Information Fusion using Belief Functions
New combination rules

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Philippe Smets (1938-2005)
Overview

1. Theory of belief functions
   - Motivations
   - Basic concepts
   - Canonical conjunctive decomposition

2. The cautious and bold rules
   - Informational orderings and the LCP
   - The cautious conjunctive rule
   - The bold disjunctive rule

3. Families of combination rules
   - T-norm-based rules
   - Uninorm-based rules
   - Applications
Belief functions
An uncertainty representation framework

- One of the main frameworks for reasoning with partial (imprecise, uncertain) knowledge, introduced by Dempster (1967) and Shafer (1976)
- Belief functions generalize:
  - probability measures;
  - crisp sets;
  - possibility measures (and fuzzy sets).
- Different semantics for belief functions:
  - Lower-upper probabilities (Dempster’s model, Hint model);
  - Random sets;
  - Degrees of belief (Transferable Belief Model - TBM).
- The latter model will be adopted in this talk.

Main features:

1. Semantics of belief functions as representing weighted opinions of rational agents, irrespective of any underlying probability model;

2. Distinction between the credal and pignistic levels, and use of the pignistic transformation for mapping belief functions to probability measures for decision-making.

3. Use of unnormalized mass functions and interpretation of $m(\emptyset)$ under the open-world assumption;
In recent years, there has been many successful applications of the TBM to information fusion problems (sensor fusion, classification, expert opinion pooling, etc.);

However, there is some lack of flexibility for combining information as compared to other theories such as Possibility Theory:

- Only two main operators:
  - TBM conjunctive rule $\cap$ (unnormalized Dempster’s rule);
  - TBM disjunctive rule $\cup$;

- Main limitations:
  - Undesirable behavior of Dempster’s rule in case of high conflict between sources;
  - These operators assume the sources to be distinct.
Many research works devoted to this problem.

Several alternatives to Dempster’s rule based on various schemes for distributing the mass $m(\emptyset)$ to various propositions (Dubois-Prade rule, Yager’s rule, etc).

Some of these rules may be more robust than Dempster’s rule in case of highly conflicting sources, but

- They lack a clear justification in the TBM;
- They are not associative (to be addressed later).
The distinctness assumption

Definition

- Real-world meaning of this notion difficult to describe
- Main idea: no elementary item of evidence should be counted twice.
  - Example: non overlapping random samples from a population;
  - Counterexample: opinions of different people based on overlapping experiences.
- The TBM conjunctive and disjunctive rules are not appropriate for handling highly overlapping evidence (they are not idempotent).
Relaxing the distinctness assumption

Main approaches

- Possible approaches for combining overlapping items of evidence:
  - Describe the nature of the interaction between sources (Dubois and Prade 1986; Smets 1986);
  - Use a combination rule tolerating redundancy in the combined information.
- Such a rule should be idempotent: $m \ast m = m$.
- Idempotent rules exist (averaging; Cattaneo, 2003; Destercke et al, 2007), but they are not associative.
The associativity requirement

- **Definition:** \((m_1 \ast m_2) \ast m_3 = m_1 \ast (m_2 \ast m_3)\) for all \(m_1, m_2, m_3\).

- **Why is associativity a desirable property?**
  - **Practical argument:** Items evidence can be combined incrementally and regardless of the order in which they are processed (provided commutativity is also verified);
  - Quasi-associativity (existence of an \(n\)-ary operator \(op(m_1, \ldots, m_n)\)) may be sufficient in that respect.

- **Conceptual argument:** \(m_1 \ast m_2\) should capture all the relevant information contained in \(m_1\) and \(m_2\); consequently it should not be necessary to keep \(m_1\) and \(m_2\) in memory for further processing.
Main results to be presented in this talk

- Two new idempotent and associative combination rules, applicable to combine possibly overlapping items of evidence:
  - the cautious conjunctive rule $\cap$
  - the bold disjunctive rule $\lor$

- These rules are derived from the Least commitment principle (an equivalent of the maximum entropy principle for belief functions).

- Each of the four rules $\cap$, $\cup$, $\land$ and $\lor$ occupies a special position in a distinct infinite family of rules with identical algebraic properties.
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Let $\Omega = \{\omega_1, \ldots, \omega_K\}$ be a finite set of answers to a given question $Q$, called a frame of discernment.

**Definition (Basic belief assignment)**

A basic belief assignment (BBA) on $\Omega$ is a mapping $m : 2^\Omega \rightarrow [0, 1]$ such that

$$\sum_{A \subseteq \Omega} m(A) = 1$$

Subsets $A$ of $\Omega$ such that $m(A) > 0$ are called focal sets of $m$. 
A BBA $m$ represents:

- the state of knowledge of a rational agent $Ag$ at a given time $t$, regarding question $Q$;
- by extension, an item of evidence that induces such a state of knowledge.

$m(A)$: part of a unit mass of belief assigned to $A$ and to no strict subset.

$m(\emptyset)$: degree of ignorance.

$m(\Omega)$: degree of conflict. Under the open-world assumption, degree of belief in the hypothesis that the true answer to question $Q$ does not lie in $\Omega$. 
Associated functions
Belief and implicability functions

**Definition (Belief function)**

\[
\text{bel}(A) = \sum_{\emptyset \neq B \subseteq A} m(B), \quad \forall A \subseteq \Omega
\]

Interpretation of \( \text{bel}(A) \): degree of belief in \( A \).

