Random fuzzy sets for uncertainty propagation

Combining variability and imprecision in numerical models

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The uncertainty propagation problem

Numerical model

A deterministic model maps uncertain inputs to quantities of interest:

$$Y = g(X_1, \ldots, X_n) = g(\mathbf{X}).$$

- The model g is assumed to be fixed and sufficiently faithful.
- Uncertainty is attached to inputs, parameters or boundary conditions.
- The goal is to quantify the induced uncertainty on Y.

$$\boxed{\text{uncertain inputs}} \quad \xrightarrow{g} \quad \boxed{\text{uncertain output}}$$

Why probability may not be enough

- Probability distributions are natural for representing random variability.
- But many input quantities are either unknown constants, or not observed often enough to specify a precise distribution.
- Expert assessments, tolerances and sparse data often provide imprecise information.
- A single probability distribution may then give a false impression of precision.

Typical situation

Some inputs can be represented by random variables, while others are only known through intervals, fuzzy constraints or partial probabilistic information.

Aleatory vs. epistemic uncertainty

Aleatory uncertainty

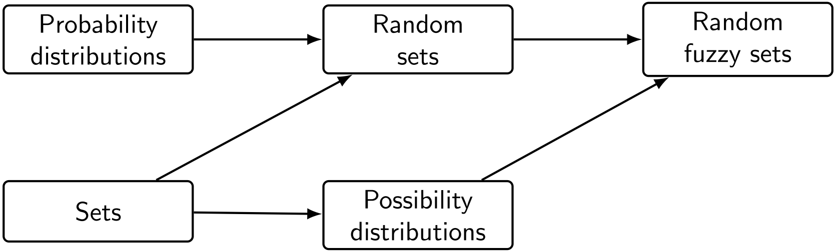
- Inherent variability
- Cannot be reduced
- Usually represented by probability

Epistemic uncertainty

- Lack of knowledge
- Can be reduced by information
- Often imprecise or incomplete

Random fuzzy sets are designed to represent both components simultaneously.

A hierarchy of uncertainty models



- Random sets extend probability and sets by allowing set-valued realisations.
- Possibility distributions encode graded compatibility.
- Random fuzzy sets combine both ideas.

Outline

1. Uncertainty representation frameworks
   - From mass functions to random sets
   - From random sets to random fuzzy sets
   - Gaussian random fuzzy models
2. Propagation of random fuzzy sets
   - Principles
   - Algorithms
   - Example: short-column design

Outline

1. Uncertainty representation frameworks
   - From mass functions to random sets
   - From random sets to random fuzzy sets
   - Gaussian random fuzzy models
2. Propagation of random fuzzy sets

Classical Dempster–Shafer theory

- Let X be a quantity with unknown value in a finite frame Θ.
- Evidence about X can be represented by a mass function assigning weights to subsets of Θ:

$$m(A) \geq 0, \qquad \sum_{A \subseteq \Theta} m(A) = 1.$$

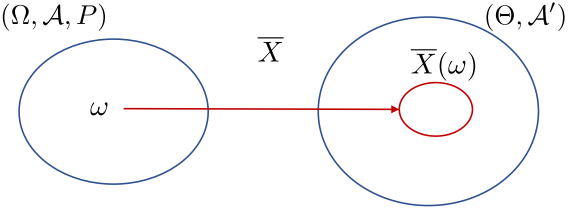
- A focal set A means: the evidence only says that the true value lies in A.
- Belief and plausibility are lower and upper probabilities:

$$\mathrm{Bel}(B) = \sum_{A \subseteq B} m(A), \qquad \mathrm{Pl}(B) = \sum_{A \cap B \neq \emptyset} m(A).$$

- Interpretation:
  - $\mathrm{Bel}(B)$ measures to what extent B is supported by the evidence;
  - $\mathrm{Bel}(B)$ measures to what extent B is compatible with the evidence.

From finite frames to infinite spaces

Many unknowns take values in $\mathbb{R}^p$, not in a finite set.



- The mass function is replaced by the more general concept of random subset of Θ.
- A random set is defined by a probability space $(\Omega, \mathcal{A}, P)$ and a set-valued mapping

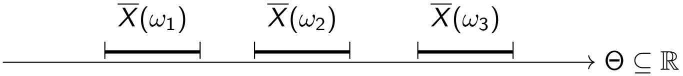
$$\overline{X} : \Omega \to \mathcal{P}(\Theta).$$

Interpretation

- Here, Ω is a set of interpretations of the evidence.
- For each elementary interpretation ω, we observe a set-valued constraint:

$$X \in \overline{X}(\omega).$$

Random set: intuition



- Each realisation is a crisp set, called a focal set.
- If focal sets are intervals, the random set is called a random interval.
- This is the continuous-space analogue of a finite mass function.

