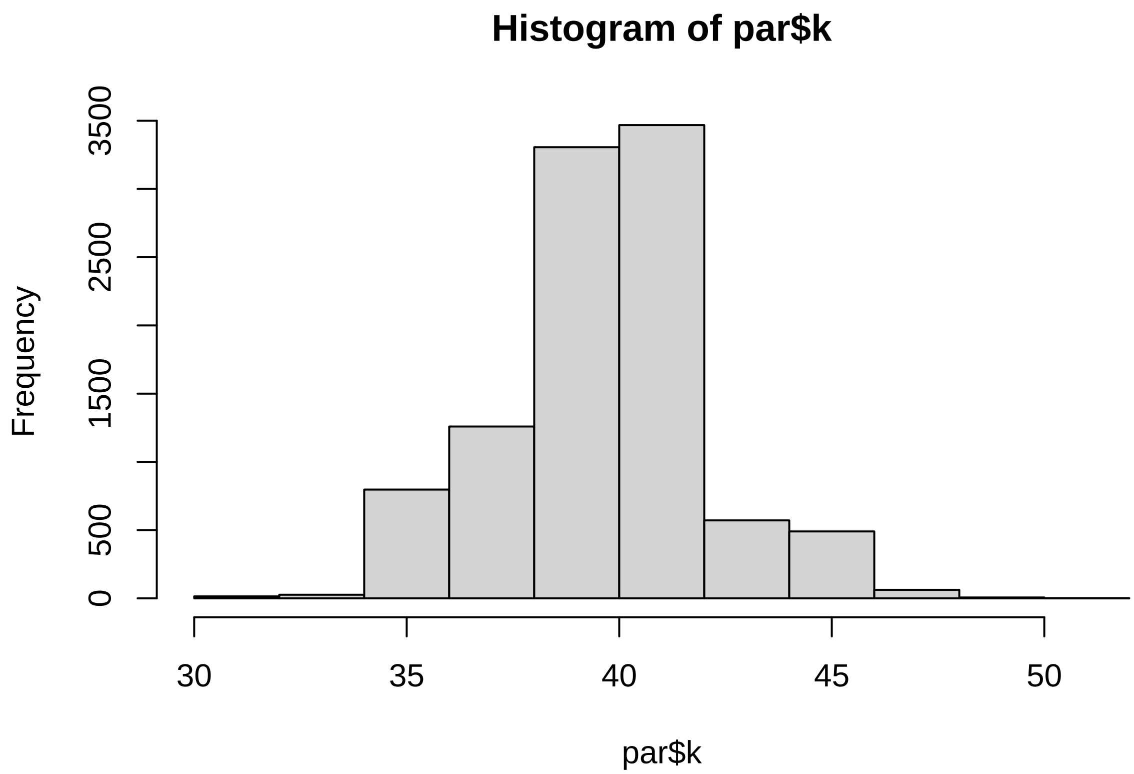


Plots for k:

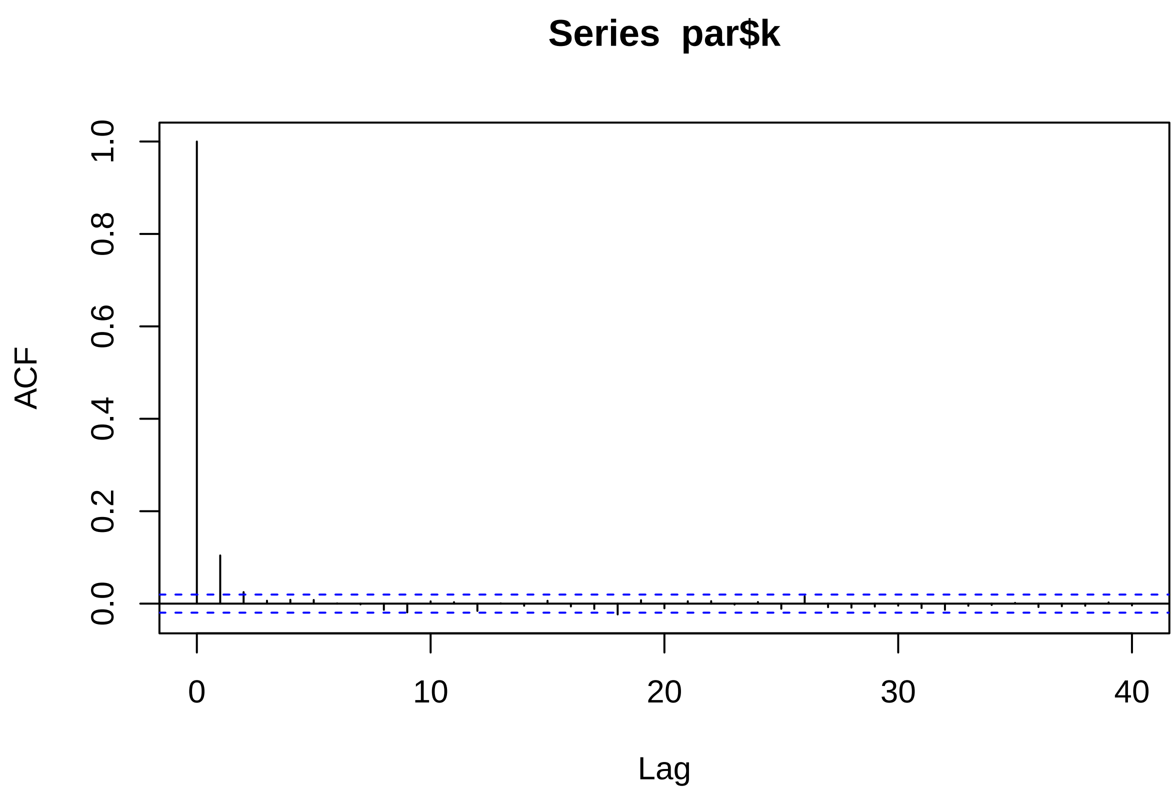
```
plot(par$k,type="l")
```




```
hist(par$k)
```


Histogram of par$k
Frequency
0
500
1500
2500
3500
30
35
40
45
50
par$k

```
acf(par$k)
```


Series par$k
ACF
0.0
0.2
0.4
0.6
0.8
1.0
0
10
20
30
40
Lag

Question e

We set the lag to 10 and the burn-in period to 1000, and we compute the number `N1` of batches:

```
L<-10 # lag
B<-1000 # burn in
N1<-floor((N-B)/L)  # number of batches
```

Estimated conditional expectation and simulated standard error for θ_1:

```
Z<-vector(N1,mode="numeric")
for(b in (1:N1)) Z[b]<-mean(par$theta1[(B+(b-1)*L+1):(B+b*L)])
se <- sd(Z)/sqrt(N1)
cat("mean=", mean(par$theta1[(B+1):N]),"se =",se)
```

```
## mean= 3.051553 se = 0.003175602
```

Estimated conditional expectation and simulated standard error for θ_2:

```
for(b in (1:N1)) Z[b]<-mean(par$theta2[(B+(b-1)*L+1):(B+b*L)])
se <- sd(Z)/sqrt(N1)
cat("mean =", mean(par$theta2[(B+1):N]),"se =", se)
```

```
## mean = 0.9144713 se = 0.001345767
```

Estimated conditional expectation and simulated standard error for k:

```
for(b in (1:N1)) Z[b]<-mean(par$k[(B+(b-1)*L+1):(B+b*L)])
se <- sd(Z)/sqrt(N1)
cat('mean =', mean(par$k[(B+1):N]),'se = ', se)
```

```
## mean = 40.15444 se =  0.02908424
```

The change is estimated to have taken place in the year

```
cat(coal$year[round(mean(par$k[(B+1):N]))])
```

```
## 1890
```