**Definition (Implicability function)**

\[
b(A) = \text{bel}(A) + m(\emptyset), \quad \forall A \subseteq \Omega
\]
**Asscoiated functions**

**Belief and implicability functions**

- **Definition (Belief function)**
  
  \[
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- **Definition (Implicability function)**
  
  \[
  b(A) = \text{bel}(A) + m(\emptyset), \quad \forall A \subseteq \Omega
  \]
Associated functions
Plausibility and commonality

Definition (Plausibility function)

\[ pl(A) = \sum_{B \cap A \neq \emptyset} m(B), \quad \forall A \subseteq \Omega \]

Definition (Commonality function)

\[ q(A) = \sum_{B \supseteq A} m(B), \quad \forall A \subseteq \Omega \]
Definition (Plausibility function)

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Definition (Commonality function)

\[ q(A) = \sum_{B \supseteq A} m(B), \quad \forall A \subseteq \Omega \]
Equivalence of representations

- Functions $bel$, $b$, $pl$, $q$, $m$ are in one-to-one correspondence.
- One can move from any representation to another using linear transformations.
- For instance:
  
  $$pl(A) = bel(\Omega) - bel(\overline{A}) = 1 - b(\overline{A}), \quad \forall A \subseteq \Omega,$$
  
  $$m(A) = \sum_{B \supseteq A} (-1)^{|B|-|A|} q(B), \quad \forall A \subseteq \Omega,$$
  
  $$m(A) = \sum_{B \subseteq A} (-1)^{|A|-|B|} b(B), \quad \forall A \subseteq \Omega,$$

- There exists at least two other equivalent representations (to be introduced later...)

T. Denœux

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**Definition (TBM conjunctive rule)**

\[
m_1 \cap_2 = m_1 \cap m_2 \text{ defined as:}
\]

\[
m_1 \cap_2(A) = \sum_{B \cap C = A} m_1(B)m_2(C), \quad \forall A \subseteq \Omega,
\]

Interpretation: \( m_1 \cap m_2 \) encodes the agent’s belief after receiving \( m_1 \) and \( m_2 \) from two sources \( S_1 \) and \( S_2 \), assuming that:

- \( S_1 \) and \( S_2 \) are distinct (Klawonn and Smets, 1992);
- both \( S_1 \) and \( S_2 \) are reliable.
Definition (TBM conjunctive rule)

\[ m_1 \odot_2 = m_1 \odot m_2 \text{ defined as:} \]

\[ m_1 \odot_2(A) = \sum_{B \cap C = A} m_1(B)m_2(C), \quad \forall A \subseteq \Omega, \]

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- \( S_1 \) and \( S_2 \) are distinct (Klawonn and Smets, 1992);
- both \( S_1 \) and \( S_2 \) are reliable.
Algebraic properties:
- Commutativity,
- Associativity
- Neutral element: vacuous BBA $m_{\Omega}$ ($m_{\Omega}(\Omega) = 1$)

$\rightarrow$ $(\mathcal{M}, \cap)$ is a commutative monoid.

Expression using the commonality functions:

$$q_1 \cap_2(A) = q_1(A) \cdot q_2(A), \quad \forall A \subseteq \Omega.$$
TBM disjunctive rule

Definition

Definition (TBM disjunctive rule)

\[ m_1 \bigcup_2 m_2 \text{ defined as:} \]

\[ m_1 \bigcup_2 (A) = \sum_{B \cup C = A} m_1(B)m_2(C), \quad \forall A \subseteq \Omega, \]

Interpretation: \( m_1 \bigcup m_2 \) encodes the agent’s belief after receiving \( m_1 \) and \( m_2 \) from two sources \( S_1 \) and \( S_2 \), assuming that:

- \( S_1 \) and \( S_2 \) are distinct (Klawonn and Smets, 1992);
- at least one of \( S_1 \) and \( S_2 \) is reliable.
**Definition (TBM disjunctive rule)**

\[ m_{1 \cup 2} = m_1 \cup m_2 \text{ defined as:} \]

\[ m_{1 \cup 2}(A) = \sum_{B \cup C = A} m_1(B)m_2(C), \quad \forall A \subseteq \Omega, \]

Interpretation: \( m_1 \cup m_2 \) encodes the agent’s belief after receiving \( m_1 \) and \( m_2 \) from two sources \( S_1 \) and \( S_2 \), assuming that:

- \( S_1 \) and \( S_2 \) are distinct (Klawonn and Smets, 1992);
- at least one of \( S_1 \) and \( S_2 \) is reliable.
TBM disjunctive rule

Properties

- **Algebraic properties:**
  - Commutativity,
  - Associativity
  - Neutral element: \( m_\emptyset \ (m_\emptyset(\emptyset) = 1) \)

\[ \rightarrow (\mathcal{M}, \cup) \text{ is a commutative monoid.} \]

- **Expression using the implicability functions:**

\[ b_1 \cup_2(A) = b_1(A) \cdot b_2(A), \quad \forall A \subseteq \Omega. \]
Complement of $m$:

$$\overline{m}(A) = m(\overline{A}), \quad \forall A \subseteq \Omega.$$ 

De Morgan laws for $\cap$ and $\cup$:

$$m_1 \cup m_2 = \overline{m_1 \cap m_2},$$
$$m_1 \cap m_2 = \overline{m_1 \cup m_2},$$

(∩ and ∪ can be interpreted as generalized intersection and union)
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Simple BBA
Definition and notation

Definition (Simple BBA)

A BBA is simple if it is of the form

\[
m(A) = 1 - w \\
m(\Omega) = w,
\]

with \( w \in [0, 1] \) and \( A \subseteq \Omega \). Notation: \( m = A^w \).

- Property: \( A^{w_1} \cap A^{w_2} = A^{w_1 w_2} \).
- Special cases:
  - Vacuous BBA: \( A^1 \) with any \( A \).
  - Categorical BBA: \( A^0 \).
- Can any BBA be decomposed as the \( \cap \)-combination of simple BBAs?
Simple BBA
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  - Categorical BBA: \( A^0 \).
- Can any BBA be decomposed as the \( \cap \)-combination of simple BBAs?
The concept of **separability** was introduced by Shafer (1976) in the case of normal BBAs. It can be adapted to subnormal BBAs as follows.