Belief and plausibility for a random set

For $B \subseteq \Theta$,

$$\text{Bel}(B) = P\{\overline{X} \subseteq B\}, \qquad \text{Pl}(B) = P\{\overline{X} \cap B \neq \emptyset\} = 1 - \text{Bel}(B^c).$$

Belief

Probability that the evidence supports B.

Plausibility

Probability that the evidence is compatible with B (does not support B^c).

Properties:

- $\text{Bel}(\varnothing) = \text{Pl}(\varnothing) = 0$
- $\text{Bel}(\Theta) = \text{Pl}(\Theta) = 1$
- $\forall B \subseteq \Theta$, $\text{Bel}(B) \leq \text{Pl}(B)$.

Random sets as evidence on infinite frames

Key equivalence

A random set is a probability distribution over crisp focal sets.

- This extends the theory of belief functions from finite frames to general spaces.
- But it still assumes that each focal element is crisp, i.e., the evidence imposes hard constraints on the unknown quantity.

What if the evidence is itself vague or graded, such as, e.g.,
"X is close to 1.5", or "X is small"?

From crisp focal sets to fuzzy focal sets

Random set

$$\overline{X}(\omega) \subseteq \Theta$$

A crisp constraint:

$$X \in \overline{X}(\omega).$$

Random fuzzy set

$$\widetilde{X}(\omega) : \Theta \rightarrow [0, 1]$$

A soft (graded, flexible) constraint:

$$X = x \text{ is possible to degree } \widetilde{X}(\omega)(x).$$

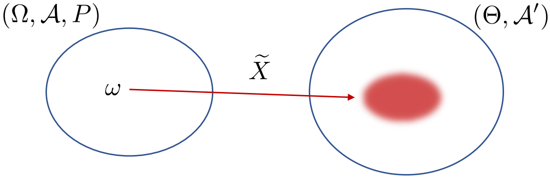
Possibility and necessity generated by a fuzzy set

- A fuzzy subset of Θ is a map $\pi : \Theta \rightarrow [0,1]$. It is normal if $\sup_{\theta\in\Theta} \pi(\theta) = 1$.
- A normal fuzzy set representing a soft constraint on an unknown X is called a possibility distribution.
- A possibility distribution induces possibility and necessity measures:

$$\Pi_\pi(B) = \sup_{x\in B} \pi(x), \qquad N_\pi(B) = 1 - \Pi_\pi(B^c).$$

  - $\Pi_\pi(B)$ measures the extent to which B is compatible with the constraint.
  - $N_\pi(B)$ measures the extent to which B is implied by the constraint.
- A fuzzy focal set $\widetilde{X}(\omega)$ therefore induces possibility and necessity measures, conditional on ω.

Definition of a random fuzzy set



Random fuzzy set

A random fuzzy set is defined by a probability space $(\Omega, \mathcal{A}, P)$ and a fuzzy-valued mapping

$$\widetilde{X} : \Omega \to \mathcal{F}(\Theta),$$

where $\widetilde{X}(\omega)$ is a fuzzy subset of Θ.

- Random sets are recovered when all fuzzy focal sets are crisp.
- Random variables are recovered when all fuzzy focal sets are points (singletons).
- Possibility distributions are recovered when there is only one non-random focal fuzzy set.

Belief and plausibility induced by a RFS

- For any $\omega \in \Omega$, $\widetilde{X}(\omega)$ is a possibility distribution for X.
- For any $B \subseteq \Theta$, under measurability conditions, $\Pi_{\widetilde{X}(\omega)}(B)$ and $\mathrm{N}_{\widetilde{X}(\omega)}(B)$ are random variables.
- We define:

$$\mathrm{Pl}_{\widetilde{X}}(B) = \mathbb{E}\left[\Pi_{\widetilde{X}}(B)\right],$$

$$\mathrm{Bel}_{\widetilde{X}}(B) = \mathbb{E}\left[\mathrm{N}_{\widetilde{X}}(B)\right] = 1 - \mathrm{Pl}_{\widetilde{X}}(B^c).$$

Interpretation

- Plausibility is the average possibility of B over all uncertain interpretations.
- Belief is the average necessity of B.

Contour function

The plausibility of a singleton gives the contour function:

$$pl_{\widetilde{X}}(x) = Pl_{\widetilde{X}}(\{x\}) = \mathbb{E}\left[\widetilde{X}(x)\right].$$

- It is the average membership degree of x over the random fuzzy focal sets.
- It generalises both probability density-like information and possibility distributions.
- It is often easier to visualise than the full random fuzzy set.

One object, several limiting cases

Model	RFS representation	Main meaning
Random variable	random singleton	precise randomness
Random set	random crisp set	imprecise evidence
Possibility distribution	fixed fuzzy set	vague evidence
Random fuzzy set	random fuzzy set	variability + imprecision + vagueness

Unifying view

RFSs extend Dempster–Shafer theory in the same way that fuzzy sets extend crisp sets: focal sets become graded.

Computation by α-cuts

α-cut

For a fuzzy focal set $\widetilde{X}(\omega)$ and $\alpha \in [0,1]$, define the α-cut of $\widetilde{X}(\omega)$ as

$$^{\alpha}\widetilde{X}(\omega) = \{x : \widetilde{X}(\omega)(x) \geq \alpha\}.$$

This is an ordinary (crisp) set.

- Each map

$$^{\alpha}\widetilde{X} : \Omega \to \mathcal{P}(\Theta)$$

  is a crisp random set.
- A RFS can be studied through the nested family of its α-cuts.
- This provides a bridge between random-set algorithms and random fuzzy sets.

Combination of random fuzzy sets

- Let $\widetilde{X}_1$ and $\widetilde{X}_2$ be two independent random fuzzy sets on Θ.
- Their combination $\widetilde{X}_1 \oplus \widetilde{X}_2$ is obtained by intersecting their fuzzy focal sets:

$$\widetilde{X}_1 \oplus \widetilde{X}_2 : (\omega_1, \omega_2) \mapsto \widetilde{X}_1(\omega_1) \cap \widetilde{X}_2(\omega_2).$$

- Intersection is defined as the normalised product:

$$(\widetilde{X}_1(\omega_1) \cap \widetilde{X}_2(\omega_2))(x) = \frac{\widetilde{X}_1(\omega_1)(x)\, \widetilde{X}_2(\omega_2)(x)}{\sup_{u \in \Theta} \widetilde{X}_1(\omega_1)(u)\, \widetilde{X}_2(\omega_2)(u)}.$$

- The product probability measure $P_1 \otimes P_2$ is updated to account for the conflict.

Main idea

$\oplus$ combines two independent random fuzzy sets by intersecting their focal fuzzy sets, and renormalising.

Marginalisation of random fuzzy sets

- Let $\widetilde{Z}$ be a random fuzzy set on a product space:

$$\widetilde{Z} : \Omega \to \mathcal{F}(\Theta_{\mathbf{X}} \times \Theta_{\mathbf{Y}}).$$

- Its marginal on $\Theta_{\mathbf{Y}}$ is the random fuzzy set

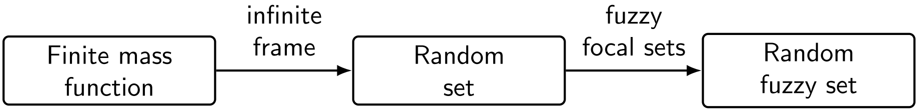
$$\widetilde{Z}^{\downarrow \Theta_{\mathbf{Y}}} : \omega \mapsto \mathrm{proj}_{\mathbf{Y}}\, \widetilde{Z}(\omega),$$

  where the projection operation is defined as

$$(\mathrm{proj}_{\mathbf{Y}}\, \widetilde{Z}(\omega))(\mathbf{y}) = \sup_{\mathbf{x} \in \Theta_{\mathbf{X}}} \widetilde{Z}(\omega)(\mathbf{x}, \mathbf{y}).$$

- Thus, marginalisation eliminates variables from each fuzzy focal set.

Conceptual summary



- Random set: probability distribution over sets.
- Random fuzzy set: probability distribution over possibility distributions.
- Belief and plausibility remain lower and upper probabilities.

Need for practical continuous models

- General RFSs are very flexible but difficult to specify and propagate.
- For numerical uncertainty propagation, we need parametric models.
- Gaussian random fuzzy numbers and vectors provide such a tractable family.
- They are indexed by three parameters representing
  - location,
  - dispersion,
  - imprecision.