**Definition (separability)**

A BBA $m$ is **separable** if it can be decomposed as the $\bigcap$ combination of simple BBAs.

This decomposition is unique as long as $m$ is **nondogmatic** ($m(\Omega) > 0$). It may be called the **canonical conjunctive decomposition** of $m$. 
If \( m \) is separable, then there exists a unique function \( w : 2^\Omega \mapsto (0, 1] \) such that

\[
m = \bigcap_{A \subset \Omega} A^{w(A)},
\]

and \( w(\Omega) = 1 \) by convention.

Function \( w \) is called the conjunctive weight function associated to \( m \). It is thus yet another representation of \( m \).

Can this representation be extended to any nondogmatic BBA?
A generalized simple BBA is a function $\mu : 2^\Omega \rightarrow \mathbb{R}$ such that

$$
\begin{align*}
\mu(A) &= 1 - w, \\
\mu(\Omega) &= w, \\
\mu(B) &= 0 \quad \forall B \in 2^\Omega \setminus \{A, \Omega\},
\end{align*}
$$

for some $A \neq \Omega$ and $w \in [0, +\infty)$. Notation: $\mu = A^w$. 

\[\text{Definition (Smets, 1995)}\]
If $w \leq 1$, $\mu$ is a simple BBA.

If $w > 1$, $\mu$ is not a BBA $\rightarrow$ inverse BBA.

Interpretation: models a state of knowledge in which we have some diffidence (disbelief) against hypothesis $A$. We need to acquire some evidence in favor of $A$ to reach a neutral state:

$$A^w \cap A^{1/w} = A^1.$$
Canonical decomposition of a nondogmatic BBA

Main result

**Theorem (Smets, 1995)**

Any nondogmatic BBA can be uniquely decomposed as the $\cap$ of generalized simple BBAs:

$$m = \bigcap_{A \subset \Omega} A^{w(A)},$$

with $w(A) \in (0, +\infty]$ for all $A \subset \Omega$.

- The canonical weight function is now from $2^{\Omega}$ to $(0, +\infty]$.
- $m$ is separable iff $w(A) \leq 1$ for all $A$. 
Theorem (Smets, 1995)

Any nondogmatic BBA can be uniquely decomposed as the \( \cap \) of generalized simple BBAs:

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m = \bigcap_{A \subset \Omega} A^{w(A)},
\]

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- The canonical weight function is now from \( 2^\Omega \) to \( (0, +\infty] \).
- \( m \) is separable iff \( w(A) \leq 1 \) for all \( A \).
Conjunctive weight function

Computation

- Computation of $w$ from $q$:

$$\ln w(A) = - \sum_{B \supseteq A} (-1)^{|B|-|A|} \ln q(B), \quad \forall A \subset \Omega.$$  

- Similarity with

$$m(A) = \sum_{B \supseteq A} (-1)^{|B|-|A|} q(B), \quad \forall A \subseteq \Omega.$$  

- Any procedure for transforming $q$ to $m$ can be used to transform $-\ln q$ to $\ln w$. 

Let $m$ be a consonant BBA, with associated possibility distribution $\pi_k = \pi(\omega_k) = q(\{\omega_k\})$, $k = 1, \ldots, K$, such that

$$1 \geq \pi_1 \geq \pi_2 \geq \ldots \geq \pi_K > 0.$$ 

The conjunctive weight function associated to $m$ is:

$$w(A) = \begin{cases} 
\frac{\pi_1}{\pi_k}, & A = \emptyset, \\
\frac{\pi_{k+1}}{\pi_k}, & A = \{\omega_1, \ldots, \omega_k\}, 1 \leq k < K, \\
1, & \text{otherwise.}
\end{cases}$$

$m$ is separable.
Let $m$ be a BBA on $\Omega$ with focal sets $A_1, \ldots, A_n$, and $\Omega$, such that $A_i \cap A_j = \emptyset$ for all $i, j \in \{1, \ldots, n\}$.

We assume that $m(\Omega) + \sum_{k=1}^{n} m(A_k) \leq 1$, so that $\emptyset$ may also be a focal set.

The conjunctive weight function associated to $m$ is:

$$ w(A) = \begin{cases} 
\frac{m(\Omega)}{m(A_k) + m(\Omega)}, & A = A_k, \\
\frac{m(\Omega) \prod_{k=1}^{n} \left(1 + \frac{m(A_k)}{m(\Omega)}\right)}{1}, & A = \emptyset, \\
1, & \text{otherwise}.
\end{cases} $$

We may have $w(\emptyset) > 1$, so that $m$ is not always separable.
Expression of the TBM conjunctive rule using $w$

Property

We have

$$m_1 \cap m_2 = \bigcap_{A \subseteq \Omega} A^{w_1(A)} \cap \bigcap_{A \subseteq \Omega} A^{w_2(A)}$$

$$= \bigcap_{A \subseteq \Omega} A^{w_1(A)w_2(A)}.$$

Consequently,

$$w_1 \cap w_2 = w_1 \cdot w_2.$$

- Similar to $q_1 \cap q_2 = q_1 \cdot q_2$. 
Several alternative representations of a BBA, including $\text{bel}$, $b$, $pl$, $q$ and $w$.

The TBM conjunctive and disjunctive rules are usually expressed in the $m$-space, but they have simpler representations in other spaces:

- $q$ and $w$ spaces for $\cap$
- $b$ space and another space to be introduced later for $\cup$.

Most attempts to generalize $\cap$ have started from its expression in the $m$ space.

Our approach will be based on the $w$ space.
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Definition (Least commitment principle)

Given two belief functions compatible with a set of constraints, the most appropriate is the least committed (informative).