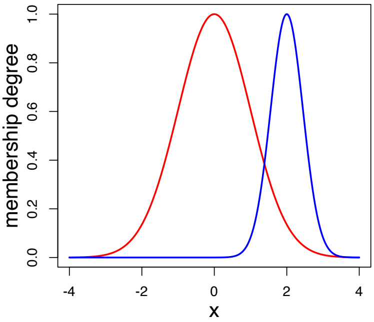
Gaussian fuzzy number

Definition

A Gaussian fuzzy number (GFN) is defined by

$$\widetilde{x}(x) = \exp\left(-\frac{h}{2}(x-m)^2\right),$$

where $m \in \mathbb{R}$ is the mode and $h \geq 0$ is a precision parameter.



We write

$$\boxed{\widetilde{x} = \mathrm{GFN}(m, h).}$$

Gaussian random fuzzy number

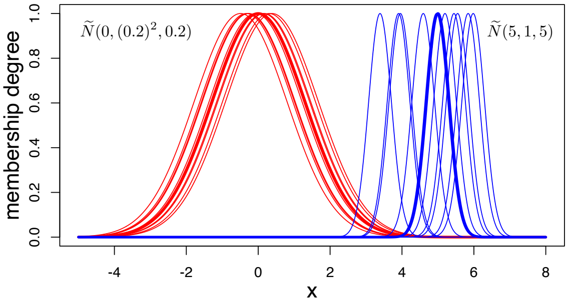
Definition

A Gaussian random fuzzy number (GRFN) is a GFN whose mode is a normal random variable

$$\widetilde{X}(M) = \text{GFN}(M, h),$$

with

$$M \sim \mathcal{N}(\mu, \sigma^2), \qquad h \geq 0.$$



We write

$$\boxed{\widetilde{X} \sim \widetilde{\mathcal{N}}(\mu, \sigma^2, h).}$$

Interpreting the parameters

- μ is the central value.
- σ^2 measures how much the modes varies across interpretations.
- h measures the specificity of each focal fuzzy number.
- Three limiting cases:

Condition	Interpretation
$\sigma^2 = 0$	Gaussian fuzzy number
$h \to \infty$	Gaussian random variable
$h = 0$	vacuous information

Message

A GRFN continuously interpolates between probability, possibility and ignorance.

Gaussian fuzzy vector

Definition

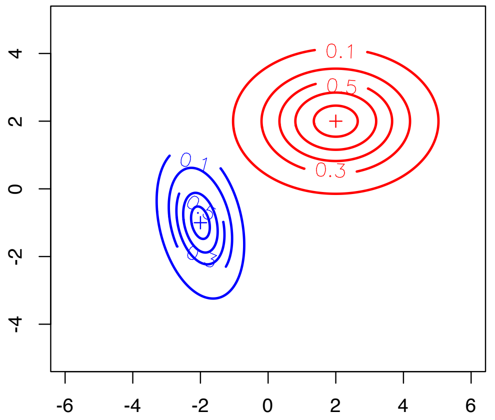
A Gaussian fuzzy vector (GFV) is defined by

$$\widetilde{\mathbf{x}}(\mathbf{x}) = \exp\left(-\frac{1}{2}(\mathbf{x}-\mathbf{m})^T\mathbf{H}(\mathbf{x}-\mathbf{m})\right),$$

where $\mathbf{m} \in \mathbb{R}^p$ is the mode and $\mathbf{H} \in \mathbb{R}^{p\times p}$ is a positive definite precision matrix.

We write

$$\boxed{\widetilde{\mathbf{x}} = \mathrm{GFV}(\mathbf{m}, \mathbf{H}).}$$



Gaussian random fuzzy vector

Definition

A Gaussian random fuzzy vector (GRFV) is a GFV whose mode is a normal random vector:

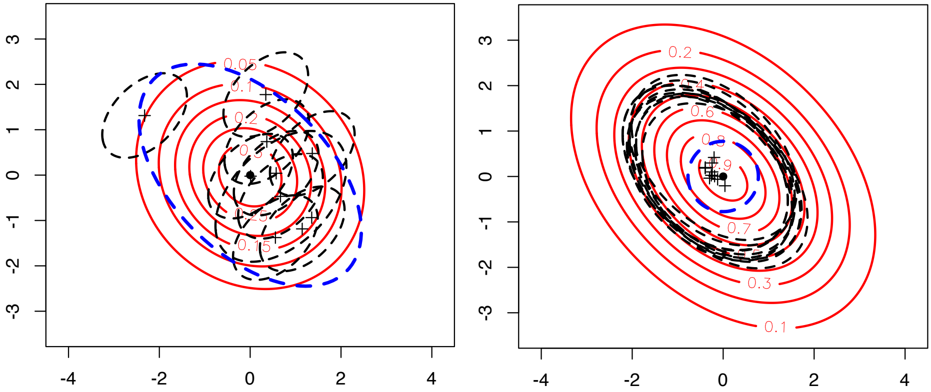
$$\widetilde{\mathbf{X}}(\mathbf{M}) = \mathrm{GFV}(\mathbf{M}, \mathbf{H}), \qquad \mathbf{M} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma}), \qquad \mathbf{H}, \boldsymbol{\Sigma} \succeq 0,$$

We write

$$\boxed{\widetilde{\mathbf{X}} \sim \widetilde{\mathcal{N}}(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \mathbf{H}).}$$

- $\boldsymbol{\Sigma}$ controls the dispersion of the random centres.
- $\mathbf{H}$ controls the shape and precision of each fuzzy focal ellipsoid.

Mental picture of a GRFV



For each GRFV, we show 0.5-level curves of ten focal sets (black broken curves), a contour plot of the contour function (red solid curves), as well as a 95% confidence ellipse for the mode $\mathbf{M} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ (blue broken curves).

Dependence and non-interactivity

- The covariance matrix $\boldsymbol{\Sigma}$ describes probabilistic dependence between random modes.
- The precision matrix $\mathbf{H}$ describes epistemic dependence within each focal fuzzy set.
- If both $\boldsymbol{\Sigma}$ and $\mathbf{H}$ are diagonal, then $\widetilde{\mathbf{X}}$ is noninteractive, i.e.,

$$\widetilde{\mathbf{X}} = \widetilde{X}_1 \oplus \ldots \oplus \widetilde{X}_n.$$

Practical point

The same model can encode both stochastic dependence and epistemic dependence.

Outline

General propagation principle

Let

$$\mathbf{Y} = g(\mathbf{X}).$$

The output random fuzzy set is obtained by combining the graph of g with the input RFS and marginalising:

$$\boxed{\widetilde{\mathbf{Y}} = (\mathrm{Gr}(g) \oplus \widetilde{\mathbf{X}})^{\downarrow \Theta_{\mathbf{Y}}}.}$$

Equivalent view

Each input fuzzy focal set is transformed through g into an output fuzzy focal set.

Propagation at the focal-set level

For a sampled mode **m**, the input focal set is

$$\widetilde{\mathbf{X}}(\mathbf{m})(\mathbf{x}) = \exp\left[-\frac{1}{2}(\mathbf{x}-\mathbf{m})^T\mathbf{H}(\mathbf{x}-\mathbf{m})\right].$$

The corresponding output focal set has membership

$$\widetilde{Y}(\mathbf{m})(\mathbf{y}) = \sup_{\mathbf{x}:g(\mathbf{x})=\mathbf{y}} \widetilde{\mathbf{X}}(\mathbf{m})(\mathbf{x}).$$

- This is the fuzzy extension principle applied inside each random realisation.
- The random modes generate a population of output fuzzy focal sets.

Affine propagation of a GRFV

Theorem

If the input is a GRFV

$$\widetilde{\mathbf{X}} \sim \widetilde{\mathcal{N}}(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \mathbf{H})$$

and g is affine

$$g(\mathbf{x}) = \mathbf{a} + \mathbf{A}\mathbf{x},$$

then the propagated RFS is again a GRFV:

$$\widetilde{\mathbf{Y}} \sim \widetilde{\mathcal{N}}\left(\mathbf{a} + \mathbf{A}\boldsymbol{\mu}, \mathbf{A}\boldsymbol{\Sigma}\mathbf{A}^T, (\mathbf{A}\mathbf{H}^{-1}\mathbf{A}^T)^{-1}\right).$$

Importance

This exact result is the basis of linearisation methods for nonlinear models.

The nonlinear challenge

For nonlinear g, the image of a Gaussian fuzzy focal set is generally not Gaussian.

- Exact propagation requires optimisation under the constraint $\mathbf{y} = g(\mathbf{x})$.
- Repeating this for many random modes can be expensive.
- Approximation methods are therefore necessary.