- Similar to the maximum entropy principle in Probability theory.
- To make this principle operational, it is necessary to define ways of comparing belief functions according to their information content: “$m_1$ is more committed than $m_2$”.
- Several such informational orderings have been proposed.
Informational Comparison of Belief Functions

Definitions

\( p_l \)-ordering: \( m_1 \sqsubseteq_{p_l} m_2 \) iff \( p_l_1(A) \leq p_l_2(A) \), for all \( A \subseteq \Omega \);

\( q \)-ordering: \( m_1 \sqsubseteq_{q} m_2 \) iff \( q_1(A) \leq q_2(A) \), for all \( A \subseteq \Omega \);

\( s \)-ordering: \( m_1 \sqsubseteq_{s} m_2 \) iff there exists a stochastic matrix \( S \) with general term \( S(A, B) \), \( A, B \in 2^\Omega \) verifying \( S(A, B) > 0 \Rightarrow A \subseteq B, A, B \subseteq \Omega \), such that

\[
m_1(A) = \sum_{B \subseteq \Omega} S(A, B) m_2(B), \quad \forall A \subseteq \Omega.
\]

\( d \)-ordering: \( m_1 \sqsubseteq_d m_2 \), iff there exists a BBA \( m \) such that \( m_1 = m \bigcap m_2 \).
Informational Comparison of Belief Functions

Properties

- \( m_1 \sqsubseteq_d m_2 \Rightarrow m_1 \sqsubseteq_s m_2 \Rightarrow \begin{cases} m_1 \sqsubseteq_{pl} m_2, \\ m_1 \sqsubseteq_q m_2, \end{cases} \)

- The vacuous BBA \( m_\Omega \) is the unique greatest element for \( \sqsubseteq_x \) with \( x \in \{pl, q, s, d\} \):

  \[
  m \sqsubseteq_x m_\Omega, \quad \forall m, \forall x \in \{pl, q, s, d\}. 
  \]

- Monotonicity of \( \forall \) with respect to \( \sqsubseteq_x \), \( x \in \{pl, q, s, d\} \):

  \[
  m_1 \sqsubseteq_x m_2 \Rightarrow m_1 \forall m_3 \sqsubseteq_x m_2 \forall m_3, \quad \forall m_1, m_2, m_3 
  \]

  \( \rightarrow (M, \forall, \sqsubseteq_x) \) is a partially ordered commutative monoid.
Two sources provide BBAs $m_1$ and $m_2$, and the sources are both considered to be reliable.

The agent’s state of belief, after receiving these two pieces of information, should be represented by a BBA $m_{12}$ more committed than $m_1$, and more committed than $m_2$.

Let $S_x(m)$ be the set of BBAs $m'$ such that $m' \sqsubseteq_x m$, for some $x \in \{pl, q, s, d\}$.

We thus have $m_{12} \in S_x(m_1)$ and $m_{12} \in S_x(m_2)$ or, equivalently, $m_{12} \in S_x(m_1) \cap S_x(m_2)$.

According to the LCP, one should select the $x$-least committed element in $S_x(m_1) \cap S_x(m_2)$, if it exists.
Cautious combination of belief functions

Problem

- The above approach works for special cases.
- Example (Dubois, Prade, Smets 2001): if \( m_1 \) and \( m_2 \) are consonant, then the \( q \)-least committed element in \( S_q(m_1) \cap S_q(m_2) \) exists and it is unique: it is the consonant BBA with commonality function \( q_{12} = q_1 \land q_2 \).
- In general, neither existence nor unicity of a solution can be guaranteed with any of the \( x \)-orderings, \( x \in \{pl, q, s, d\} \).
- We need to define a new ordering relation.
The \( w \)-ordering

**Definition (\( w \)-ordering)**

Let \( m_1 \) and \( m_2 \) be two nondogmatic BBAs. 
\( m_1 \sqsubseteq_w m_2 \) iff \( w_1(A) \leq w_2(A) \), for all \( A \subset \Omega \).

- Interpretation: \( m_1 = m \cap m_2 \) for some separable BBA \( m \).
- \( m_1 \sqsubseteq_w m_2 \Rightarrow m_1 \sqsubseteq_d m_2 \Rightarrow m_1 \sqsubseteq_s m_2 \Rightarrow \begin{cases} m_1 \sqsubseteq_{pl} m_2 \\ m_1 \sqsubseteq_{q} m_2, \end{cases} \)
- No greatest element, but \( m_\Omega \) is the unique maximal element: \( m_\Omega \sqsubseteq_w m \Rightarrow m = m_\Omega \).
- Monotonicity of \( \cap \): 
  \( m_1 \sqsubseteq_w m_2 \Rightarrow m_1 \cap m_3 \sqsubseteq_w m_2 \cap m_3, \quad \forall m_1, m_2, m_3 \)
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The cautious conjunctive rule

Definition

**Theorem**

*Let* \( m_1 \) *and* \( m_2 \) *be two nondogmatic BBAs. The* \( w \)-least committed element in* \( S_w(m_1) \cap S_w(m_2) \) *exists and is unique. It is defined by the following weight function:*

\[
w_1 \land_2 (A) = w_1(A) \land w_2(A), \quad \forall A \subset \Omega.
\]

**Definition (cautious conjunctive rule)**

\[
m_1 \land m_2 = \bigsqcap_{A \subset \Omega} A^{w_1(A) \land w_2(A)}.
\]
The cautious conjunctive rule
Definition

**Theorem**

Let \( m_1 \) and \( m_2 \) be two nondogmatic BBAs. The \( w \)-least committed element in \( S_w(m_1) \cap S_w(m_2) \) exists and is unique. It is defined by the following weight function:

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\]
### Cautious rule computation

<table>
<thead>
<tr>
<th>m-space</th>
<th>w-space</th>
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<tbody>
<tr>
<td>$m_1$</td>
<td>$w_1$</td>
</tr>
<tr>
<td>$m_2$</td>
<td>$w_2$</td>
</tr>
<tr>
<td>$m_1 \land m_2$</td>
<td>$w_1 \land w_2$</td>
</tr>
</tbody>
</table>
The cautious conjunctive rule

Properties

Commutativity: \( \forall m_1, m_2, \ m_1 \bigotimes m_2 = m_2 \bigotimes m_1 \)

Associativity: \( \forall m_1, m_2, m_3, \)

\[
m_1 \bigotimes (m_2 \bigotimes m_3) = (m_1 \bigotimes m_2) \bigotimes m_3
\]

No neutral element: \( m_\Omega \bigotimes m = m \) iff \( m \) is separable.