Method	Basic idea
Global linearisation	one tangent approximation
Local linearisation	one tangent approximation per focal set
Direct method	constrained optimisation
Sample-based method	propagate a cloud of points

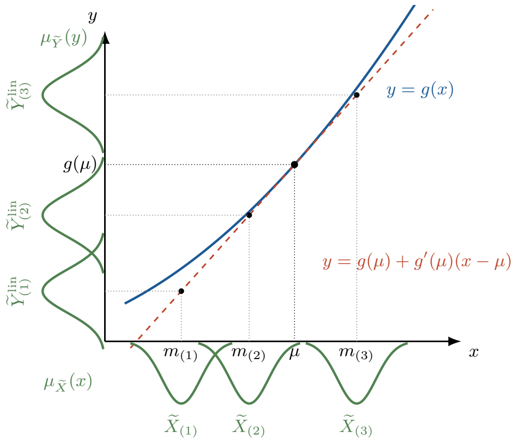
Global linearisation

For scalar output,

$$g(\mathbf{x}) \approx g(\boldsymbol{\mu}) + \nabla g(\boldsymbol{\mu})^T(\mathbf{x}-\boldsymbol{\mu}).$$

If $\widetilde{\mathbf{X}} \sim \widetilde{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \mathbf{H})$, then

$$\widetilde{Y} \overset{\text{app.}}{\sim} \widetilde{\mathcal{N}}\left(g(\boldsymbol{\mu}), \nabla g(\boldsymbol{\mu})^T \boldsymbol{\Sigma} \nabla g(\boldsymbol{\mu}), \left(\nabla g(\boldsymbol{\mu})^T \mathbf{H}^{-1} \nabla g(\boldsymbol{\mu})\right)^{-1}\right).$$

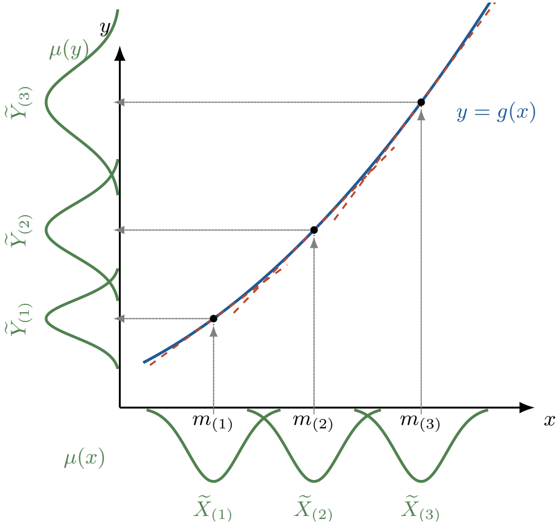


Pros and cons

Very fast, but can be misleading when g is nonlinear over the relevant input region.

Local linearisation

1. Sample modes
$$\mathbf{m}_{(1)}, \ldots, \mathbf{m}_{(N)} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma}).$$
2. For each sampled mode, linearise g at $\mathbf{m}_{(j)}$.
3. For scalar output, set
$$y^*_{(j)} = g(\mathbf{m}_{(j)}),$$
$$h^*_{(j)} = \left(\nabla g(\mathbf{m}_{(j)})^T \mathbf{H}^{-1} \nabla g(\mathbf{m}_{(j)})\right)^{-1}.$$
4. Approximate the output by fuzzy focal sets
$$\mathrm{GFN}(y^*_{(j)}, h^*_{(j)}), \qquad j = 1, \ldots, N.$$



Direct method

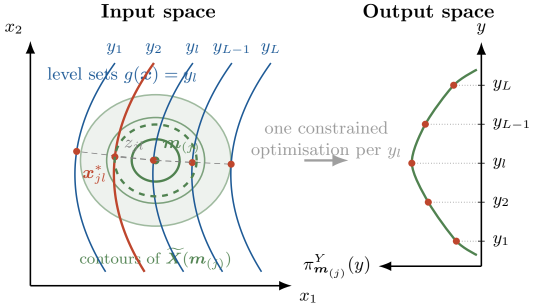
Choose a grid $y_1, \ldots, y_L$ on the output space. For each sampled mode $\mathbf{m}_{(j)}$ and each grid point y_l, compute

$$\widetilde{Y}(\mathbf{m}_{(j)})(y_l) = \sup_{\mathbf{x}: g(\mathbf{x})=y_l} \widetilde{\mathbf{X}}(\mathbf{m}_{(j)})(\mathbf{x})$$

by solving

$$\min_{\mathbf{x}} (\mathbf{x} - \mathbf{m}_{(j)})^T \mathbf{H} (\mathbf{x} - \mathbf{m}_{(j)})$$

subject to $g(\mathbf{x}) = y_l$. Then,



$$\widetilde{Y}(\mathbf{m}_{(j)})(y_l) = \exp\left(-\frac{1}{2} z_{jl}\right), \quad \text{where } z_{jl} \text{ is the optimum.}$$

Feature

Accurate in principle, but computationally expensive (NL constrained optimisation problems).

Sample-based method

1. Generate input points $\mathbf{x}_{(q)}$ in a domain $\mathcal{D}$.
2. Propagate them once:
$$y_{(q)} = g(\mathbf{x}_{(q)}), \qquad q = 1, \ldots, N'.$$
3. For each sampled focal-set mode $\mathbf{m}_{(j)}$, compute
$$\widetilde{\pi}^{Y}_{jq} = \widetilde{\mathbf{X}}(\mathbf{m}_{(j)})(\mathbf{x}_{(q)}).$$
4. Divide the output domain into bins $B_1, \ldots, B_L$ centred at $y_1, \ldots, y_L$ and compute
$$\max_{y_{(q)} \in B_l} \widetilde{\pi}^{Y}_{jq} \approx \widetilde{Y}(\mathbf{m}_{(j)})(y_l).$$

Use case

Useful when constrained optimisation is hard but model evaluations are cheap, and the input dimension is not too large.

Choosing a propagation method

Method	Main advantage	Main limitation
Global linearisation	very fast	may underestimate uncertainty
Local linearisation	good compromise	still approximate
Direct method	accurate in principle	many optimisations
Sample-based	practical with simulations	curse of dimensionality

Practical rule

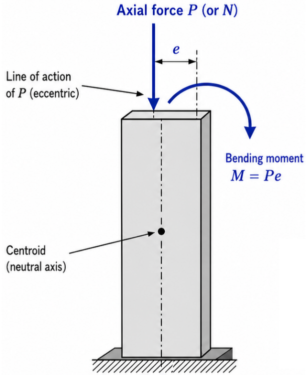
Start with global or local linearisation; use direct or sample-based methods, if computationally feasible, to check important risk estimates.

Example: short-column design

- A short column has rectangular cross-section with width b and depth h. It is subject to axial force and bending moment.
- The limit-state function is

$$g(\mathbf{X}) = 1 - \frac{4X_2}{bh^2 e^{X_3}} - \frac{X_1^2}{b^2 h^2 e^{2X_3}},$$

  where
  - X_1 = axial force P,
  - X_2 = the bending moment M ,
  - X_3 = logarithm of yield stress S,
- The system is safe iff $g(\mathbf{X}) \geq 0$.



Aim

Estimate the plausibility of non-compliance when both variability and imprecision affect the input variables.

Short column design: probabilistic analysis

- In [Kuschel & Rackwitz, *Mathematical Methods of Operations Research* **46**(3), 1997], the authors assume
$$P \sim \mathcal{N}(500, 100^2), \qquad M \sim \mathcal{N}(2000, 400^2),$$
and S is lognormal with mean 5 and standard deviation 0.5.
- Design problem:
$$\min_{b,h} bh$$
subject to
$$5 \leq b \leq 15, \quad 15 \leq h \leq 25,$$
$$P(g(\mathbf{X}) < 0) \leq 0.00621.$$
- Solution: $(b, h) = (8.668, 25.0)$.
- Impact of epistemic uncertainties?

RFS input model

The input vector is represented by

$$\widetilde{\mathbf{X}} \sim \widetilde{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \mathbf{H}),$$

with

$$\boldsymbol{\mu} = \left(500,\ 2000,\ \ln(5) - \frac{1}{2}\ln(1.01)\right)^T,$$

$$\boldsymbol{\Sigma} = \begin{pmatrix} 10000 & 20000 & 0 \\ 20000 & 160000 & 0 \\ 0 & 0 & \ln(1.01) \end{pmatrix},$$

$$\mathbf{H} = \begin{pmatrix} h_1 & 0 & 0 \\ 0 & h_2 & 0 \\ 0 & 0 & h_3 \end{pmatrix},$$

where $1/\sqrt{h_i}$ is chosen as a percentage of the corresponding mean (5% or 10%).

Non-compliance plausibility

For given design dimensions, let $R_{\mathrm{N}C}$ denote the non-compliance region in the output space.

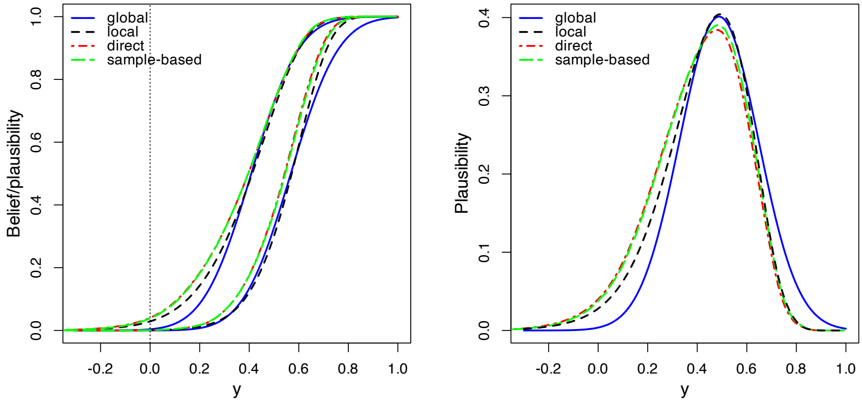
$$\mathrm{pl}_0 = \mathrm{Pl}_{\widetilde{\mathbf{Y}}}(R_{\mathrm{N}C}).$$