Monotonicity:

\[
m_1 \sqsubseteq_w m_2 \Rightarrow m_1 \bigotimes m_3 \sqsubseteq_w m_2 \bigotimes m_3, \quad \forall m_1, m_2, m_3.
\]

\( (M_{nd}, \bigotimes, \sqsubseteq_w) \) is a partially ordered commutative semigroup.
The cautious conjunctive rule

Properties related to the combination of non distinct evidence

Idempotence: $\forall m, \ m \otimes m = m$

Distributivity $\cap$ with respect to $\otimes$:

$$(m_1 \cap m_2) \otimes (m_1 \cap m_3) = m_1 \cap (m_2 \otimes m_3), \ \forall m_1, m_2, m_3.$$ 

→ Item of evidence $m_1$ is not counted twice!
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   - Uninorm-based rules
   - Applications
The agent receives two BBAs $m_1$ and $m_2$ from two sources, at least one of which is considered to be reliable.

The resulting BBA should be less committed than $m_1$ and $m_2$.

Formally, $m_{12} \in G_x(m_1) \cap G_x(m_2)$, for some $x \in \{w, d, s, pl, q\}$, with $G_x(m) =$ set of BBAs less committed than $m$ according to $\sqsubseteq_x$.

Most commitment principle: we should choose in $G_x(m_1) \cap G_x(m_2)$ the most committed BBA according to $\sqsubseteq_x$ (if it exists).
Bold disjunctive combination of belief functions

Search for a suitable informational oredring

- With $x = w$, this approach leads to a mass function $m_{12}$ defined by $w_{12} = w_1 \lor w_2$.
- OK with separable BBAs, but $w_1 \lor w_2$ does not always correspond to a belief function.
- We need yet another ordering relation...
Let $m$ be a subnormal BBA. Its complement $\overline{m}$ is nondogmatic and can be decomposed as

$$\overline{m} = \bigcap_{A \subset \Omega} A^{\overline{w}(A)}.$$

Consequently, $m$ can be written

$$m = \bigcap_{A \subset \Omega} A^{\overline{w}(A)} = \bigcup_{A \subset \Omega} A^{\overline{w}(A)}.$$

Each BBA $A^{\overline{w}(A)}$ is the complement of a generalized simple BBA. Its focal sets are $\overline{A}$ and $\emptyset$. Notation: $\overline{A}_{\nu(\overline{A})}$, with $\nu(\overline{A}) = \overline{w}(A)$. 

---

T. Denœux

Information Fusion using Belief Functions: New Rules 50/82
Theorem

Any subnormal BBA $m$ can be uniquely decomposed as the $\bigcup$-combination of generalized BBAs $A_{v(A)}$ assigning a mass $v(A) > 0$ to $\emptyset$, and a mass $1 - v(A)$ to $A$, for all $A \subseteq \Omega$, $A \neq \emptyset$:

$$m = \bigcup_{A \neq \emptyset} A_{v(A)} \cdot$$

(1)

Definition (Disjunctive weight function)

Function $v : 2^\Omega \setminus \{\emptyset\} \rightarrow (0, +\infty)$ will be referred to as the disjunctive weight function.
Disjunctive weight function

Properties

- **Duality with** $w$: $v(A) = \overline{w(\overline{A})}$, $\forall A \neq \emptyset$ (similar to $b(A) = \overline{q(\overline{A})}$).

- **Computation from** $b$:

$$\ln v(A) = - \sum_{B \subseteq A} (-1)^{|A|-|B|} \ln b(B).$$

- **Similarity with**

$$m(A) = \sum_{B \subseteq A} (-1)^{|A|-|B|} b(B), \quad \forall A \subseteq \Omega.$$

- **TBM disjunctive rule**:

$$v_1 \cup_2 = v_1 \cdot v_2.$$
The $\nu$-ordering
Definition and properties

**Definition (\nu-ordering)**

Let $m_1$ and $m_2$ be two subnormal BBAs. $m_1 \sqsubseteq_\nu m_2$ iff $\nu_1(A) \geq \nu_2(A)$, for all $A \neq \emptyset$.

- Interpretation: $m_2 = m \ominus m_1$ for some BBA $m$ such that $\overline{m}$ is separable.
- $m_1 \sqsubseteq_\nu m_2 \Rightarrow m_1 \sqsubseteq_s m_2$.
- No smallest element, but $m_{\emptyset}$ is the unique minimal element: $m \sqsubseteq_\nu m_{\emptyset} \Rightarrow m = m_{\emptyset}$.
- Monotonicity of $\ominus$:
  $m_1 \sqsubseteq_\nu m_2 \Rightarrow m_1 \ominus m_3 \sqsubseteq_\nu m_2 \ominus m_3$, $\forall m_1, m_2, m_3$. 

T. Denœux

Information Fusion using Belief Functions: New Rules
The \( \nu \)-ordering
Definition and properties

**Definition (\( \nu \)-ordering)**

Let \( m_1 \) and \( m_2 \) be two subnormal BBAs. \( m_1 \sqsubseteq \nu m_2 \) iff \( \nu_1(A) \geq \nu_2(A) \), for all \( A \neq \emptyset \).