- In a purely probabilistic analysis, one estimates a probability of failure.
- In a RFS analysis, one estimates a plausibility of failure.
- This is an upper probability that accounts for epistemic imprecision.

Interpretation

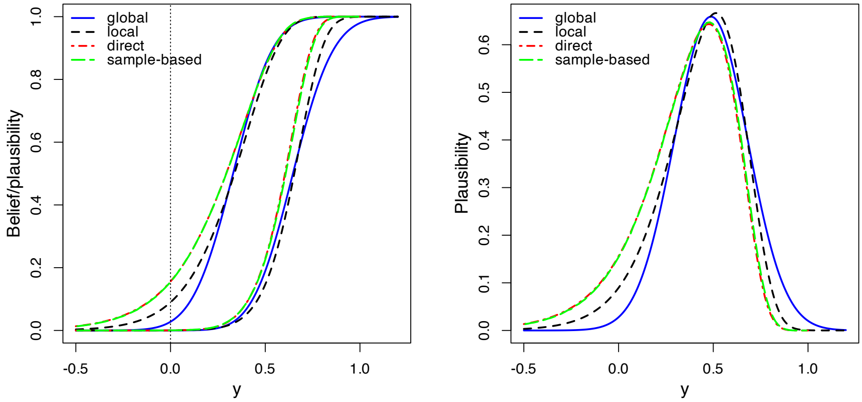
A small value of pl_0 means that failure is not only unlikely, but also implausible under the available information.

Estimated CDFs and contour functions ($x = 5$)



Output lower and upper CDFs (a) and contour function (b) for $(b, h) = (8.668, 25.0)$, estimated by the global and local linearisation, direct, and sample-based methods, for $x = 5$.

Estimated CDFs and contour functions ($x = 10$)



Output lower and upper CDFs (a) and contour function (b) for $(b, h) = (8.668, 25.0)$, estimated by the global and local linearisation, direct, and sample-based methods, for $x = 10$.

Comparison of RFS propagation methods

Imprecision	Quantity	Global	Local	Direct	Sample
5%	time (s)	0	0.023	45.717	1.065
5%	$\widehat{\mathrm{pl}}_0$	0.00375	0.0289	0.0404	0.0387
5%	error (%)	90.7	28.4	0	4.17
10%	time (s)	0	0.022	50.511	1.171
10%	$\widehat{\mathrm{pl}}_0$	0.0278	0.0898	0.156	0.157
10%	error (%)	82.2	42.4	0	0.563

- Global linearisation severely underestimates plausibility.
- Sample-based propagation is close to the direct method here.

What the example shows

- Increasing epistemic imprecision increases the plausibility of non-compliance.
- The global linear approximation can be overconfident for nonlinear models.
- Local linearisation is cheap and useful, but not always accurate enough.
- The sample-based method gives a practical alternative to repeated constrained optimisation.

Design interpretation

A robust design problem can be formulated as

$$\min_{b,h} \; bh$$

subject to

$$\mathrm{Pl}_{\widetilde{\mathbf{Y}}}(R_{\mathrm{N}C}) \leq \eta.$$

- bh measures material or cost.
- η is an admissible plausibility of non-compliance.
- The constraint is stronger than a purely probabilistic failure constraint when epistemic uncertainty is present.

Take-home messages

1. Random fuzzy sets generalise random variables, random sets and possibility distributions.
2. They represent aleatory uncertainty through random focal elements and epistemic uncertainty through fuzziness.
3. Gaussian random fuzzy numbers and vectors provide tractable continuous models.
4. Propagation can be done by linearisation, optimisation or sampling.
5. Output uncertainty is expressed by belief, plausibility, contours and lower/upper distribution functions.

Final perspective

From

"What is the probability of failure?"

to

"How plausible is failure, given aleatory and epistemic uncertainties?"

Seminar message

Random fuzzy sets provide a unified and operational framework for propagating mixed uncertainty through numerical models.

References

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