- Interpretation: \( m_2 = m \bigcup m_1 \) for some BBA \( m \) such that \( \overline{m} \) is separable.
- \( m_1 \sqsubseteq \nu m_2 \Rightarrow m_1 \sqsubseteq_s m_2 \).
- No smallest element, but \( m_\emptyset \) is the unique minimal element: \( m \sqsubseteq \nu m_\emptyset \Rightarrow m = m_\emptyset \).
- Monotonicity of \( \bigcup \): \( m_1 \sqsubseteq \nu m_2 \Rightarrow m_1 \bigcup m_3 \sqsubseteq \nu m_2 \bigcup m_3, \quad \forall m_1, m_2, m_3 \)
The bold disjunctive rule

Definition

**Theorem**

Let $m_1$ and $m_2$ be two subnormal BBAs. The $v$-most committed element in $G_v(m_1) \cap G_v(m_2)$ exists and is unique. It is defined by the following disjunctive weight function:

$$v_1 \bigoplus v_2 (A) = v_1 (A) \land v_2 (A), \quad \forall A \in 2^\Omega \setminus \emptyset.$$  

**Definition (Bold disjunctive rule)**

$$m_1 \bigoplus m_2 = \bigcup_{A \neq \emptyset} A v_1 (A) \land v_2 (A).$$
### The bold disjunctive rule

**Computation**

<table>
<thead>
<tr>
<th>m-space</th>
<th>v-space</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m_1$</td>
<td>$v_1$</td>
</tr>
<tr>
<td>$m_2$</td>
<td>$v_2$</td>
</tr>
<tr>
<td>$m_1 \lor m_2$</td>
<td>$v_1 \land v_2$</td>
</tr>
</tbody>
</table>
The bold disjunctive rule

Properties

Commutativity: \( \forall m_1, m_2, m_1 \lor m_2 = m_2 \lor m_1 \)

Associativity: \( \forall m_1, m_2, m_3, m_1 \lor (m_2 \lor m_3) = (m_1 \lor m_2) \lor m_3 \)

No neutral element: \( m_{\emptyset} \lor m = m \) iff \( m \) is separable.

Monotonicity:

\[
    m_1 \subseteq_v m_2 \Rightarrow m_1 \lor m_3 \subseteq_v m_2 \lor m_3, \quad \forall m_1, m_2, m_3.
\]

\( \rightarrow (\mathcal{M}_s, \lor, \subseteq_v) \) is a partially ordered commutative semigroup.
The bold disjunctive rule

Properties (continued)

Idempotence:  \( \forall m, m \lor m = m; \)

Distributivity of \( \cup \) with respect to \( \lor \):

\[
( m_1 \cup m_2 ) \lor ( m_1 \cup m_3 ) = m_1 \cup ( m_2 \lor m_3 ), \quad \forall m_1, m_2, m_3.
\]

\( \rightarrow \) Item of evidence \( m_1 \) is not counted twice.

De Morgan laws:

\[
\begin{align*}
\overline{m_1 \lor m_2} & = \overline{m_1} \land \overline{m_2} \\
\overline{m_1 \land m_2} & = \overline{m_1} \lor \overline{m_2}
\end{align*}
\]
Generalizing the cautious and bold rules

<table>
<thead>
<tr>
<th>conjunctive weights $w$</th>
<th>product $\cap$</th>
<th>minimum $\wedge$</th>
<th>$\ast$</th>
</tr>
</thead>
<tbody>
<tr>
<td>disjunctive weights $v$</td>
<td>$\cup$</td>
<td>$\vee$</td>
<td>$\ast$</td>
</tr>
</tbody>
</table>

- Properties of the minimum and the product on $\mathbb{R}^+ = (0, +\infty]$:
  - Commutativity, associativity;
  - Monotonicity: $x \leq y \Rightarrow x \wedge z \leq y \wedge z$, $\forall x, y, z \in (0, +\infty]$.
- Neutral element:
  - $+\infty$ for the minimum $\rightarrow$ t-norm;
  - 1 for the product $\rightarrow$ uninorm.
- Generalization to other t-norms and uninorms?
Overview

1. Theory of belief functions
   - Motivations
   - Basic concepts
   - Canonical conjunctive decomposition

2. The cautious and bold rules
   - Informational orderings and the LCP
   - The cautious conjunctive rule
   - The bold disjunctive rule

3. Families of combination rules
   - T-norm-based rules
   - Uninorm-based rules
   - Applications
Proposition

Let $*$ be a positive $t$-norm on $(0, +\infty]$. Then, for any conjunctive weight functions $w_1$ and $w_2$, the function $w_1*2$ defined by:

$$w_1*2(A) = w_1(A) * w_2(A), \forall A \subset \Omega,$$

is a conjunctive weight function associated to some nondogmatic BBA $m_1*2$.

Definition (T-norm-based conjunctive rule)

$$m_1 \ast_w m_2 = \bigcap_{A \subset \Omega} A^{w_1(A) * w_2(A)}.$$
Let $\mathcal{M}_{nd}$ be the set of nondogmatic BBAs, and $\otimes_w$ the conjunctive rule based on t-norm $\ast$. Then $(\mathcal{M}_{nd}, \otimes_w, \sqsubseteq_w)$ is a commutative, partially ordered semigroup.

The minimum is the largest t-norm on $(0, +\infty]$. Consequently:

**Proposition**

*Among all t-norm based conjunctive operators, the cautious rule is the $w$-least committed:*

\[ m_1 \otimes_w m_2 \sqsubseteq_w m_1 \bigwedge m_2, \quad \forall m_1, m_2. \]
T-norm based disjunctive rules
Definition and properties

- Let \( \ast \) be a t-norm on \((0, +\infty] \). The disjunctive rule associated to \( \ast \) is

\[
m_1 \circledast_v m_2 = \bigcup_{\emptyset \neq A \subseteq \Omega} A_{v_1}(A) \ast_{v_2}(A).
\]

- \((\mathcal{M}_s, \circledast_v, \sqsubseteq_v)\) is a commutative, partially ordered semigroup.

- Among all t-norm based disjunctive operators, the bold rule is the \( v \)-most committed.

- De Morgan laws:

\[
\overline{m_1 \circledast_w m_2} = \overline{m_1} \circledast_v \overline{m_2}
\]

\[
\overline{m_1 \circledast_v m_2} = \overline{m_1} \circledast_w \overline{m_2}
\]
Proposition

Let $\top$ be a positive t-norm on $[0, 1]$, and let $\bot$ be a t-conorm on $[0, 1]$. Then the operator $\ast_{\top, \bot}$ defined by

$$
x \ast_{\top, \bot} y = \begin{cases} 
x \top y & \text{if } x \lor y \leq 1, \\
\left(\frac{1}{x} \bot \frac{1}{y}\right)^{-1} & \text{if } x \land y > 1, \\
x \land y & \text{otherwise,}
\end{cases}
$$

for all $x, y \in (0, +\infty]$ is a t-norm on $(0, +\infty]$.

→ For each pair $(\top, \bot)$, there is a pair of dual conjunctive and disjunctive rules generalizing the cautious and bold rules, respectively.
Overview

1. Theory of belief functions
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   - Uninorm-based rules
   - Applications
Uninorm-based conjunctive rules

**Definition**

Let $\circ$ be a uninorm on $(0, +\infty]$ with 1 as neutral element, such that $x \circ y \leq xy$ for all $x, y \in (0, +\infty]$. Then, for any $w$ functions $w_1$ and $w_2$, the function $w_{1\circ2}$ defined by:

$$w_{1\circ2}(A) = w_1(A) \circ w_2(A), \forall A \subset \Omega,$$

is a $w$ function associated to some nondogmatic BBA $m_{1\circ2}$.

**Proposition**

Let $\circ$ be a uninorm on $(0, +\infty]$ with 1 as neutral element, such that $x \circ y \leq xy$ for all $x, y \in (0, +\infty]$. Then, for any $w$ functions $w_1$ and $w_2$, the function $w_{1\circ2}$ defined by:

$$w_{1\circ2}(A) = w_1(A) \circ w_2(A), \forall A \subset \Omega,$$

is a $w$ function associated to some nondogmatic BBA $m_{1\circ2}$.

**Definition (Uninorm-based conjunctive rule)**

Let $\circ$ be a uninorm on $(0, +\infty]$ verifying the above condition.

$$m_1 \odot_w m_2 = \bigcap_{A \subset \Omega} A^{w_1(A) \circ w_2(A)}.$$
Proposition

Let $\mathcal{M}_{nd}$ be the set of nondogmatic BBAs, and $\circ_w$ the conjunctive rule based on uninorm $\circ$ with one as neutral element, and verifying $x \circ y \leq xy$ for all $x, y \in (0, +\infty]$. Then $(\mathcal{M}_{nd}, \circ_w, \sqsubseteq_w)$ is a commutative, partially ordered monoid, with the vacuous BBA as neutral element.

**Question:** Can we relax the condition $x \circ y \leq xy$ for all $x, y \in (0, +\infty]$, and get an operator $\circ_w$ that is not more committed than $\cap$?
Theorem (Pichon and Denœux, 2007)

Let $\circ$ be a binary operator on $(0, +\infty]$ such that

1. $x \circ 1 = 1 \circ x = x$ for all $x$ and
2. $x \circ y > xy$ for some $x, y > 0$.

Then, there exists two BBAs $m_1$ and $m_2$ such that $w_1 \circ w_2$ is not a valid $w$ function.

Corollary

Consequence: among all uninorm-norm based conjunctive operators, the TBM conjunctive rule is the $w$-least committed:

$$m_1 \bigcirc_w m_2 \preceq_w m_1 \bigcap m_2, \quad \forall m_1, m_2, \forall \bigcirc_w.$$
Uninorm-based disjunctive rules
Definition and properties

- Let $\circ$ be a uninorm on $(0, +\infty]$ with 1 as neutral element, such that $x \circ y \leq xy$ for all $x, y \in (0, +\infty]$. The disjunctive rule associated to $\circ$ is defined as:

$$m_1 \circ_v m_2 = \bigcup_{A \subseteq \Omega} A_{v_1}(A) \circ v_2(A).$$

- $(\mathcal{M}_s, \circ_v, \sqsubseteq_v)$ is a commutative, partially ordered monoid, with $m_{\emptyset}$ as neutral element.

- Among all uninorm-norm based disjunctive operators, the TBM disjunctive rule is the $v$-most committed.

- De Morgan laws:

$$\overline{m_1 \circ_w m_2} = \overline{m_1} \circ_v \overline{m_2}$$
$$\overline{m_1 \circ_v m_2} = \overline{m_1} \circ_w \overline{m_2}$$
Construction of uninorms on $[0, +\infty]$ 

**Proposition**

Let $\top$ be a positive t-norm on $[0, 1]$ verifying $x \top y \leq xy$ for all $x, y \in [0, 1]$, and let $\top'$ be a t-norm on $[0, 1]$ verifying $x \top y \geq xy$ for all $x, y \in [0, 1]$. Then the operator defined by

$$x \circ_{\top, \top'} y = \begin{cases} 
    x \top y & \text{if } x \lor y \leq 1, \\
    \left(\frac{1}{x} \top' \frac{1}{y}\right)^{-1} & \text{if } x \land y \geq 1, \\
    x \land y & \text{otherwise,}
\end{cases}$$

for all $x, y \in (0, +\infty]$ is a uninorm on $(0, +\infty]$ verifying $x \circ_{\top, \top'} y \leq xy$ for all $x, y > 0$.

→ For each pair $(\top, \top')$, there is a pair of dual conjunctive and disjunctive uninorm-based rules.
Let $\top$ and $\top'$ be t-norms on $[0, 1]$, and $\bot$ be a t-conorm on $[0, 1]$.

One can build:

- a t-norm $\ast_{\top, \bot}$ on $(0, +\infty]$;
- a uninorm $\odot_{\top, \top'}$ on $(0, +\infty]$.

The corresponding t-norm and uninorm based conjunctive rules $\ast_w$ and $\odot_w$ coincide on separable BBAs.

Consequence: to define a rule for combining separable BBAs, one only needs to define a t-norm $\top$. 

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Coincidence for separable BBAs
Summary

- We now have four infinite families of rules:
  - conjunctive and disjunctive t-norm-based rules;
  - conjunctive and disjunctive uninorm-based rules.
- In each of these families, one rule plays a special role and is well justified by the LCP:
  - the $\wedge$ and $\cap$ rules are the w-least-committed conjunctive rules in the t-norm-based and uninorm-based families, respectively;
  - the $\lor$ and $\cup$ rules are the v-most committed disjunctive rules in the t-norm-based and uninorm-based families, respectively.
- The justification of the other rules is less clear but...
- Can they be useful in practice?
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   - Applications
Application to classification

The problem

Let us consider a classification problem where objects are described by feature vectors $\mathbf{x} \in \mathbb{R}^p$ and belong to one of $K$ groups in $\Omega = \{\omega_1, \ldots, \omega_K\}$.

Learning set $\mathcal{L} = \{(\mathbf{x}_1, z_1), \ldots, (\mathbf{x}_n, z_n)\}$, where $z_i \in \Omega$ denotes the class of object $i$.

Problem: predict the class of a new object described by feature vector $\mathbf{x}$.

Application of new combination rules to:

- combine neighborhood information in the evidential $k$ nearest neighbor rule;
- combine outputs from classifiers built from different features.
Example 1: evidential $k$-NN rule

**Principle**

- The evidence of example $i$ is represented by a simple BBA $m_i$ on $\Omega$ defined by
  \[ m_i = \{ z_i \} \varphi(d_i) \]
  where $d_i$ is the distance between $x$ and $x_i$, and $\varphi$ is an increasing function from $\mathbb{R}^+$ to $[0, 1]$.
- The evidence of the $k$ nearest neighbors of $x$ in $\mathcal{L}$ is pooled using the TBM conjunctive rule:
  \[ m = \bigcap_{i \in N_k(x)} \{ z_i \} \varphi(d_i). \]
- Generalization: replace $\bigcap$ by another conjunctive operator $\ast_w$ defined by a t-norm taken in a parameterized family ranging from the product to the minimum (e.g. Dubois-Prade, Frank).
Results
Heart disease and USPS datasets

Heart disease data set

USPS data set

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Results
Ionosphere and Letter recognition datasets

![Graph of Ionosphere data set](image)

- **Ionosphere data set**
  - Cautious rule
  - TBM conj. rule

![Graph of Letter recognition data set](image)

- **Letter recognition data set**
  - Cautious rule
  - TBM conj. rule
Example 2: classifier fusion

Principle

- One separate classifier for each feature $x_j$.
- Classifier using input feature $x_j$ produces a BBA $m_j$.
- Method:
  - logistic regression;
  - posterior probabilities transformed into consonant BBAs using the isopignistic transformation.
- Classifier outputs combined using t-norm based conjunctive operators.
- T-norm on $[0, 1]$ taken in Frank’s family.
Results

Segment data

Cross-validation error rate vs. s

- Cautious rule
- TBM conj. rule

Waveform data

Cross-validation error rate vs. s

- Cautious rule
- TBM conj. rule
Summary
Four basic rules

- Two new dual commutative, associative et idempotent rules:
  - cautious conjunctive rule $w_1 \ominus_2 = w_1 \land w_2$;
  - bold disjunctive rule $v_1 \oslash_2 = v_1 \land v_2$.

- Both rules are derived from the Least commitment principle, for some (different) informational ordering relations.

- With the TBM conjunctive and disjunctive rules, we now have four basic rules:

<table>
<thead>
<tr>
<th>sources</th>
<th>all reliable</th>
<th>at least one reliable</th>
</tr>
</thead>
<tbody>
<tr>
<td>distinct</td>
<td>$\cap$</td>
<td>$\cup$</td>
</tr>
<tr>
<td>non distinct</td>
<td>$\wedge$</td>
<td>$\lor$</td>
</tr>
</tbody>
</table>
The $\lor$ and $\cap$ rules have fundamentally different algebraic properties:

- The $\lor$ rule is based on a t-norm on $(0, +\infty]$ and has no neutral element;
- The $\cap$ rule is based on a uninorm on $(0, +\infty]$ and has a neutral element (the vacuous BBA).

Similarly, the $\lor$ and $\lor$ rules are based, respectively, on a t-norm and a uninorm; $\lor$ has a neutral element, whereas $\lor$ has not.

The pairs $\lor$-$\lor$ and $\lor$-$\lor$ are dual to each other and are related by De Morgan laws.
Summary

T-norm and uninorm-based rules

- To each of the four basic rules corresponds one infinite family of combination rules:
  - the t-norm-based conjunctive and disjunctive families;
  - the uninorm-based conjunctive and disjunctive families.
  → at least as much flexibility and diversity as in Possibility theory!
- Each of the four basic rules occupies a special position in its family:
  - The $\land$ and $\cap$ rules are the least committed elements;
  - The $\lor$ and $\cup$ rules are the most committed elements.
- Preliminary experiments suggest that the use of general t-norm and uninorm-based rules may improve the performances of information fusion systems.
References

Ph. Smets.

T. Denœux.
Conjunctive and Disjunctive Combination of Belief Functions Induced by Non Distinct Bodies of Evidence. *Artificial Intelligence (In press)*, 2007.